

## ON TWO IDENTITIES FOR I-FUNCTION

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ABSTRACT. In this research note, two interesting identities involving I-function of one variable introduced by Rathie have been derived. These results enable us to split a particular I-function into the sum of four I-functions. A few new as well as known special cases of our main results have been obtained.

Keywords: I-function, Mellin-Barnes integral.

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### 1. INTRODUCTION

The I-function introduced by A.K.Rathie[3] is defined and represented by the following Mellin Barnes type contour integral:

$$\begin{aligned} I_{p,q}^{m,n}(z) &\equiv I_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_1, e_1, A_1), \dots, (a_p, e_p, A_p) \\ (b_1, f_1, B_1), \dots, (b_q, f_q, B_q) \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \theta(s) z^s ds \end{aligned} \tag{1}$$

where

$$\theta(s) = \frac{\prod_{j=1}^m \Gamma^{B_j}(b_j - f_j s) \prod_{j=1}^n \Gamma^{A_j}(1 - a_j + e_j s)}{\prod_{j=m+1}^q \Gamma^{B_j}(1 - b_j + f_j s) \prod_{j=n+1}^p \Gamma^{A_j}(a_j - e_j s)} \tag{2}$$

Also

- (i)  $i = \sqrt{-1}$ ;
- (ii)  $z \neq 0$ ;
- (iii)  $m, n, p, q$  are integers satisfying  $0 \leq m \leq q, 0 \leq n \leq p$ ;
- (iv)  $\mathcal{L}$  is a suitable contour in the complex plane;
- (v) an empty product is to be interpreted as unity;
- (vi)  $e_j, j = 1, \dots, p; f_j, j = 1, \dots, q; A_j, j = 1, \dots, p; \text{ and } B_j, j = 1, \dots, q$  are positive numbers;

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- (vii)  $a_j, j = 1, \dots, p$  and  $b_j, j = 1, \dots, q$  are complex numbers such that no singularity of  $\Gamma^{B_j}(b_j - f_j s), j = 1, \dots, m$ , coincides with any singularity of  $\Gamma^{A_j}(1 - a_j + e_j s), j = 1, \dots, n$ . In general these singularities are not poles.
- (viii) The contour  $\mathcal{L}$  goes from  $\sigma - i\infty$  to  $\sigma + i\infty$  ( $\sigma$  real) so that all the singularities of  $\Gamma^{B_j}(b_j - f_j s), j = 1, \dots, m$ , lie to the right of  $\mathcal{L}$ , and all the singularities of  $\Gamma^{A_j}(1 - a_j + e_j s), j = 1, \dots, n$ , lie to the left of  $\mathcal{L}$ .

In short, (1) will be denoted by

$$I_{p,q}^{m,n} \left[ z \left| \begin{matrix} {}_1(a_j, e_j, A_j)_p \\ {}_1(b_j, f_j, B_j)_q \end{matrix} \right. \right]$$

The function defined by (1) is convergent if

$$\Delta > 0, \quad |arg(z)| < \frac{1}{2}\Delta\pi, \tag{3}$$

where

$$\Delta = \sum_{j=1}^m B_j f_j - \sum_{j=m+1}^q B_j f_j + \sum_{j=1}^n A_j e_j - \sum_{j=n+1}^p A_j e_j. \tag{4}$$

When  $A_1 = A_2 = \dots = A_p = 1 = B_1 = B_2 = \dots = B_q$ , (1) reduces to the H-function introduced by Fox[2] and studied by Braaksma[1].

## 2. MAIN RESULTS

The identities for the I-function to be established in this note are the following.

### Result 1.

$$\begin{aligned} & (2\pi i) I_{p+2, q+2}^{m+1, n+1} \left[ z \left| \begin{matrix} (\beta, \delta, 1), {}_1(a_j, e_j, A_j)_p, (\alpha, \lambda, 1) \\ (\beta, \delta, 1), {}_1(b_j, f_j, B_j)_q, (\alpha, \lambda, 1) \end{matrix} \right. \right] \\ &= e^{i\pi(\alpha+\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{-i\pi(\lambda+\delta)} \left| \begin{matrix} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{matrix} \right. \right] \\ &+ e^{i\pi(\alpha-\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{-i\pi(\lambda-\delta)} \left| \begin{matrix} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{matrix} \right. \right] \\ &- e^{-i\pi(\alpha-\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{i\pi(\lambda-\delta)} \left| \begin{matrix} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{matrix} \right. \right] \\ &- e^{-i\pi(\alpha+\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{i\pi(\lambda+\delta)} \left| \begin{matrix} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{matrix} \right. \right] \end{aligned} \tag{5}$$

*Proof.* In order to establish the identity (5), we proceed as follows.

Denoting the left-hand of (5) by S, expressing the I-function with the help of its definition we have,

$$S = (2\pi i) \frac{1}{2\pi i} \int_L \theta(s) z^s \frac{\Gamma(\beta - \delta s) \Gamma(1 - \beta + \delta s)}{\Gamma(\alpha - \lambda s) \Gamma(1 - \alpha + \lambda s)} ds \tag{6}$$

where  $\theta(s)$  is given by (2).

Using the result

$$\Gamma(\beta - \delta s) \Gamma(1 - \beta + \delta s) = 2\pi \frac{\Gamma(2\beta - 2\delta s) \Gamma(1 - 2\beta + 2\delta s)}{\Gamma(\frac{1}{2} + \beta - \delta s) \Gamma(\frac{1}{2} - \beta + \delta s)} \tag{7}$$

(6) can be written as

$$S = \int_L \theta(s) z^s \frac{\Gamma(2\beta - 2\delta s) \Gamma(1 - 2\beta + 2\delta s) \Gamma(\frac{1}{2} + \alpha - \lambda s) \Gamma(\frac{1}{2} - \alpha + \lambda s)}{\Gamma(\frac{1}{2} + \beta - \delta s) \Gamma(\frac{1}{2} - \beta - \delta s) \Gamma(2\alpha - 2\lambda s) \Gamma(1 - 2\alpha + 2\lambda s)} ds \tag{8}$$

Using the results

$$\cos\pi z = \frac{\pi}{\Gamma\left(\frac{1}{2} - z\right) \Gamma\left(\frac{1}{2} + z\right)} = \frac{e^{i\pi z} + e^{-i\pi z}}{2} \quad (9)$$

and

$$\sin\pi z = \frac{\pi}{\Gamma(z)\Gamma(1-z)} = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \quad (10)$$

and after some algebra, we have

$$\begin{aligned} S &= \frac{1}{2\pi i} \int_L \theta(s) z^s \Gamma(2\beta - 2\delta s) \Gamma(1 - 2\beta + 2\delta s) \\ &\quad \cdot \left( e^{i\pi(\alpha - \lambda s)} - e^{-i\pi(\alpha - \lambda s)} \right) \left( e^{i\pi(\beta - \delta s)} + e^{-i\pi(\beta - \delta s)} \right) ds \\ &= \frac{1}{2\pi i} \int_L \theta(s) z^s \Gamma(2\beta - 2\delta s) \Gamma(1 - 2\beta + 2\delta s) \\ &\quad \cdot \left\{ e^{i\pi(\alpha + \beta - \lambda s - \delta s)} + e^{i\pi(\alpha - \beta - \lambda s + \delta s)} \right. \\ &\quad \left. - e^{-i\pi(\alpha - \beta - \lambda s + \delta s)} - e^{-i\pi(\alpha + \beta - \lambda s - \delta s)} \right\} ds \end{aligned} \quad (11)$$

Now, breaking in to four parts and after some simplification, using the definition of I-function, we easily arrive at the right-hand side of (5).

This completes the proof of the identity (5).  $\square$

## Result 2.

$$\begin{aligned} & \mathbb{I}_{p+2, q+2}^{m+1, n+1} \left[ z \left| \begin{array}{l} (\beta, \delta, A), {}_1(a_j, e_j, A_j)_p, (\alpha, \lambda, A) \\ (\beta, \delta, A), {}_1(b_j, f_j, B_j)_q, (\alpha, \lambda, A) \end{array} \right. \right] \\ &= \mathbb{I}_{p+4, q+4}^{m+2, n+2} \left[ z \left| \begin{array}{l} (2\beta, 2\delta, A), \left(\frac{1}{2} + \alpha, \lambda, A\right), {}_1(a_j, e_j, A_j)_p, (2\alpha, 2\lambda, A), \left(\frac{1}{2} + \beta, \delta, A\right) \\ (2\beta, 2\delta, A), \left(\frac{1}{2} + \alpha, \lambda, A\right), {}_1(b_j, f_j, B_j)_q, (2\alpha, 2\lambda, A), \left(\frac{1}{2} + \beta, \delta, A\right) \end{array} \right. \right] \end{aligned} \quad (12)$$

*Proof.* In order to establish the identity (12), we proceed as follows.

Denoting the left-hand of (12) by S, expressing the I-function with the help of its definition we have,

$$S = \frac{1}{2\pi i} \int_L \theta(s) z^s \frac{\Gamma^A(\beta - \delta s) \Gamma^A(1 - \beta + \delta s)}{\Gamma^A(\alpha - \lambda s) \Gamma^A(1 - \alpha + \lambda s)} ds \quad (13)$$

Using the result (7) and after some algebra, we have

$$\begin{aligned} S &= \frac{1}{2\pi i} \int_L \left\{ \theta(s) z^s \frac{\Gamma^A(1 - 2\beta + 2\delta s) \Gamma^A(2\beta - 2\delta s)}{\Gamma^A\left(\frac{1}{2} + \beta - \delta s\right) \Gamma^A\left(\frac{1}{2} - \beta + \delta s\right)} \right. \\ &\quad \left. \times \frac{\Gamma^A\left(\frac{1}{2} - \alpha + \lambda s\right) \Gamma^A\left(\frac{1}{2} + \alpha - \lambda s\right)}{\Gamma^A(2\alpha - 2\lambda s) \Gamma^A(1 - 2\alpha + 2\lambda s)} \right\} ds \end{aligned} \quad (14)$$

After some simplification, using the definition of I-function, we easily arrive at the right-hand side of (12).

This completes the proof of the identity (12).  $\square$

3. SPECIAL CASES

(a) In (5), if we take  $\delta = 0$ , we get, after some simplification,

$$\begin{aligned} & \mathbb{I}_{p+1, q+1}^{m, n} \left[ z \left| \begin{array}{l} {}_1(a_j, e_j, A_j)_p, (\alpha, \lambda, 1) \\ {}_1(b_j, f_j, B_j)_q, (\alpha, \lambda, 1) \end{array} \right. \right] \\ &= \frac{1}{2\pi i} \left\{ e^{i\pi\alpha} \mathbb{I}_{p, q}^{m, n} \left[ ze^{-i\pi\lambda} \left| \begin{array}{l} {}_1(a_j, e_j, A_j)_p \\ {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \right. \\ & \quad \left. - e^{-i\pi\alpha} \mathbb{I}_{p, q}^{m, n} \left[ ze^{i\pi\lambda} \left| \begin{array}{l} {}_1(a_j, e_j, A_j)_p \\ {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \right\} \end{aligned} \tag{15}$$

Further in (15), if we take  $A_j = 1(j = 1, \dots, p)$  and  $B_j = 1(j = 1, \dots, q)$ , it reduces to the H-function identity obtained by Rathie[5].

(b) In (5), if we take  $\lambda = 0$ , we get, after some simplification,

$$\begin{aligned} & \mathbb{I}_{p+1, q+1}^{m+1, n+1} \left[ z \left| \begin{array}{l} (\alpha, \lambda, 1), {}_1(a_j, e_j, A_j)_p \\ (\alpha, \lambda, 1), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \\ &= e^{i\pi\alpha} \mathbb{I}_{p+1, q+1}^{m+1, n+1} \left[ ze^{-i\pi\lambda} \left| \begin{array}{l} (2\alpha, 2\lambda, 1), {}_1(a_j, e_j, A_j)_p \\ (2\alpha, 2\lambda, 1), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \\ & \quad + e^{-i\pi\alpha} \mathbb{I}_{p+1, q+1}^{m+1, n+1} \left[ ze^{i\pi\lambda} \left| \begin{array}{l} (2\alpha, 2\lambda, 1), {}_1(a_j, e_j, A_j)_p \\ (2\alpha, 2\lambda, 1), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \end{aligned} \tag{16}$$

Further in (16), if we take  $A_j = 1(j = 1, \dots, p)$  and  $B_j = 1(j = 1, \dots, q)$ , it reduces to the H-function identity obtained recently by Rathie et al.[6].

(c) In (5), if we take  $A_j = 1(j = 1, \dots, p)$  and  $B_j = 1(j = 1, \dots, q)$ , it reduces to the H-function identity obtained recently by Rathie[4].

(d) In (12), if we take  $\delta = 0$  we get

$$\begin{aligned} & \mathbb{I}_{p+1, q+1}^{m, n} \left[ z \left| \begin{array}{l} {}_1(a_j, e_j, A_j)_p, (\alpha, \lambda, A) \\ {}_1(b_j, f_j, B_j)_q, (\alpha, \lambda, A) \end{array} \right. \right] \\ &= \frac{1}{(2\pi)^A} \mathbb{I}_{p+2, q+2}^{m+1, n+1} \left[ z \left| \begin{array}{l} (\frac{1}{2} + \alpha, \lambda, A), {}_1(a_j, e_j, A_j)_p, (2\alpha, 2\lambda, A) \\ (\frac{1}{2} + \alpha, \lambda, A), {}_1(b_j, f_j, B_j)_q, (2\alpha, 2\lambda, A) \end{array} \right. \right] \end{aligned} \tag{17}$$

In (17), if we take  $A_j = 1(j = 1, \dots, p)$ ,  $B_j = 1(j = 1, \dots, q)$  and  $A = 1$ , it reduces to the H-function identity obtained by Rathie[4].

(e) In (12), if we take  $\lambda = 0$ , we get

$$\begin{aligned} & \mathbb{I}_{p+1, q+1}^{m, n} \left[ z \left| \begin{array}{l} (\beta, \delta, A), {}_1(a_j, e_j, A_j)_p \\ (\beta, \delta, A), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \\ &= (2\pi)^A \mathbb{I}_{p+2, q+2}^{m+1, n+1} \left[ z \left| \begin{array}{l} (2\beta, 2\delta, A), {}_1(a_j, e_j, A_j)_p, (\frac{1}{2} + \beta, \delta, A) \\ (2\beta, 2\delta, A), {}_1(b_j, f_j, B_j)_q, (\frac{1}{2} + \beta, \delta, A) \end{array} \right. \right] \end{aligned} \tag{18}$$

In (18), if we take  $A_j = 1(j = 1, \dots, p)$ ,  $B_j = 1(j = 1, \dots, q)$  and  $A = 1$ , it reduces to the H-function identity obtained by Rathie[4].

(f) In (12), if we take  $A_j = 1(j = 1, \dots, p)$ ,  $B_j = 1(j = 1, \dots, q)$  and  $A = 1$ , it reduces to the H-function identity obtained by Rathie[4].

(g) In the LHS of (12), if we put  $A=1$  and multiply by  $2\pi i$  and equate with the LHS of (5), we get an interesting result as below.

$$\begin{aligned}
 (2\pi i) I_{p+4, q+4}^{m+2, n+2} \left[ z \left| \begin{array}{l} (2\beta, 2\delta, 1), \left(\frac{1}{2} + \alpha, \lambda, 1\right), {}_1(a_j, e_j, A_j)_p, (2\alpha, 2\lambda, 1), \left(\frac{1}{2} + \beta, \delta, 1\right) \\ (2\beta, 2\delta, 1), \left(\frac{1}{2} + \alpha, \lambda, 1\right), {}_1(b_j, f_j, B_j)_q, (2\alpha, 2\lambda, 1), \left(\frac{1}{2} + \beta, \delta, 1\right) \end{array} \right. \right] \\
 = e^{i\pi(\alpha+\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{-i\pi(\lambda+\delta)} \left| \begin{array}{l} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \\
 + e^{i\pi(\alpha-\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{-i\pi(\lambda-\delta)} \left| \begin{array}{l} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \\
 - e^{-i\pi(\alpha-\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{i\pi(\lambda-\delta)} \left| \begin{array}{l} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \\
 - e^{-i\pi(\alpha+\beta)} I_{p+1, q+1}^{m+1, n+1} \left[ ze^{i\pi(\lambda+\delta)} \left| \begin{array}{l} (2\beta, 2\delta, 1), {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1), {}_1(b_j, f_j, B_j)_q \end{array} \right. \right] \quad (19)
 \end{aligned}$$

#### 4. ANOTHER PROOF OF (19)

Denoting the left-hand of (19) by  $S$ , expressing the I-function with the help of its definition we have,

$$S = (2\pi i) \frac{1}{2\pi i} \int_L \theta(s) z^s \frac{\Gamma(1-2\beta+2\delta s) \Gamma(\frac{1}{2}-\alpha+\lambda s) \Gamma(2\beta-2\delta s) \Gamma(\frac{1}{2}+\alpha-\lambda s)}{\Gamma(2\alpha-2\lambda s) \Gamma(\frac{1}{2}+\beta-\delta s) \Gamma(1-2\alpha+2\lambda s) \Gamma(\frac{1}{2}-\beta+\delta s)} ds \quad (20)$$

Using the results (7), (9), (10) and after some algebra, we have

$$\begin{aligned}
 S &= \frac{1}{2\pi i} \int_L \theta(s) z^s \Gamma(2\beta-2\delta s) \Gamma(1-2\beta+2\delta s) \\
 &\quad \cdot \left( e^{i\pi(\alpha-\lambda s)} - e^{-i\pi(\alpha-\lambda s)} \right) \left( e^{i\pi(\beta-\delta s)} + e^{-i\pi(\beta-\delta s)} \right) ds \\
 &= \frac{1}{2\pi i} \int_L \left\{ \theta(s) z^s \Gamma(2\beta-2\delta s) \Gamma(1-2\beta+2\delta s) \right. \\
 &\quad \cdot \left. \left\{ e^{i\pi(\alpha+\beta-\lambda s-\delta s)} + e^{i\pi(\alpha-\beta-\lambda s+\delta s)} \right. \right. \\
 &\quad \left. \left. - e^{-i\pi(\alpha-\beta-\lambda s+\delta s)} - e^{-i\pi(\alpha+\beta-\lambda s-\delta s)} \right\} \right\} ds \quad (21)
 \end{aligned}$$

Now, breaking in to four parts and after some simplification, using the definition of I-function, we easily arrive at the right-hand side of (19).

Since I-function is the most generalized function among the functions of one variable studied so far, so by specializing the paramaters therein it reduces to H-function, G-function, Generalized Hypergeometric function  ${}_pF_q$  and other elementary functions and hence we can obtain corresponding results. However we do not mention here due to lack of space.

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