TWMS J. App. Eng. Math. V.10, N.2, 2020, pp. 428-433

ON TWO IDENTITIES FOR I-FUNCTION

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ABSTRACT. In this research note, two interesting identities involving I-function of one variable introduced by Rathie have been derived. These results enable us to split a particular I-function into the sum of four I-functions. A few new as well as known special cases of our main results have been obtained.

Keywords: I-function, Mellin-Barnes integral.

AMS Subject Classification: 33C60

1. Introduction

The I-function introduced by A.K.Rathie[3] is defined and represented by the following Mellin Barnes type contour integral:

$$\mathrm{I}_{\mathrm{p,q}}^{\mathrm{m,n}}(z) \equiv \mathrm{I}_{\mathrm{p,q}}^{\mathrm{m,n}}\left[z \;\middle|\; \begin{matrix} (a_1, e_1, A_1), \ldots, (a_p, e_p, A_p)) \\ (b_1, f_1, B_1), \ldots, (b_q, f_q, B_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \theta(s) z^s ds \tag{1}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma^{B_j}(b_j - f_j s) \prod_{j=1}^{n} \Gamma^{A_j}(1 - a_j + e_j s)}{\prod_{j=m+1}^{q} \Gamma^{B_j}(1 - b_j + f_j s) \prod_{j=n+1}^{p} \Gamma^{A_j}(a_j - e_j s)} \tag{2}$$

Also

(i) $i = \sqrt{-1}$;
(ii) $z \neq 0$;
(iii) m, n, p, q are integers satisfying $0 \leq m \leq q$, $0 \leq n \leq p$;
(iv) $\mathcal{L}$ is a suitable contour in the complex plane;
(v) an empty product is to be interpreted as unity;
(vi) e_j, $j = 1, \ldots, p$; f_j, $j = 1, \ldots, q$; A_j, $j = 1, \ldots, p$; and B_j, $j = 1, \ldots, q$ are positive numbers;

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§ Manuscript received: August 23, 2018; accepted: October 2, 2018.

(vii) a_j, $j = 1,\ldots,p$ and b_j, $j = 1,\ldots,q$ are complex numbers such that no singularity of $\Gamma^{B_j}(b_j - f_j s)$, $j = 1,\ldots,m$, coincides with any singularity of $\Gamma^{A_j}(1 - a_j + e_j s)$, $j = 1,\ldots,n$. In general these singularities are not poles.

(viii) The contour $\mathcal{L}$ goes from $\sigma - i\infty$ to $\sigma + i\infty$ (σ real) so that all the singularities of $\Gamma^{B_j}(b_j - f_j s)$, $j = 1,\ldots,m$, lie to the right of $\mathcal{L}$, and all the singularities of $\Gamma^{A_j}(1 - a_j + e_j s)$, $j = 1,\ldots,n$, lie to the left of $\mathcal{L}$.

In short, (1) will be denoted by

$$\mathrm{I}^{\,\mathrm{m,\,n}}_{\mathrm{p,\,q}}\left[\, z \;\middle|\; \begin{matrix} {}_1(a_j, e_j, A_j)_p \\ {}_1(b_j, f_j, B_j)_q \end{matrix} \right]$$

The function defined by (1) is convergent if

$$\Delta > 0, \quad |arg(z)| < \frac{1}{2}\Delta\pi, \tag{3}$$

where

$$\Delta = \sum_{j=1}^{m} B_j f_j - \sum_{j=m+1}^{q} B_j f_j + \sum_{j=1}^{n} A_j e_j - \sum_{j=n+1}^{p} A_j e_j. \tag{4}$$

When $A_1 = A_2 = \cdots = A_p = 1 = B_1 = B_2 = \cdots = B_q$, (1) reduces to the H-function introduced by Fox[2] and studied by Braaksma[1].

2. Main Results

The identities for the I-function to be established in this note are the following.

Result 1.

$$\begin{aligned}
&(2\pi i)\, \mathrm{I}^{\,m+1,\, n+1}_{p+2,\, q+2}\left[z \;\middle|\; \begin{matrix} (\beta,\delta,1),\ {}_1(a_j, e_j, A_j)_p, (\alpha,\lambda,1) \\ (\beta,\delta,1),\ {}_1(b_j, f_j, B_j)_q, (\alpha,\lambda,1) \end{matrix} \right] \\
&= e^{i\pi(\alpha+\beta)}\, \mathrm{I}^{\,m+1,\, n+1}_{p+1,\, q+1}\left[z e^{-i\pi(\lambda+\delta)} \;\middle|\; \begin{matrix} (2\beta, 2\delta, 1),\ {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1),\ {}_1(b_j, f_j, B_j)_q \end{matrix} \right] \\
&+ e^{i\pi(\alpha-\beta)}\, \mathrm{I}^{\,m+1,\, n+1}_{p+1,\, q+1}\left[z e^{-i\pi(\lambda-\delta)} \;\middle|\; \begin{matrix} (2\beta, 2\delta, 1),\ {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1),\ {}_1(b_j, f_j, B_j)_q \end{matrix} \right] \\
&- e^{-i\pi(\alpha-\beta)}\, \mathrm{I}^{\,m+1,\, n+1}_{p+1,\, q+1}\left[z e^{i\pi(\lambda-\delta)} \;\middle|\; \begin{matrix} (2\beta, 2\delta, 1),\ {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1),\ {}_1(b_j, f_j, B_j)_q \end{matrix} \right] \\
&- e^{-i\pi(\alpha+\beta)}\, \mathrm{I}^{\,m+1,\, n+1}_{p+1,\, q+1}\left[z e^{i\pi(\lambda+\delta)} \;\middle|\; \begin{matrix} (2\beta, 2\delta, 1),\ {}_1(a_j, e_j, A_j)_p \\ (2\beta, 2\delta, 1),\ {}_1(b_j, f_j, B_j)_q \end{matrix} \right]
\end{aligned} \tag{5}$$

Proof. In order to establish the identity (5), we proceed as follows.
Denoting the left-hand of (5) by S, expressing the I-function with the help of its definition we have,

$$S = (2\pi i)\, \frac{1}{2\pi i} \int_L \theta(s)\, z^s\, \frac{\Gamma(\beta - \delta s)\, \Gamma(1 - \beta + \delta s)}{\Gamma(\alpha - \lambda s)\, \Gamma(1 - \alpha + \lambda s)}\, ds \tag{6}$$

where $\theta(s)$ is given by (2).
Using the result

$$\Gamma(\beta - \delta s)\, \Gamma(1 - \beta + \delta s) = 2\pi\, \frac{\Gamma(2\beta - 2\delta s)\, \Gamma(1 - 2\beta + 2\delta s)}{\Gamma(\frac{1}{2} + \beta - \delta s)\, \Gamma(\frac{1}{2} - \beta + \delta s)} \tag{7}$$

(6) can be written as

$$S = \int_L \theta(s)\, z^s\, \frac{\Gamma(2\beta - 2\delta s)\, \Gamma(1 - 2\beta + 2\delta s)\, \Gamma(\frac{1}{2} + \alpha - \lambda s)\, \Gamma(\frac{1}{2} - \alpha + \lambda s)}{\Gamma(\frac{1}{2} + \beta - \delta s)\, \Gamma(\frac{1}{2} - \beta - \delta s)\, \Gamma(2\alpha - 2\lambda s)\, \Gamma(1 - 2\alpha + 2\lambda s)}\, ds \tag{8}$$

Using the results

$$cos\pi z = \frac{\pi}{\Gamma\left(\frac{1}{2}-z\right)\ \Gamma\left(\frac{1}{2}+z\right)} = \frac{e^{i\pi z}+e^{-i\pi z}}{2} \tag{9}$$

and

$$sin\pi z = \frac{\pi}{\Gamma(z)\Gamma(1-z)} = \frac{e^{i\pi z}-e^{-i\pi z}}{2i} \tag{10}$$

and after some algebra, we have

$$\begin{aligned} S &= \frac{1}{2\pi i}\int_L \theta(s)\ z^s\ \Gamma(2\beta-2\delta s)\ \Gamma(1-2\beta+2\delta s) \\ &\quad .\left(e^{i\pi(\alpha-\lambda s)}-e^{-i\pi(\alpha-\lambda s)}\right)\ \left(e^{i\pi(\beta-\delta s)}+e^{-i\pi(\beta-\delta s)}\right) ds \\ &= \frac{1}{2\pi i}\int_L \theta(s)\ z^s\ \Gamma(2\beta-2\delta s)\ \Gamma(1-2\beta+2\delta s) \\ &\quad .\Big\{e^{i\pi(\alpha+\beta-\lambda s-\delta s)}+e^{i\pi(\alpha-\beta-\lambda s+\delta s)} \\ &\quad -e^{-i\pi(\alpha-\beta-\lambda s+\delta s)}-e^{-i\pi(\alpha+\beta-\lambda s-\delta s)}\Big\}\ ds \end{aligned} \tag{11}$$

Now, breaking in to four parts and after some simplification, using the definition of I-function, we easily arrive at the right-hand side of (5).
This completes the proof of the identity (5). □

Result 2.

$$\begin{aligned} &\mathrm{I}^{\,m+1,\,n+1}_{\,p+2,\,q+2}\left[z\,\middle|\,\begin{matrix}(\beta,\delta,A),\ {}_1(a_j,e_j,A_j)_p,(\alpha,\lambda,A)\\(\beta,\delta,A),\ {}_1(b_j,f_j,B_j)_q,(\alpha,\lambda,A)\end{matrix}\right] \\ &= \mathrm{I}^{\,m+2,\,n+2}_{\,p+4,\,q+4}\left[z\,\middle|\,\begin{matrix}(2\beta,2\delta,A),\left(\frac{1}{2}+\alpha,\lambda,A\right),{}_1(a_j,e_j,A_j)_p,(2\alpha,2\lambda,A),\left(\frac{1}{2}+\beta,\delta,A\right)\\(2\beta,2\delta,A),\ \left(\frac{1}{2}+\alpha,\lambda,A\right),{}_1(b_j,f_j,B_j)_q,(2\alpha,2\lambda,A),\ \left(\frac{1}{2}+\beta,\delta,A\right)\end{matrix}\right] \end{aligned} \tag{12}$$

Proof. In order to establish the identity (12), we proceed as follows.
Denoting the left-hand of (12) by S, expressing the I-function with the help of its definition we have,

$$S = \frac{1}{2\pi i}\int_L \theta(s)\ z^s\ \frac{\Gamma^A(\beta-\delta s)\ \Gamma^A(1-\beta+\delta s)}{\Gamma^A(\alpha-\lambda s)\ \Gamma^A(1-\alpha+\lambda s)}\ ds \tag{13}$$

Using the result (7) and after some algebra, we have

$$\begin{aligned} S = \frac{1}{2\pi i}\int_L \Bigg\{\theta(s)\ z^s\ &\frac{\Gamma^A(1-2\beta+2\delta s)\ \Gamma^A(2\beta-2\delta s)}{\Gamma^A(\frac{1}{2}+\beta-\delta s)\ \Gamma^A(\frac{1}{2}-\beta+\delta s)} \\ &\times\ \frac{\Gamma^A(\frac{1}{2}-\alpha+\lambda s)\ \Gamma^A(\frac{1}{2}+\alpha-\lambda s)}{\Gamma^A(2\alpha-2\lambda s)\ \Gamma^A(1-2\alpha+2\lambda s)}\Bigg\}\ ds \end{aligned} \tag{14}$$

After some simplification, using the definition of I-function, we easily arrive at the right-hand side of (12).
This completes the proof of the identity (12). □

3. Special Cases

(a) In (5), if we take $\delta = 0$, we get, after some simplification,

$$\mathrm{I}^{\,m,\,n}_{\,p+1,\,q+1}\left[z\left|\begin{array}{l} {}_1(a_j,e_j,A_j)_p,(\alpha,\lambda,1)\\ {}_1(b_j,f_j,B_j)_q,(\alpha,\lambda,1)\end{array}\right.\right] = \frac{1}{2\pi i}\left\{e^{i\pi\alpha}\,\mathrm{I}^{\,m,\,n}_{\,p,\,q}\left[ze^{-i\pi\lambda}\left|\begin{array}{l} {}_1(a_j,e_j,A_j)_p\\ {}_1(b_j,f_j,B_j)_q\end{array}\right.\right] - e^{-i\pi\alpha}\,\mathrm{I}^{\,m,\,n}_{\,p,\,q}\left[ze^{i\pi\lambda}\left|\begin{array}{l} {}_1(a_j,e_j,A_j)_p\\ {}_1(b_j,f_j,B_j)_q\end{array}\right.\right]\right\} \tag{15}$$

Further in (15), if we take $A_j = 1(j = 1,\ldots,p)$ and $B_j = 1(j = 1,\ldots,q)$, it reduces to the H-function identity obtained by Rathie[5].

(b) In (5), if we take $\lambda = 0$, we get, after some simplification,

$$\mathrm{I}^{\,m+1,\,n+1}_{\,p+1,\,q+1}\left[z\left|\begin{array}{l} (\alpha,\lambda,1),\ {}_1(a_j,e_j,A_j)_p\\ (\alpha,\lambda,1),\ {}_1(b_j,f_j,B_j)_q\end{array}\right.\right] = e^{i\pi\alpha}\,\mathrm{I}^{\,m+1,\,n+1}_{\,p+1,\,q+1}\left[ze^{-i\pi\lambda}\left|\begin{array}{l} (2\alpha,2\lambda,1),\ {}_1(a_j,e_j,A_j)_p\\ (2\alpha,2\lambda,1),\ {}_1(b_j,f_j,B_j)_q\end{array}\right.\right] + e^{-i\pi\alpha}\,\mathrm{I}^{\,m+1,\,n+1}_{\,p+1,\,q+1}\left[ze^{i\pi\lambda}\left|\begin{array}{l} (2\alpha,2\lambda,1),\ {}_1(a_j,e_j,A_j)_p\\ (2\alpha,2\lambda,1),\ {}_1(b_j,f_j,B_j)_q\end{array}\right.\right] \tag{16}$$

Further in (16), if we take $A_j = 1(j = 1,\ldots,p)$ and $B_j = 1(j = 1,\ldots,q)$, it reduces to the H-function identity obtained recently by Rathie et al.[6].

(c) In (5), if we take $A_j = 1(j = 1,\ldots,p)$ and $B_j = 1(j = 1,\ldots,q)$, it reduces to the H-function identity obtained recently by Rathie[4].

(d) In (12), if we take $\delta = 0$ we get

$$\mathrm{I}^{\,m,\,n}_{\,p+1,\,q+1}\left[z\left|\begin{array}{l} {}_1(a_j,e_j,A_j)_p,(\alpha,\lambda,A)\\ {}_1(b_j,f_j,B_j)_q,(\alpha,\lambda,A)\end{array}\right.\right] = \frac{1}{(2\pi)^A}\,\mathrm{I}^{\,m+1,\,n+1}_{\,p+2,\,q+2}\left[z\left|\begin{array}{l} \left(\frac{1}{2}+\alpha,\lambda,A\right),\ {}_1(a_j,e_j,A_j)_p,(2\alpha,2\lambda,A)\\ \left(\frac{1}{2}+\alpha,\lambda,A\right),\ {}_1(b_j,f_j,B_j)_q,(2\alpha,2\lambda,A),\end{array}\right.\right] \tag{17}$$

In (17), if we take $A_j = 1(j = 1,\ldots,p)$, $B_j = 1(j = 1,\ldots,q)$ and A = 1, it reduces to the H-function identity obtained by Rathie[4].

(e) In (12), if we take $\lambda = 0$, we get

$$\mathrm{I}^{\,m,\,n}_{\,p+1,\,q+1}\left[z\left|\begin{array}{l} (\beta,\delta,A),\ {}_1(a_j,e_j,A_j)_p\\ (\beta,\delta,A),\ {}_1(b_j,f_j,B_j)_q\end{array}\right.\right] = (2\pi)^A\,\mathrm{I}^{\,m+1,\,n+1}_{\,p+2,\,q+2}\left[z\left|\begin{array}{l} (2\beta,2\delta,A),\ {}_1(a_j,e_j,A_j)_p,\left(\frac{1}{2}+\beta,\delta,A\right)\\ (2\beta,2\delta,A),\ {}_1(b_j,f_j,B_j)_q,\left(\frac{1}{2}+\beta,\delta,A\right)\end{array}\right.\right] \tag{18}$$

In (18), if we take $A_j = 1(j = 1,\ldots,p)$, $B_j = 1(j = 1,\ldots,q)$ and A = 1, it reduces to the H-function identity obtained by Rathie[4].

(f) In (12), if we take $A_j = 1(j = 1,\ldots,p)$, $B_j = 1(j = 1,\ldots,q)$ and A = 1, it reduces to the H-function identity obtained by Rathie[4].

(g) In the LHS of (12), if we put A=1 and multiply by $2\pi i$ and equate with the LHS of (5), we get an interesting result as below.

$$(2\pi i)\, \mathrm{I}^{m+2,\,n+2}_{p+4,\,q+4}\left[z \,\middle|\, \begin{matrix} (2\beta,2\delta,1), \left(\frac{1}{2}+\alpha,\lambda,1\right), {}_1(a_j,e_j,A_j)_p, (2\alpha,2\lambda,1), \left(\frac{1}{2}+\beta,\delta,1\right) \\ (2\beta,2\delta,1), \left(\frac{1}{2}+\alpha,\lambda,1\right), {}_1(b_j,f_j,B_j)_q, (2\alpha,2\lambda,1), \left(\frac{1}{2}+\beta,\delta,1\right) \end{matrix}\right]$$

$$= e^{i\pi(\alpha+\beta)}\, \mathrm{I}^{m+1,\,n+1}_{p+1,\,q+1}\left[ze^{-i\pi(\lambda+\delta)} \,\middle|\, \begin{matrix} (2\beta,2\delta,1),\ {}_1(a_j,e_j,A_j)_p \\ (2\beta,2\delta,1),\ {}_1(b_j,f_j,B_j)_q \end{matrix}\right]$$

$$+\ e^{i\pi(\alpha-\beta)}\, \mathrm{I}^{m+1,\,n+1}_{p+1,\,q+1}\left[ze^{-i\pi(\lambda-\delta)} \,\middle|\, \begin{matrix} (2\beta,2\delta,1),\ {}_1(a_j,e_j,A_j)_p \\ (2\beta,2\delta,1),\ {}_1(b_j,f_j,B_j)_q \end{matrix}\right]$$

$$-\ e^{-i\pi(\alpha-\beta)}\, \mathrm{I}^{m+1,\,n+1}_{p+1,\,q+1}\left[ze^{i\pi(\lambda-\delta)} \,\middle|\, \begin{matrix} (2\beta,2\delta,1),\ {}_1(a_j,e_j,A_j)_p \\ (2\beta,2\delta,1),\ {}_1(b_j,f_j,B_j)_q \end{matrix}\right]$$

$$-\ e^{-i\pi(\alpha+\beta)}\, \mathrm{I}^{m+1,\,n+1}_{p+1,\,q+1}\left[ze^{i\pi(\lambda+\delta)} \,\middle|\, \begin{matrix} (2\beta,2\delta,1),\ {}_1(a_j,e_j,A_j)_p \\ (2\beta,2\delta,1),\ {}_1(b_j,f_j,B_j)_q \end{matrix}\right] \quad (19)$$

4. Another proof of (19)

Denoting the left-hand of (19) by S, expressing the I-function with the help of its definition we have,

$$S = (2\pi i)\frac{1}{2\pi i}\int_L \theta(s)\, z^s\, \frac{\Gamma(1-2\beta+2\delta s)\,\Gamma(\frac{1}{2}-\alpha+\lambda s)\,\Gamma(2\beta-2\delta s)\,\Gamma(\frac{1}{2}+\alpha-\lambda s)}{\Gamma(2\alpha-2\lambda s)\,\Gamma(\frac{1}{2}+\beta-\delta s)\Gamma(1-2\alpha+2\lambda s)\,\Gamma(\frac{1}{2}-\beta+\delta s)}\, ds \quad (20)$$

Using the results (7), (9), (10) and after some algebra, we have

$$S = \frac{1}{2\pi i}\int_L \theta(s)\, z^s\, \Gamma(2\beta-2\delta s)\,\Gamma(1-2\beta+2\delta s)$$
$$\cdot\left(e^{i\pi(\alpha-\lambda s)} - e^{-i\pi(\alpha-\lambda s)}\right)\left(e^{i\pi(\beta-\delta s)} + e^{-i\pi(\beta-\delta s)}\right) ds$$
$$= \frac{1}{2\pi i}\int_L \{\theta(s)\, z^s\, \Gamma(2\beta-2\delta s)\,\Gamma(1-2\beta+2\delta s)$$
$$\cdot\Big\{e^{i\pi(\alpha+\beta-\lambda s-\delta s)} + e^{i\pi(\alpha-\beta-\lambda s+\delta s)}$$
$$- e^{-i\pi(\alpha-\beta-\lambda s+\delta s)} - e^{-i\pi(\alpha+\beta-\lambda s-\delta s)}\Big\}\, ds \quad (21)$$

Now, breaking in to four parts and after some simplification, using the definition of I-function, we easily arrive at the right-hand side of (19).

Since I-function is the most generalized function among the functions of one variable studied so far, so by specializing the paramaters therein it reduces to H-function, G-function, Generalized Hypergeometric function ${}_pF_q$ and other elementary functions and hence we can obtain corresponding results. However we do not mention here due to lack of space.

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