



AJ INSTITUTE OF ENGINEERING & TECHNOLOGY

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Mathematics -I for Civil Stream - Lecture Notes

Subject Code : BMATC101

Module 1- Calculus

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Module - 1 Calculus

Syllabus : Introduction to polar coordinates and curvature relating to Computer Science and engineering:

Polar coordinates, Polar curves, angle between the radius vector and tangent, angle between two curves. Pedal equations. Curvature and Radius of curvature - Cartesian, Parametric, Polar and Pedal forms. Problems.

Prerequisites

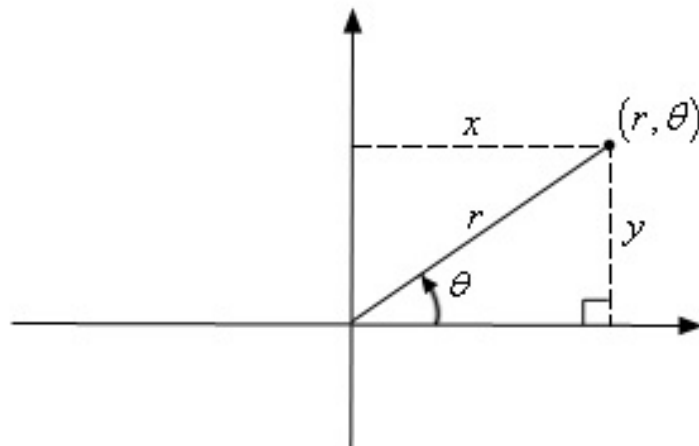
It will be helpful if you can recall the formulas of differentiation, trigonometric functions and allied angles.

Notations :

The first order derivative of $y \implies y'$ or y_1 or $\frac{dy}{dx}$ or $D(y)$

The Second order derivative of $y \implies y''$ or y_2 or $\frac{d^2y}{dx^2}$ or $D^2(y)$

Polar curves :



We are familiar with Cartesian coordinate system for specifying a point in the xy - plane. Another useful system for similar purpose is Polar coordinate system, in which each point P on a plane is determined by a distance r from a fixed point O that is called the **pole** (or origin) and an angle θ measured from x -axis. The point P is represented by the ordered pair (r, θ) where r and θ are called **polar coordinates**.

If the equation of a curve is specified in terms of r and θ , then the curve is referred to as polar curve.

The transformation from Cartesian to polar is given by

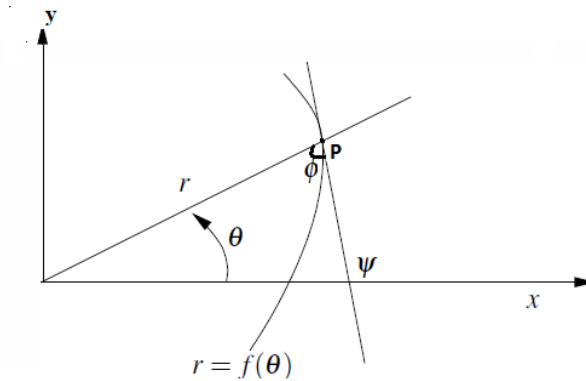
$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

The transformation from Cartesian to polar is given by

$$x = r \cos \theta \quad y = r \sin \theta$$

1.1 Angle between the radius vector and the tangent

Let Φ be the angle between the radius vector OP and the tangent AB at the point 'P' on the polar curve $r = f(\theta)$.



Let Ψ be the angle made by the tangent with positive direction of $x - axis$.

From the figure,

$$\Psi = \theta + \Phi$$

$$\tan \Psi = \tan(\theta + \Phi) = \frac{\tan \theta + \tan \Phi}{1 - \tan \theta \tan \Phi}$$

$$i.e. \frac{dy}{dx} = \frac{\tan \theta + \tan \Phi}{1 - \tan \theta \tan \Phi} \dots\dots (1)$$

On the other hand, we have $x = r \cos \theta ; y = r \sin \theta$ as the relation between polar and Cartesian Coordinates.

differentiating these, w.r.t θ , we obtain

$$\frac{dx}{d\theta} = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

$$\text{and } \frac{dy}{d\theta} = r \cos \theta + \sin \theta \frac{dr}{d\theta}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}} \end{aligned}$$

dividing the Numerator and Denominator by $\cos \theta \frac{dr}{d\theta}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{r}{\frac{dr}{d\theta}} + \tan \theta}{-\frac{r}{\frac{dr}{d\theta}} \tan \theta + 1} \\ &= \frac{\tan \theta + \frac{r}{\frac{dr}{d\theta}}}{1 - \frac{r}{\frac{dr}{d\theta}} \tan \theta} \dots\dots (2) \end{aligned}$$

Comparing equations (1) and (2), we get

$$\tan \Phi = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{r_1} = r \frac{d\theta}{dr}$$

Note : We can also find angle Φ by using

$$\cot\Phi = \frac{1}{r} \frac{dr}{d\theta}$$

Problem 1.1.1. Find the angle between the radius vector and the tangent to the curve $r = a(1 + \cos\theta)$. Also find slope of the tangent at $\theta = \frac{\pi}{3}$

Solution :

$$r = a(1 + \cos\theta) \dots\dots (1)$$

Applying log on both sides

$$\log(r) = \log(a) + \log(1 + \cos\theta)$$

differentiate w.r.to θ ,

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta}$$

$$\cot\Phi = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$i.e.\cot\Phi = -\tan\frac{\theta}{2} = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \Phi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\text{At } \theta = \frac{\pi}{3}, \Phi = \frac{\pi}{2} + \frac{\pi}{6}$$

$$\text{Slope} = \tan\Psi = \tan(\theta + \Phi) = \tan\pi = 0$$

Exercise 1.1

Find the angle between radius vector and tangent for the following curves.

1. $r = a(1 + \sin\theta)$

Ans : $\Phi = \frac{\pi}{4} + \frac{\theta}{4}$

2. $r^2\cos2\theta = a^2$

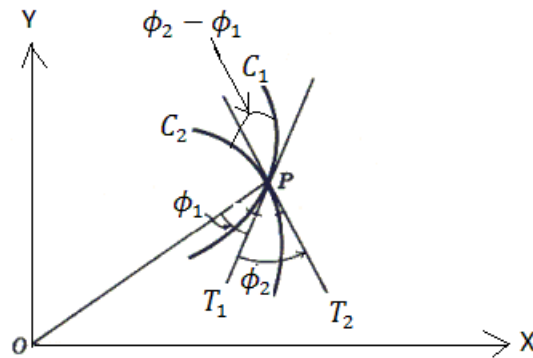
Ans : $\Phi = \frac{\pi}{2} - 2\theta$

3. $\frac{2a}{r} = 1 - \cos\theta$

Ans : $\Phi = \frac{-\theta}{2}$

1.2 Angle of intersection of two polar curves

If Φ_1 and Φ_2 are the angles between the common radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$, then the angle of intersection of the curves is given by $|\phi_1 - \Phi_2|$.



Note 1. Two curves intersect orthogonally if any one of the following conditions are satisfied.

a) $|\Phi_1 - \Phi_2| = \frac{\pi}{2}$

or

b) $\tan\Phi_1 \tan\Phi_2 = -1$

or

c) $\cot\Phi_1 \cot\Phi_2 = -1$

Note 2. If the angles Φ_1 and Φ_2 can not be obtained explicitly, then angle of intersection can be found by using the formula

$$|\tan(\Phi_1 - \Phi_2)| = \frac{\tan\Phi_1 - \tan\Phi_2}{1 + \tan\Phi_1 \tan\Phi_2}$$

Problem 1.2.1. Show that the following curves intersect each other orthogonally. $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$ (VTU 2015, July 2003)

Solution : Given

$$r = a(1 + \cos\theta)$$

Applying log on both sides

$$\log(r) = \log(a) + \log(1 + \cos\theta)$$

differentiate w.r.to θ ,

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta}$$

$$\cot\Phi_1 = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$i.e.\cot\Phi_1 = -\tan\frac{\theta}{2} = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \Phi_1 = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\text{Now consider the curve, } r = b(1 - \cos\theta)$$

Applying log on both sides

$$\log(r) = \log(b) + \log(1 - \cos\theta)$$

differentiate w.r.to θ ,

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}$$

$$\cot\Phi_1 = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\text{i.e. } \cot\Phi_2 = \cot\frac{\theta}{2}$$

$$\therefore \Phi_2 = \frac{\theta}{2}$$

$$\text{Consider } |\Phi_1 - \Phi_2| = \left| \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2} \right| = \frac{\pi}{2}$$

This shows that the given curves intersect each other orthogonally.

Problem 1.2.2. Show that the following curves intersect each other orthogonally. $r = a\theta$ and $r = \frac{a}{\theta}$

Solution :

$$\text{Consider } r = a\theta$$

Applying log on both sides

$$\log r = \log(a) + \log\theta$$

differentiate w.r.to θ

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\theta}$$

$$\text{i.e. } \cot\Phi_1 = \frac{1}{\theta}$$

$$\implies \tan\Phi_1 = \theta$$

$$\text{Now consider } r = \frac{a}{\theta}$$

Applying log on both sides

$$\log r = \log(a) - \log\theta$$

differentiate w.r.to θ

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-1}{\theta}$$

$$\cot \Phi_2 = \frac{-1}{\theta}$$

$$i.e. \tan \Phi_2 = -\theta$$

Using $r = a\theta$ and $r = \frac{a}{\theta}$, we can write

$$a\theta = \frac{a}{\theta}$$

$$i.e. \theta^2 = 1$$

$$\implies \theta = \pm 1$$

Case 1 : If $\theta = 1$, then $\tan \Phi_1 = 1$ and $\tan \Phi_2 = -1$

$$\therefore \tan \Phi_1 \tan \Phi_2 = -1$$

Case 2 : If $\theta = -1$, then $\tan \Phi_1 = -1$ and $\tan \Phi_2 = 1$

$$\therefore \tan \Phi_1 \tan \Phi_2 = -1$$

This shows that the given curves intersect orthogonally.

Problem 1.2.3. Show that the following curves intersect each other orthogonally. $r = a(1 - \sin \theta)$, $r = b(1 + \sin \theta)$

$$r = a(1 - \sin \theta)$$

Solution : Diff. w.r.to θ we get, $r_1 = a(-\cos \theta)$

$$\therefore \tan \phi_1 = \frac{r}{r_1} = -\frac{(1 - \sin \theta)}{\cos \theta}$$

For the curve, $r = b(1 + \sin \theta)$

Diff. w.r.to θ we get, $r_1 = b(\cos \theta)$

$$\therefore \tan \phi_2 = \frac{r}{r_1} = \frac{(1 + \sin \theta)}{\cos \theta}$$

$$\tan \phi_1 \cdot \tan \phi_2 = -\frac{(1 - \sin^2 \theta)}{\cos^2 \theta} = -1$$

This shows that the given curves intersect orthogonally.

Problem 1.2.4. Show that the following curves intersect each other orthogonally. Show that the curves $r^m = a^m \cos m\theta$ and $r^m = a^m \sin m\theta$

Solution: Equations of curves are $r^m = a^m \cos m\theta$ (1) $r^m = a^m \sin m\theta$ (2) Take natural logarithm of both sides of (1)

$$m \log r = m \log a + \log(\cos m\theta)$$

Differentiate both sides w. r. t θ

$$\begin{aligned}\frac{m}{r}r_1 &= -\frac{m \sin m\theta}{\cos m\theta} \\ \therefore \frac{r}{r_1} &= -\cot m\theta \\ \tan \phi_1 &= \frac{r}{r_1} = -\cot m\theta\end{aligned}$$

Take natural logarithm of both sides of (2) $m \log r = m \log a + \log(\sin m\theta)$ Differentiate both sides w. r. t θ

$$\begin{aligned}\frac{m}{r}r_1 &= \frac{m \cos m\theta}{\sin m\theta} \\ \therefore \frac{r}{r_1} &= \tan m\theta \\ \tan \phi_2 &= \frac{r}{r_1} = \tan m\theta\end{aligned}$$

$$\therefore \tan \phi_1 \cdot \tan \phi_2 = -\cot m\theta \times \tan m\theta = -1$$

\therefore curves (1) and (2) cut orthogonally.

Problem 1.2.5. Find the angle between the following curves $r = \sin \theta + \cos \theta$, $r = 2 \sin \theta$

Sol :

$$r = \sin \theta + \cos \theta$$

Diff. w.r.to θ we get,

$$r_1 = \cos \theta - \sin \theta$$

$$\begin{aligned}\therefore \tan \phi_1 &= \frac{r}{r_1} = \frac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta} \\ &= \frac{1 + \tan \theta}{1 - \tan \theta} \quad (\text{By dividing each term in Numerator and Denominator by } \cos \theta) \\ &= \tan \left(\theta + \frac{\pi}{4} \right) \\ \Rightarrow \phi_1 &= \theta + \frac{\pi}{4}\end{aligned}$$

$$r = 2 \sin \theta$$

Diff. W.r.to θ we get,

$$r_1 = 2 \cos \theta$$

$$\begin{aligned}\therefore \tan \phi_2 &= \frac{r}{r_1} = \frac{\sin \theta}{\cos \theta} = \tan \theta \\ \Rightarrow \phi_2 &= \theta\end{aligned}$$

Angle of intersection is

$$|\phi_1 - \phi_2| = \left| \theta + \frac{\pi}{4} - \theta \right| = \frac{\pi}{4}$$

Problem 1.2.6. Find the angle between the following curves : $r = a \log \theta$ and $r = \frac{a}{\log \theta}$ (VTU Jan 2015, July 2005, Aug 2001)

Consider the curve $r = a \log \theta$ Applying log on both sides

$$\log r = \log(a) + \log(\log \theta)$$

$$\text{differentiate w.r.to } \theta, \frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\theta \log \theta}$$

$$\cot \Phi_1 = \frac{1}{\theta \log \theta}$$

$$\therefore \tan \Phi_1 = \theta \log \theta$$

$$\text{Consider the curve } r = \frac{a}{\log \theta}$$

Applying log on both sides

$$\log r = \log(a) - \log(\log \theta)$$

$$\text{differentiate w.r.to } \theta, \frac{1}{r} \frac{dr}{d\theta} = \frac{-1}{\theta \log \theta}$$

$$\cot \Phi_2 = \frac{-1}{\theta \log \theta}$$

$$\therefore \tan \Phi_2 = -\theta \log \theta$$

Using the given curves $r = a \log \theta$ and $r = \frac{a}{\log \theta}$, we can write

$$a \log \theta = \frac{a}{\log \theta}$$

$$(\log \theta)^2 = 1$$

i.e. $\theta = e$ (assuming positive value)

Consider

$$|\tan(\Phi_1 - \Phi_2)| = \frac{\tan \Phi_1 - \tan \Phi_2}{1 + \tan \Phi_1 \tan \Phi_2} = \frac{2e}{1 - e^2}$$

$$\therefore |\Phi_1 - \Phi_2| = \tan^{-1} \left(\frac{2e}{1 - e^2} \right) = 2 \tan^{-1} e$$

Exercise 1.2

1. Show that the following curves intersect each other orthogonally.

(a) $r = a(1 + \cos \theta)$, $r = a(1 - \cos \theta)$ (VTU Jan 2016, July 2015, 2011)

(b) $r = a(1 + \sin\theta), r = a(1 - \sin\theta)$

(c) $r^n = a^n \cos n\theta, r^n = b^n \sin n\theta$

2. Find the angle of intersection of the following curves :

(a) $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$ (VTU Model 2015) Ans : $\frac{\pi}{3}$

(b) $r = \sin\theta + \cos\theta$ and $r = 2 \sin\theta$ (VTU July 2004, 2002) Ans : $\frac{\pi}{4}$

(c) $r = \frac{a}{1 + \cos\theta}$ and $r = \frac{b}{1 - \cos\theta}$ (VTU Jan 2017, Jun 2012, June 2009, Jul 2008) Ans : $\frac{\pi}{2}$

(d) $r^n = a^n (\cos n\theta + \sin n\theta)$ and $r^n = a^n \sin n\theta$, Ans : $\frac{\pi}{4}$

(e) $r^n = a^n \cos n\theta$ and $r^n \sin n\theta = b^n$, Ans : $\frac{\pi}{2}$

(f) $r^m = a^m \cos m\theta$ and $r^m = b^m \sin m\theta$, Ans : $\frac{\pi}{2}$

(g) $r = 2 \sin\theta$ and $r = 2(\sin\theta + \cos\theta)$ (VTU July 2017).

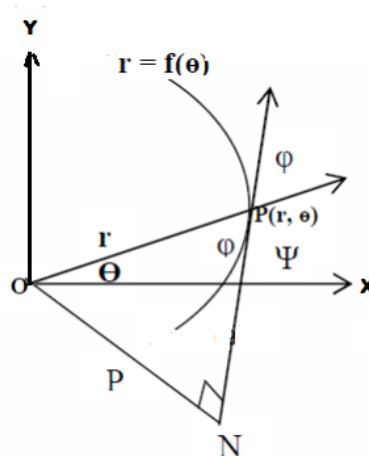
(h) $r = a(1 + \sin\theta)$ and $r = b(1 - \sin\theta)$ (VTU July 2017)

The length p of perpendicular from pole to the tangent in a polar curve

The length p of perpendicular from pole to the tangent at a point (r, θ) in a polar curve is given by

$$p = r \sin \Phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

Proof :



From the right angled triangle OPN,

$$\sin \phi = \frac{ON}{OP}$$

$$i.e.. \sin\phi = \frac{p}{r} \implies \boxed{p = r \sin\phi}$$

$$\text{Consider } \frac{1}{p} = \frac{1}{r \sin\phi} = \frac{1}{r} \operatorname{cosec}\phi$$

$$\begin{aligned} \therefore \frac{1}{p^2} &= \frac{1}{r^2} \operatorname{cosec}^2\phi = \frac{1}{r^2} (1 + \cot^2\phi) \\ &= \frac{1}{r^2} \left[1 + \left(\frac{1}{r} \frac{dr}{d\theta} \right)^2 \right] \end{aligned}$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2}$$

1.3 Pedal equations ($p - r$ equations)

For a plane curve $r = f(\theta)$ and a given point $P(r, \theta)$ on this curve, the pedal equation of the curve is a relation between r and p where r is the distance from O (origin) to a point $P(r, \theta)$, and p is the perpendicular distance from O to the tangent to the curve at the point P .

Working rules to find pedal equations :

Let the polar equation of any curve be,

$$f(r, \theta) = 0 \quad (1)$$

Let Φ be the angle between the radius vector and the tangent, then we know that,

$$\tan(\Phi) = r \cdot \frac{d\theta}{dr} \quad (2)$$

and perpendicular distance from O to the tangent is given by using the formula

$$p = r \sin\Phi \quad (3)$$

or

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \quad (4)$$

Now, if we eliminate θ between the equations (1) and (3) or (1) and (4), then we shall get an equation in terms of p and r and thus will be required an equation of the curve.

Problem 1.3.1. Find the pedal equation for the curve $\frac{2a}{r} = 1 - \cos\theta$

Solution : Given equation can be written as

$$r = \frac{2a}{1 - \cos\theta} = \frac{2a}{2\sin^2\frac{\theta}{2}} = a\operatorname{cosec}^2\frac{\theta}{2} \quad (1)$$

$$\begin{aligned} \frac{dr}{d\theta} &= a2\operatorname{cosec}^2\frac{\theta}{2} \left(-\operatorname{cosec}\frac{\theta}{2}\right) \cot\frac{\theta}{2} \cdot \frac{1}{2} \\ &= -a\operatorname{cosec}^2\frac{\theta}{2} \cot\frac{\theta}{2} \end{aligned}$$

$$\begin{aligned} \tan\Phi &= r \frac{d\theta}{dr} = a\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{-a\operatorname{cosec}^2\frac{\theta}{2} \cot\frac{\theta}{2}} \\ &= \tan\frac{\theta}{2} = \tan\left(\pi - \frac{\theta}{2}\right) \end{aligned}$$

$$\implies \Phi = \left(\pi - \frac{\theta}{2}\right)$$

$$\text{Now } p = r \sin\Phi = r \sin\left(\pi - \frac{\theta}{2}\right) = r \sin\frac{\theta}{2}$$

$$\text{Squaring } p^2 = r^2 \sin^2\frac{\theta}{2} \implies \sin^2\frac{\theta}{2} = \frac{p^2}{r^2}$$

Now (1) can be written as

$$r = \frac{2a}{2\sin^2\frac{\theta}{2}} = \frac{a}{\frac{p^2}{r^2}} = \frac{ar^2}{p^2}$$

$\implies p^2 = ar$ This is the required pedal equation.

Problem 1.3.2. Find the pedal equation of the curve $r^n = a^n \sin n\theta$

Solution : Equation of the curve is $r^n = a^n \sin n\theta$

Take the natural logarithm of both sides

$$n \log r = n \log a + \log(\sin n\theta)$$

Differentiate both sides w.r.t. θ

$$\begin{aligned} n \frac{1}{r} r_1 &= \frac{n \cos n\theta}{\sin n\theta} \\ \therefore \tan \phi &= \frac{r}{r_1} = \frac{\sin n\theta}{\cos n\theta} = \tan n\theta \end{aligned}$$

$$\therefore \phi = n\theta$$

$$\therefore p = r \sin \phi = r \sin n\theta = r \frac{r^n}{a^n} \text{ from (1)}$$

$$\therefore pa^n = r^{n+1}$$

which is the pedal equation of the curve.

Problem 1.3.3. Find the pedal equation for the polar curve $r^m = a^m(\cos(m\theta) + \sin(m\theta))$.

For this curve in the usual way, taking log on both sides,

$$m \log r = \log(a^m) + \log(\cos(m\theta) + \sin(m\theta))$$

Diff. w.r.to θ ,

$$m \frac{1}{r} r_1 = 0 + \frac{1}{\cos(m\theta) + \sin(m\theta)} \cdot (-m \sin m\theta + m \cos m\theta)$$

$$\cot \phi = \frac{(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$

$$\tan \phi = \frac{(\cos m\theta + \sin m\theta)}{(\cos m\theta - \sin m\theta)}$$

$$= \frac{1 + \tan m\theta}{1 - \tan m\theta} \text{ (By dividing each term in Numerator and Denominator by } \cos m\theta)$$

$$= \tan\left(\frac{\pi}{4} + m\theta\right)$$

$$\Rightarrow \phi = \frac{\pi}{4} + m\theta$$

Pedal equation is

$$\begin{aligned} p &= r \sin \phi = r \sin\left(\frac{\pi}{4} + m\theta\right) \\ &= r \left(\sin\left(\frac{\pi}{4}\right) \cos \theta + \cos\left(\frac{\pi}{4}\right) \sin \theta \right) \\ &= \frac{r}{\sqrt{2}} (\cos(m\theta) + \sin(m\theta)) \\ &= \frac{r}{\sqrt{2}} \left(\frac{r^m}{a^m} \right) \text{ (from the given equation)} \\ &= \frac{r^{m+1}}{a^m \sqrt{2}} \end{aligned}$$

Exercise 1.3

Find the pedal equations of the following curves.

1. $r^n = \operatorname{sech} n\theta$ (VTU Jan 2015)

Ans : $\frac{1}{p^2} = \frac{-1}{r^2} (r^{2n} - 2)$

2. $r^m = a^m \sin m\theta + b^m \cos m\theta$ (VTU July 2015, Jan-2005)

Ans : $p^2 = \frac{r^{2(m+1)}}{a^{2m} + b^{2m}}$

3. $\frac{\ell}{r} = 1 + e \cos \theta$

Ans : $\frac{1}{p^2} = 1 - \frac{1}{e^2} + \frac{2}{lr}$

4. $r = ae^{\theta \cot \alpha}$

Ans : $r^2 = p^2 \operatorname{cosec}^2 \alpha$

5. $r^n = a^n \cos n\theta$ (VTU July 2017, Jan 2016, Model 2014, Jan 2014, June 2014, Jan 2010, May 2001, Aug 2000)

Ans : $pa^n = r^{n+1}$

6. $r = a\theta$

Ans : $p = \frac{r^2}{\sqrt{r^2 + a^2}}$

7. $r = a(1 + \cos\theta)$

Ans : $r^3 = 2ap^2$

8. $r^m = a^m(\sin m\theta + \cos m\theta)$ (VTU 2010)

Ans : $r^{m+1} = \sqrt{2}a^m p$

9. $r^m \cos m\theta = a^m$ (VTU July 2016)

Ans : $pr^{m-1} = a^m$

10. $r^n = a^n \sin n\theta$

Ans : $pa^n = r^{n+1}$

11. $\frac{2a}{r} = 1 - \cos\theta$ (VTU Jun 2011)

Ans : $p^2 = ar$

12. $r = a + b\cos\theta$

(VTU July 2017)

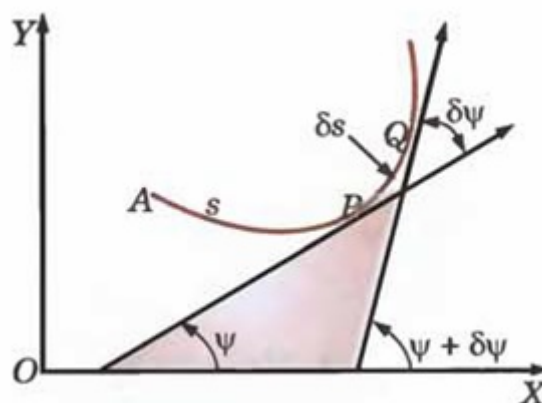
13. $\frac{2a}{r} = 1 + \cos\theta$

(VTU Jan 2017)

Curvature.

The curvature is the concept in geometry that indicates the change in direction of the curve at a certain point. While the radius of curvature gives the radius of the approximate circle that matches the curve at a particular point.

Consider a smooth curve C in XY-plane and let P, Q be any two neighbouring points on it. Let $arc AP = s$ and $arc PQ = \Delta s$. Let the tangents drawn to the curve at P, Q respectively make angles Ψ and $\Psi + \Delta\Psi$ with X-axis i.e., the angle between the tangents at P and Q is $\Delta\Psi$. While moving from P to Q through a distance Δs , the tangent has turned through the angle $\Delta\Psi$. This is called the bending of the arc PQ. Geometrically, a change in Ψ represents the bending of the curve C and the ratio $\frac{\Delta\Psi}{\Delta s}$ represents the ratio of bending of C between the point P & Q and the arc length between them.



Rate of bending of Curve at P is

$$\frac{d\Psi}{ds} = \lim_{Q \rightarrow P} \frac{\Delta\Psi}{\Delta s}$$

Curvature is a numerical measure of bending of the curve. Or, more simply, it measures the rate of change of direction of the curve, and it is denoted by κ (kappa). Thus

$$\kappa = \frac{d\Psi}{ds}$$

Note : Intuitively the curvature of a plane curve at a point P can be thought of as the curvature of that circle which approximates the curve most closely near that point. The curvature of a circle is directly defined by the reciprocal of the length of its radius. The shorter the radius, the greater the curvature of the arc in the vicinity of any point P on it. The longer the radius, the bigger the circle, and the less the curvature of the arc in the vicinity of any point P on it. For a very large circle the curvature of an arc at some point P approaches that of a straight line and the curvature of a straight line is zero since there exist no bending.

1.4 Radius of Curvature

If $\kappa \neq 0$, then $\frac{1}{\kappa}$ is called the radius of curvature and is denoted by ρ .

ie.

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\Psi}$$

The sign of $\frac{d\Psi}{ds}$ indicates the convexity and concavity of the curve in the neighbourhood of the point.

Radius of curvature in Cartesian form

Suppose $y = f(x)$ is the Cartesian equation of a curve, then we know that

$$\tan \psi = dy/dx = y_1$$

or

$$\psi = \tan^{-1}(y_1)$$

Differentiating both sides w.r.t. x ,

$$\begin{aligned}\frac{d\psi}{dx} &= \frac{1}{1 + y_1^2} \frac{d(y_1)}{dx} \\ &= \frac{y_2}{1 + y_1^2} \\ \text{Curvature } K &= \frac{d\psi}{ds} \\ &= \frac{d\psi}{dx} \times \frac{dx}{ds}\end{aligned}$$

From the derivative of arc length, we have

$$\begin{aligned}\frac{dS}{dx} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\ \text{i.e. } \frac{dx}{ds} &= \frac{1}{\sqrt{(1 + y_1^2)}} \\ \therefore K &= \frac{d\psi}{dx} \times \frac{dx}{ds} \\ &= \frac{y_2}{1 + y_1^2} \times \frac{1}{\sqrt{(1 + y_1^2)}} \\ &= \frac{y_2}{(1 + y_1^2)^{3/2}} \\ \therefore \text{Radius of curvature } \rho &= \frac{1}{K} \\ &= \frac{(1 + y_1^2)^{3/2}}{y_2}\end{aligned}$$

the expression for radius of curvature is,

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\text{i.e. } \rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$$

Alternate formula for radius of curvature :

An alternate formula for radius of curvature is

$$\rho = \frac{\left(1 + \left(\frac{dx}{dy}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

This formula can be used at a point where $\frac{dy}{dx}$ doesn't exist such as a point on a curve where the tangent line is parallel to the y-axis. (i.e. when $\frac{dy}{dx} = \infty$)

Radius of curvature in Parametric form:

For the curve $x = x(t)$ and $y = y(t)$, where t is the parameter,

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Where $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, \ddot{y} = \frac{d^2y}{dt^2}$.

Note : When parametric form $x = x(t)$ and $y = y(t)$ is given, we can also find the radius of curvature using

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$$

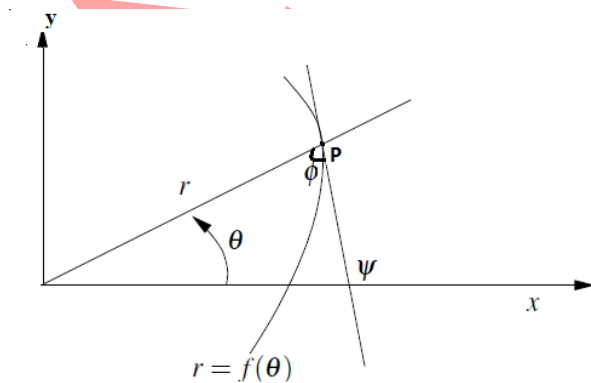
where

$$y_1 = \frac{dy/dt}{dx/dt}$$

and

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}(y_1) \times \frac{dt}{dx}$$

Radius of curvature in polar form



For the curve $r = f(\theta)$ i.e., the curve in polar coordinates

With the usual notations, we have

$$\psi = \theta + \phi$$

Differentiating w.r.t. s ,

$$\begin{aligned}\kappa &= \frac{d\psi}{ds} \\ &= \frac{d\theta}{ds} + \frac{d\phi}{ds} \\ &= \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} \\ &= \frac{d\theta}{ds} \left(1 + \frac{d\phi}{d\theta} \right)\end{aligned}\quad (1)$$

Also we know that

$$\begin{aligned}\tan \phi &= r \frac{d\theta}{dr} = \frac{r}{r_1} \\ \text{or } \phi &= \tan^{-1} \left(\frac{r}{r_1} \right)\end{aligned}$$

Differentiating w.r.t. θ ,

$$\begin{aligned}\frac{d\phi}{d\theta} &= \frac{1}{1 + (r/r_1)^2} \times \frac{r_1 \cdot r_1 - r r_2}{r_1^2} \\ &= \frac{r_1^2 - r r_2}{r^2 + r_1^2}\end{aligned}\quad (2)$$

Also,

$$\frac{ds}{d\theta} = \sqrt{(r^2 + r_1^2)} \quad (3)$$

Substituting the value from (2) and (3) in (1),

$$\begin{aligned}\kappa &= \frac{1}{\sqrt{r^2 + r_1^2}} + \left(1 + \frac{r_1^2 - r_2}{r^2 + r_1^2} \right) \\ &= \frac{r^2 + r_1^2 - r r_2}{(r^2 + r_1^2)^{3/2}}\end{aligned}$$

$$\therefore \rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + r_1^2 - r r_2}$$

where $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$

Radius of Curvature in Pedal form

For the curve in terms of p and r (i.e., the curve in pedal form):

$$\rho = r \frac{dr}{dp}$$

Problem 1.4.1. Find the radius of curvature for $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where it meets the line $y = x$.

Solution : On the line $y = x$, we have $\sqrt{x} + \sqrt{x} = \sqrt{a}$, i.e. $2\sqrt{x} = \sqrt{a}$ or $x = \frac{a}{4}$

Let

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

differentiating w.r.to x ,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y_1 = 0$$

$$i.e.y_1 = -\frac{\sqrt{y}}{\sqrt{x}} \dots\dots (1)$$

$$\therefore y_1|_{\frac{a}{4}, \frac{a}{4}} = -1$$

differentiating (1) w.r.to x ,

$$y_2 = \frac{\sqrt{x} \frac{1}{2\sqrt{y}}y_1 - \sqrt{y} \frac{1}{2\sqrt{x}}}{x}$$

$$y_2|_{\frac{a}{4}, \frac{a}{4}} = \frac{4}{a}$$

$$\therefore \rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1 + (-1)^2)^{\frac{3}{2}}}{\frac{4}{a}} = \frac{a}{\sqrt{2}}$$

Problem 1.4.2. Find ρ at the point $(\frac{3a}{2}, \frac{3a}{2})$ of the Folium $x^3 + y^3 = 3axy$ (VTU July 2017, July 2016, Model 2014, 2015, 2008)

Solution : Given

$$x^3 + y^3 = 3axy \tag{1}$$

Differentiating (1) w.r.to x , we get

$$x^2 + yy_1 = a(xy_1 + y) = (y^2 - ax)y_1 = ay - x^2$$

$$i.e.y_1 = \frac{ay - x^2}{y^2 - ax} \dots\dots (2)$$

$$y_1|_{(\frac{3a}{2}, \frac{3a}{2})} = -1$$

Differentiating (2) w.r.to x , we get

$$y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$y_2|_{(\frac{3a}{2}, \frac{3a}{2})} = \frac{-32}{3a} \text{ on simplification}$$

$$\therefore \text{Radius of Curvature, } \rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1 + 1)^{\frac{3}{2}}}{\frac{-32}{3a}} = \frac{3a}{8\sqrt{2}}$$

Problem 1.4.3. Find the radius of curvature for the curve $x = a \log(\sec t + \tan t), y = a \sec t$ (VTU July 2015)

Solution:

$$\text{Given } x = a \log(\sec t + \tan t) \text{ --- (1)}$$

$$\text{and } y = a \sec t \text{ --- (2)}$$

differentiating (1) w.r.to t , we get

$$\frac{dx}{dt} = a \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} = a \sec t$$

differentiating (2) w.r.to t , we get

$$\frac{dy}{dt} = a \sec t \tan t$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \tan t \text{ --- (3)}$$

differentiating (3) w.r.to x , we get

$$y_2 = \sec^2 t \frac{dt}{dx} = \frac{1}{a} \sec t$$

$$\therefore \text{Radius of Curvature, } \rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1 + \tan^2 t)^{\frac{3}{2}}}{\frac{1}{a} \sec t} = a \sec^2 t$$

Problem 1.4.4. Show that the R.O.C at any point of the cardioid $r = a(1 - \cos \theta)$ varies as \sqrt{r} . (VTU 2003)

Solution : Given that

$$r = a(1 - \cos \theta) \text{ --- (1)}$$

differentiating (1) w.r.to θ , we get

$$r_1 = a \sin \theta \text{ and } r_2 = a \cos \theta$$

\therefore Radius of Curvature ,

$$\begin{aligned} \rho &= \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \\ &= \frac{(a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta)^{\frac{3}{2}}}{a^2(1 - \cos \theta)^2 + 2a^2 \sin^2 \theta - a^2(1 - \cos \theta) \cos \theta} \\ &= \frac{2\sqrt{2}a}{3} \sqrt{r} \text{ on simplification} \end{aligned}$$

$$\therefore \rho \propto \sqrt{r}$$

Problem 1.4.5. Find the radius of the curvature of $r^n = a^n \cos n\theta$

Solution : Given curve is $r^n = a^n \cos n\theta$ Diff. w.r.to θ we get,

$$nr^{n-1}r_1 = -na^n \sin n\theta$$

$$n \frac{r^n}{r} r_1 = -na^n \sin n\theta$$

$$r_1 = \frac{-na^n \sin n\theta r}{a^n \cos n\theta} \quad (\text{from the given equation})$$

$$\therefore r_1 = -r \tan n\theta$$

$$r_2 = -r_1 \tan n\theta - nr \sec^2 n\theta$$

$$= r \tan^2 n\theta - nr \sec^2 n\theta$$

$$\begin{aligned} \rho &= \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \\ &= \frac{r^3 (1 + \tan^2 n\theta)^{\frac{3}{2}}}{r^2 + 2r^2 \tan^2 n\theta - r(r \tan^2 n\theta - nr \sec^2 n\theta)} \\ &= \frac{r \sec^3 n\theta}{1 + \tan^2 n\theta + n \sec^2 n\theta} \\ &= \frac{n+1}{r \sec n\theta} \\ &= \frac{n+1}{ra^n} = \frac{a^n}{(n+1)r^n} = \frac{a^n}{(n+1)r^{n-1}} \end{aligned}$$

Problem 1.4.6. Find the radius of the curvature of $r^3 = 2ap^2$

Solution : Here $r^3 = 2ap^2$

Differentiating w.r.t. p, we get

$$\begin{aligned} 3r^2 \cdot \frac{dr}{dp} &= 4ap \\ \Rightarrow \frac{dr}{dp} &= \frac{4ap}{3r^2} \end{aligned}$$

Hence,

$$\rho = r \cdot \frac{dr}{dp} = r \cdot \frac{4ap}{3r^2} = \frac{4ap}{3r}$$

$$\begin{aligned} \text{where } p &= \left(\frac{r^3}{2a} \right)^{\frac{1}{2}} \\ \rho &= \frac{4a \cdot \left(\frac{r^3}{2a} \right)^{\frac{1}{2}}}{3r} \\ &= \frac{4ar^{\frac{3}{2}}}{3r\sqrt{2a}} = \frac{2\sqrt{2ar}}{3} \end{aligned}$$

Problem 1.4.7. Find the radius of the curvature of $p^2 = ar$

Solution : Here $p^2 = ar$

Differentiating w.r.t. p , we get

$$\begin{aligned} 2p &= a \cdot \frac{dr}{dp} \\ \Rightarrow \frac{dr}{dp} &= \frac{2p}{a} \end{aligned}$$

$$\text{where } p = \sqrt{ar}. \quad \rho = r \frac{dr}{dp} = r \cdot \frac{2\sqrt{ar}}{a} = \frac{2r^{\frac{3}{2}}}{\sqrt{a}}$$

Problem 1.4.8. Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a, 0)$.

Solution :

$$y^2 = \frac{a^2(a-x)}{x}$$

$$y^2x = a^3 - a^2x$$

Differentiating w.r.t x

$$2xyy_1 + y^2 = -a^2 \quad \Rightarrow \quad y_1 = -\frac{a^2 + y^2}{2xy}$$

at $(a, 0)$, $y_1 = \infty$

$$\text{therefore } \rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2} = \frac{1}{x_2}$$

$$x_1 = \frac{dx}{dy} = -\frac{2xy}{y^2 + a^2}$$

then $x_1 = 0$ at $(a, 0)$

$$(y^2 + a^2) x_1 = -2xy$$

differentiating w.r.t y

$$(y^2 + a^2) x_2 + 2yx_1 = -2x - 2yx_1$$

$$\Rightarrow (y^2 + a^2) x_2 = -2x - 2yx_1 - 2yx_1$$

Then

$$x_2 = -\frac{2}{a} \text{ at } (a, 0)$$

$$\therefore \rho = \frac{1}{x_2} = -\frac{a}{2}$$

\therefore The radius of curvature of the given curve is $\frac{a}{2}$.

Problem 1.4.9. Find ρ at any point on $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$

Sol : Here $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$

Differentiating w.r.t. θ

$$\frac{dx}{d\theta} = a(1 + \cos \theta), \frac{dy}{d\theta} = a \sin \theta$$

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$= \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

$$y_1 = \tan \frac{\theta}{2}$$

Again differentiating w.r.t. θ

$$y_2 = \frac{d}{dx} \left(\tan \frac{\theta}{2} \right)$$

$$= \frac{d}{d\theta} \left(\tan \frac{\theta}{2} \right) \times \frac{d\theta}{dx}$$

$$= \sec^2 \left(\frac{\theta}{2} \right) \times \frac{1}{2} \times \frac{1}{a(1 + \cos \theta)}$$

$$= \frac{\sec^2 \frac{\theta}{2}}{2a \times 2 \cos^2 \frac{\theta}{2}}$$

$$y_2 = \frac{1}{4a \cos^4 \frac{\theta}{2}}$$

$$\rho = \frac{\{1 + y_1^2\}^{\frac{3}{2}}}{y_2}$$

$$= \frac{\{1 + \tan^2 \frac{\theta}{2}\}^{\frac{3}{2}}}{\left\{ \frac{1}{4a \cos^4 \frac{\theta}{2}} \right\}}$$

$$= \left\{ \sec^2 \left(\frac{\theta}{2} \right) \right\}^{\frac{3}{2}} \times 4a \cos^4 \left(\frac{\theta}{2} \right)$$

$$= \frac{1}{\cos^3 \left(\frac{\theta}{2} \right)} \times 4a \cos^4 \left(\frac{\theta}{2} \right)$$

$$= 4a \cos \left(\frac{\theta}{2} \right).$$

Problem 1.4.10. Find the radius of curvature at any point on the curve $y = a \log \sec(x/a)$.

$$\text{Sol : Radius of curvature } \rho = \frac{\{1+y_1^2\}^{3/2}}{y_2}$$

$$\text{Here, } y = a \log \sec \left(\frac{x}{a} \right)$$

$$y_1 = a \times \frac{1}{\sec \left(\frac{x}{a} \right)} \cdot \sec \left(\frac{x}{a} \right) \tan \left(\frac{x}{a} \right) \cdot \frac{1}{a}$$

$$y_1 = \tan \left(\frac{x}{a} \right)$$

$$y_2 = \sec^2 \left(\frac{x}{a} \right) \cdot \frac{1}{a}$$

$$\begin{aligned} \text{Hence } \rho &= \frac{\{1 + \tan^2 \left(\frac{x}{a} \right)\}^{3/2}}{\frac{1}{a} \sec^2 \left(\frac{x}{a} \right)} \\ &= \frac{\{\sec^2 \left(\frac{x}{a} \right)\}^{3/2}}{\frac{1}{a} \sec^2 \left(\frac{x}{a} \right)} \\ &= \frac{a \sec^3 \left(\frac{x}{a} \right)}{\sec^2 \left(\frac{x}{a} \right)} \\ &= a \sec \left(\frac{x}{a} \right) \end{aligned}$$

$$\therefore \text{Radius of curvature} = a \sec(x/a)$$

Exercise 1.4

- Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ where the curve meets the x-axis. (VTU Jan 2014, 2000) Ans : $\frac{a}{2}$
- Find the radius of curvature at $(a, 0)$ of the curve $y = x^3(x - a)$ (VTU 2010)
- Find ρ at $(-2a, 2a)$ for the curve $x^2y = a(x^2 + y^2)$ Ans : $2a$
- Find the radius of curvature of $a^2y = x^3 - a^3$ at the point where the curve cuts x-axis. (VTU July 2014) Ans : $\frac{5\sqrt{10}a}{3}$
- Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1,1) (VTU Jan 2017, July 2016) Ans : $\frac{-\sqrt{2}}{3}$
- For the curve $y = \frac{ax}{(a+x)}$, show that. $\left(\frac{2\rho}{a}\right)^{\frac{2}{3}} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$ (VTU July 2017, 2008)
- Prove that the radius of curvature ρ at any point (x,y) on the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ is given by $\frac{2(ax+by)^{\frac{3}{2}}}{ab}$ (VTU Jan 2014)
- Show that the radius of curvature at $x = \frac{\pi}{2}$ of the curve $y = 4\sin x - \sin 2x$ is $\frac{5\sqrt{5}}{4}$ (VTU 2009)

9. Show that the radius of curvature of the curve $x^3 + y^3 = 3xy$ at $(\frac{3}{2}, \frac{3}{2})$ is $\frac{-3}{8\sqrt{2}}$ (VTU Jan 2015)
10. Show that for the rectangular hyperbola $xy = c^2$, $\rho = \frac{(x^2+y^2)^{\frac{2}{3}}}{2xy}$
11. Show that the radius of curvature of the curve $y = 4\sin x - \sin 2x$ at $x = \frac{\pi}{2}$ is $\frac{5\sqrt{5}}{4}$
12. Find the radius of curvature of $xy^2 = a^3 - x^3$ at $(a, 0)$.
Hint : Here $y_1 = \infty$ Hence find $x_1 = \frac{dx}{dy} = 0$ and $x_2 = \frac{d^2x}{dy^2} = \frac{-2}{3a}$ and use the formula
 $\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}$ Ans : $\frac{3a}{2}$
13. If ρ is the R.O.C at the point P on the parabola $y^2 = 4ax$ & S be its focus, then show that ρ^2 varies with $(SP)^3$
14. Find the radius of curvature for the curve $r^2 = a^2 \cos 2\theta$. (VTU July 2017)
15. Find the radius of curvature at any point of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ Ans : $3a \sin \theta \cos \theta$
16. P.T the R.O.C at any point of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is 3 times the length of the perpendicular from the origin to the tangent at that point.
17. Show that the R.O.C at the end of the major axis $(a, 0)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the semi - latus rectum.
18. Find the radius of curvature for the curve $x = a(\cos t + t \sin t), y = a(\sin t - t \cos t)$ Ans : at
19. Find the radius of curvature for the curve $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$ (VTU 2003)
Ans : $4a \sin(\frac{\theta}{2})$
20. Show that the radius of curvature at any point of the cycloid, $x = a(\theta + \sin \theta), y = a(1 - \cos \theta)$ is $4a \cos \frac{\theta}{2}$ (VTU Jan 2016)
21. Find the radius of curvature for the curve $x = a(\cos t + \log \tan \frac{t}{2}), y = a \sin t$ (VTU July 2014)
Ans : $acott$
22. Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin \theta), y = a(1 - \cos \theta)$ is $4a \cos(\frac{\theta}{2})$ (VTU 2011)
23. For the curve $r = a(1 + \cos \theta)$, show that $\frac{\rho^2}{r}$ is a constant.
24. For the curve $r(1 - \cos \theta) = 2a$, show that ρ^2 varies as r^3 .

25. For the curve $r^m = a^m \cos m\theta$, show that $\rho = \frac{a^m}{(m+1)r^{m-1}}$
26. If ρ_1 & ρ_2 are radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, then show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$
27. Find the radius of curvature for the curve $r^n = a^n \cos n\theta$. (VTU Jan 2017)
28. Show that the R.O.C at any point of the lemniscate $r^2 = a^2 \cos 2\theta$ is inversely proportional to r
29. Find ρ for the curve $\frac{2a}{r} = (1 - \cos\theta)$
30. Show that for the equiangular spiral $r = ae^{\theta \cot \alpha}$, $\frac{\rho}{r}$ is a constant.
31. Find the radius of curvature of the polar curve $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1} \left(\frac{a}{r} \right)$
32. Find the radius of curvature for the curve $r^n = a^n \sin n\theta$
33. Show that the R.O.C at any point of the cardioid $r^2 \sec 2\theta = a^2$ is $\rho = \frac{a^2}{3r}$
34. Find the radius of curvature of $y = c \cosh \left(\frac{x}{c} \right)$ at the point where it crosses the y-axis. Ans : c
35. Find the radius of curvature of the curve $xy = c^2$ at (c, c) Ans : $\sqrt{2}c$
36. Find the radius of curvature of the curve $y^2 = x^2 \frac{(a+x)}{(a-x)}$ at the point $(-a, 0)$ Ans : $\frac{a}{4}$
37. Find the radius of curvature at 't' on $x = e^t \cos t, y = e^t \sin t$ Ans : $\sqrt{2}t$
38. Find the radius of curvature of $r = ae^{\theta \cot \alpha}$ Ans : $\rho = r \operatorname{cosec} \alpha$
39. Find the radius of the curvature of each of the following curves:
- $r^3 = 2ap^2$ (Cardioid)
 - $p^2 = ar$
 - $pr = a^2$
 - $r^3 = a^2p$
 - $p^2(a^2 + b^2 - r^2) = a^2b^2$
 - $1/p^2 = 1/a^2 + 1/b^2 - r^2/a^2b^2$
40. For the curve $p = \frac{r^{n+1}}{a^n}$, Show that ρ varies inversely as $(n - 1)$ th power of the radius vector.



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Mathematics-I for Civil Stream - Lecture Notes

Subject Code : BMATC101

**Module 2- Series Expansion and Multivariable
Calculus**

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Module 2 - Series Expansion and Multivariable Calculus

Syllabus

Taylor's and Maclaurin's series expansion for one variable (Statement only) – problems. Indeterminate forms - L'Hospital's rule. Problems. Partial differentiation, total derivative - differentiation of composite functions. Jacobian and problems. Maxima and minima for a function of two variables. Problems.

2.1 Taylor's series expansion of a function of single variable:

If $y = f(x)$ be a function, then the Taylor's series expansion for $y = f(x)$ about $x = a$ is given by

$$f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \dots \quad (1)$$

where $f'(a), f''(a), f'''(a)$ — stands for successive derivatives of $f(x)$ at $x = a$.

2.2 Maclaurin’s series :

If $y = f(x)$ be a function, then the Maclaurin’s series expansion for $y = f(x)$ about is given by

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

Problem 2.2.1. Expand $y = \cos x$ by Maclaurin’s series.

Solution: By Maclaurin’s series expansion

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \tag{1}$$

$$\begin{aligned} f(x) &= \cos x, & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x, & f'(0) &= 0 \\ f''(x) &= -\cos x, & f''(0) &= 1 \\ f'''(x) &= \sin x, & f'''(0) &= 0 \\ f''''(x) &= \cos x, & f''''(0) &= 1 \end{aligned}$$

Substituting in (1), we get

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Problem 2.2.2. Expand $\sqrt{(1 + \sin 2x)}$ by Maclaurin’s series. (VTU July 2021, June 2019, June 2018, Model 2018, Jan 2016, July 2014)

Solution: By Maclaurin’s series expansion

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \tag{1}$$

$$f(x) = \sqrt{(1 + \sin 2x)} = \sqrt{(\sin x + \cos x)^2} = \sin x + \cos x$$

$$f(0) = \sin 0 + \cos 0 = 1$$

$$f'(x) = \cos x - \sin x \qquad f'(0) = 1$$

$$f''(x) = -\sin x - \cos x \qquad f''(0) = -1$$

$$f'''(x) = -\cos x + \sin x \qquad f'''(0) = -1$$

$$f''''(x) = \sin x + \cos x$$

$$f''''(0) = 1$$

$$\therefore \sqrt{(1 + \sin 2x)} = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Problem 2.2.3. Obtain the Maclaurin series of $f(x) = \log \sec x$ upto 6th degree term. (VTU Jan 2018, July 2017, 2009)

Solution : $y = f(x) = \log \sec x$

$$\therefore f(0) = \log 1 = 0$$

$$y_1 = f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x \qquad \therefore f'(0) = 0$$

$$y_2 = f''(x) = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2 \qquad \therefore f''(0) = 1$$

$$y_3 = 2y_1y_2 \qquad \therefore f'''(0) = 2(0)(1) = 0$$

$$y_4 = 2y_2^2 + 2y_1y_3 \qquad \therefore f^{iv}(0) = 2(1)^2 + 2(0)(0) = 2$$

$$y_5 = 4y_2y_3 + 2y_2y_3 + 2y_1y_4 = 6y_2y_3 + 2y_1y_4 \qquad \therefore f^v(0) = 6(1)(0) + 2(0)(2) = 0$$

$$y_6 = 6y_3^2 + 8y_2y_4 + 2y_1y_5 \qquad \therefore f^{vi}(0) = 6(0) + 8(1)(2) + 2(0) = 16$$

\therefore By Maclaurin series expansion

$$\begin{aligned} \ln \sec x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \\ &= \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \dots \end{aligned}$$

Problem 2.2.4. Obtain the Maclaurin series of $\log(1 + \cos x)$ upto 6th degree term. (VTU June 2019, July 2017)

Solution:

$$y(x) = \log(1 + \cos x)$$

$$y(0) = \log(1 + 1) = \log 2$$

Diff y with respect to x.

$$y_1(x) = \frac{(0 - \sin x)}{(1 + \cos x)}$$

$$(1 + \cos x)y_1(x) = -\sin x.$$

$$\text{Put } x = 0$$

$$y_1(0) = 0$$

Diff again w.r.to x ,

$$(1 + \cos x)(y_2) + y_1(-\sin x) = -\cos x$$

Put $x = 0$, we get $y_2(0) = \frac{-1}{2}$

Diff again w.r.to x ,

$$(1 + \cos x)y_3 - \sin x(y_2) - \sin x(y_2) - \cos x(y_1) = \sin x$$

Put $x = 0$, we get $y_3(0) = 0$

Diff again w.r.to x ,

$$(1 + \cos x)y_4 - \sin x(y_3) - 2[\sin x(y_3) + \cos x(y_2)]$$

$$- [\cos x(y_2) - \sin x(y_1)] = \cos x$$

Put $x = 0$, we get $y_4(0) = \frac{-1}{4}$

\therefore By Maclaurin series expansion

$$\begin{aligned} \ln \sec x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \dots \\ &= \log 2 - \frac{x^2}{4} - \frac{x^4}{96} + \dots \end{aligned}$$

Problem 2.2.5. Using Maclaurin's series Prove that $\sqrt{1 + \cos 2x} = \sqrt{2} \left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots \right]$

(VTU Jan 2021)

$$y = \sqrt{1 + \cos 2x} = \sqrt{2 \cos^2 x}$$

$$\Rightarrow y = \sqrt{2} \cos x, \quad \Rightarrow y(0) = \sqrt{2} \cos(0) = \sqrt{2}$$

$$y'(x) = \sqrt{2} \times (-\sin x) \quad \Rightarrow y'(0) = -\sqrt{2} \sin(0) = -\sqrt{2}(0) = 0$$

$$\therefore y''(x) = -\sqrt{2} \times \cos x = -\sqrt{2} \cos x, \quad \Rightarrow y''(0) = -\sqrt{2} \cos 0 = -\sqrt{2}$$

$$y'''(x) = -\sqrt{2}(-\sin x) = \sqrt{2} \sin x$$

$$\Rightarrow y'''(0) = \sqrt{2} \sin(0) = 0$$

Maclaurin's series expansion is given by

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \dots$$

$$\Rightarrow \sqrt{1 + \cos 2x} = \sqrt{2} + \frac{x}{1!}(0) + \frac{x^2}{2!}(-\sqrt{2}) + \frac{x^3}{3!}(0) + \dots$$

$$= \sqrt{1 + \cos 2x} = \sqrt{2} + \frac{x^2}{2!}(-\sqrt{2}) - 0 + \dots$$

$$= \sqrt{1 + \cos 2x} = \sqrt{2} - \frac{\sqrt{2}x^2}{2} + \dots$$

2.3 Indeterminate Forms:

If $f(x)$ and $g(x)$ be two functions such that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exists, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}.$$

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

Which do not have any definite value. Such an expression is called indeterminate form.

The other indeterminate forms are $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 1^∞ , 0^0 , ∞^0 ,

L'Hospital's Rule:

If two functions $f(x)$ and $g(x)$ are such that

(i) $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$

(ii) $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$

Then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Note : If $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ is in $\frac{0}{0}$ form, we have to apply L'Hospital's rule again.i.e.

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$$

and so on.

Note:

(i) L'Hospital's rule is applicable only when the expression has the form $\frac{0}{0}$, or $\frac{\infty}{\infty}$ in the limit.

(ii) Note that we do not take the derivative of the ratio using the quotient rule, but rather separately find the derivatives of the numerator and denominator functions, then find the limit of their ratio.(Do not confuse L'Hopital's Rule with the quotient rule for derivatives.)

(iii) Sometimes we need to repeat L'Hospital's Rule more than once till we get definite value of the limit.

(iv) **Important Limits :**

$$(a) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$(b) \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

$$(c) \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$(d) \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1$$

$$(e) \lim_{x \rightarrow 0} (1 + nx)^{\frac{1}{x}} = e^n. \text{ In Particular, } \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$$

$$(f) \lim_{x \rightarrow \infty} \left(1 + \frac{n}{x}\right)^x = e^n. \text{ In Particular, } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$(g) \lim_{x \rightarrow \infty} e^x = \infty \text{ and } \lim_{x \rightarrow \infty} e^{-x} = 0$$

$$(h) \lim_{x \rightarrow 0} \ln(x) = -\infty$$

$$(i) \lim_{x \rightarrow 1} \log x = 0$$

$$(j) \lim_{x \rightarrow \infty} \log x = \infty$$

$$(k) \log(e) = 1$$

The indeterminate form $0 \cdot \infty$:

If $\lim_{x \rightarrow a} f(x)g(x)$ assumes the indeterminate form $0 \cdot \infty$ then the product may be converted to one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing $f(x)g(x)$ as

$$f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}} \text{ or } \frac{g(x)}{\frac{1}{f(x)}}$$

and then we can apply L'Hospital's rule to get the limit.

Problem 2.3.1. Evaluate the following limits. $\lim_{x \rightarrow 0} \log_{\sin x} \sin 2x$

Solution:

$$\begin{aligned} k &= \lim_{x \rightarrow 0} \log_{\sin x} \sin 2x \\ &= \lim_{x \rightarrow 0} \frac{\log \sin 2x}{\log \sin x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \end{aligned}$$

By applying L'Hospital's rule,

$$k = \lim_{x \rightarrow 0} \frac{2\cot 2x}{\cot x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= 2 \lim_{x \rightarrow 0} \frac{\tan x}{\tan 2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

Again by applying L'Hospital's rule,

$$= 2 \lim_{x \rightarrow 0} \frac{\sec^2 x}{2\sec^2 2x}$$

$$= 1$$

Problem 2.3.2. Evaluate $\lim_{x \rightarrow 0} \log_{\tan x} \tan 2x$

Solution :

$$\lim_{x \rightarrow 0} \log_{\tan x} \tan 2x = \lim_{x \rightarrow 0} \frac{\log \tan 2x}{\log \tan x} : \frac{\infty}{\infty}$$

Applying L'Hospital's rule

$$= \lim_{x \rightarrow 0} \frac{1}{\tan 2x} \cdot 2 \cdot \sec^2 2x \cdot \frac{\tan x}{\sec^2 x} : \frac{0}{0} \text{ form}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x \cdot \cos x \cdot 2}{\sin 2x + \cos 2x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \cdot \frac{1}{2} \sin 2x}{\frac{1}{2} \sin 4x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \frac{4x}{\sin 4x}$$

$$= 1 \cdot 1$$

$$= 1$$

Problem 2.3.3. Evaluate $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$

Solution: Let

$$k = \lim_{x \rightarrow 0} \frac{a^x - b^x}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

By applying L'Hospital's rule ,

$$= \lim_{x \rightarrow 0} \frac{a^x \log a - b^x \log b}{1}$$

$$= \log a - \log b$$

$$= \log \frac{(a)}{(b)}$$

The indeterminate form $\infty - \infty$:

If $\lim_{x \rightarrow a} f(x) - g(x)$ assumes the indeterminate form $\infty - \infty$, can be transformed into a type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by rewriting the expression as one function.

Problem 2.3.4. Evaluate $\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right)$

Solution:

$$\begin{aligned}
 K &= \lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right) \\
 &= \lim_{x \rightarrow 1} \frac{x \cdot \ln(x) - x + 1}{(x-1) \cdot \ln(x)} \\
 &= \lim_{x \rightarrow 1} \frac{\ln(x)}{\frac{x-1}{x} + \ln(x)} \\
 &= \lim_{x \rightarrow 1} \frac{x \cdot \ln(x)}{x-1 + x \cdot \ln(x)} \\
 &= \lim_{x \rightarrow 1} \frac{1 + \ln(x)}{1 + 1 + \ln(x)} \\
 &= \lim_{x \rightarrow 1} \frac{1 + \ln(x)}{2 + \ln(x)} \\
 &= \frac{1}{2}
 \end{aligned}$$

Problem 2.3.5. Evaluate $\lim_{x \rightarrow 2} \left[\frac{1}{(x-2)} - \frac{1}{\log(x-1)} \right]$

Solution :

$$k = \lim_{x \rightarrow 2} \left[\frac{1}{(x-2)} - \frac{1}{\log(x-1)} \right] \quad (\infty - \infty \text{ form})$$

$$k = \lim_{x \rightarrow 2} \left[\frac{\log(x-1) - (x-2)}{(x-2)\log(x-1)} \right] \quad \left(\frac{0}{0} \text{ form} \right)$$

By applying L'Hospital's rule,

$$k = \lim_{x \rightarrow 2} \left[\frac{2-x}{(x-2) + (x-1)\log(x-1)} \right] \quad \left(\frac{0}{0} \text{ form} \right)$$

Again applying L'Hospital's rule,

$$\begin{aligned}
 k &= \lim_{x \rightarrow 2} \left[\frac{-1}{2 + \log(x-1)} \right] \\
 &= \frac{-1}{2}
 \end{aligned}$$

The indeterminate forms $1^\infty, 0^0, \infty^0$:

If $\lim_{x \rightarrow a} f(x)^{g(x)}$ assumes the indeterminate form $1^\infty, 0^0$ or ∞^0 , then let $y = \lim_{x \rightarrow a} f(x)^{g(x)}$ and taking logarithm on both sides, we get $\log y = \lim_{x \rightarrow a} g(x) (\log(f(x)))$. This limit will be in any one of the forms $\frac{0}{0}, \frac{\infty}{\infty}$ or $0 \cdot \infty$ and can be evaluated easily. If this limit is K then the required limit is obtained as $y = \lim_{x \rightarrow a} f(x)^{g(x)} = e^K$

Problem 2.3.6. Evaluate $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$ (VTU Model 2022, Jan 2021, Jan 2020, June 2019, Jan 2018, July 2015, Model 2014)

Solution: Let

$$k = \lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} \quad (1^\infty \text{ form})$$

Apply log on both sides

$$\begin{aligned} \log k &= \log \lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0} \log \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0} \frac{1}{x} \log \left(\frac{a^x + b^x + c^x}{3} \right) \\ &= \lim_{x \rightarrow 0} \frac{\log(a^x + b^x + c^x) - \log 3}{x} \quad \left(\frac{0}{0} \text{ form} \right) \end{aligned}$$

By applying L'Hospital's rule,

$$\begin{aligned} \log k &= \lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x} \\ &= \frac{(\log a + \log b + \log c)}{3} \\ &= \frac{\log(abc)}{3} \end{aligned}$$

$$i.e. \log k = \log(abc)^{\frac{1}{3}}$$

$$\therefore k = (abc)^{\frac{1}{3}}$$

Problem 2.3.7. Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} (\tan x^{\cos x})$

Solution : Let

$$k = \lim_{x \rightarrow \frac{\pi}{2}} (\tan x^{\cos x}) \quad (\infty^0 \text{ form})$$

Apply log on both sides

$$\log k = \log \lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{\cos x}$$

$$\begin{aligned} \log k &= \lim_{x \rightarrow \frac{\pi}{2}} \log (\tan x)^{\cos x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \cos x \log (\tan x) \quad (0 * \infty \text{ form}) \end{aligned}$$

By applying L'Hospital's rule,

$$\log k = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\log \tan x}{\sec x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

By simplifying we get,

$$\log k = \lim_{x \rightarrow \frac{\pi}{2}} \cot x \operatorname{cosec} x$$

$$i.e. \log k = 0 \implies k = e^0 = 1$$

Problem 2.3.8. Prove that $\lim_{x \rightarrow a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$.

Solution : Let

$$L = \lim_{x \rightarrow a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$$

$$\log L = \lim_{x \rightarrow a} \tan \left(\frac{\pi x}{2a}\right) \log \left(2 - \frac{x}{a}\right)$$

$$= \lim_{x \rightarrow a} \frac{\log \left(2 - \frac{x}{a}\right)}{\cot \left(\frac{\pi x}{2a}\right)} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \lim_{x \rightarrow a} \frac{1}{\left(2 - \frac{x}{a}\right)} \left(-\frac{1}{a}\right) \frac{1}{\left(-\operatorname{cosec}^2 \frac{\pi x}{2a}\right) \left(\frac{\pi}{2a}\right)} \quad [\text{Applying L'Hospital's rule}]$$

$$= \frac{2}{\pi}$$

$$\log L = \frac{2}{\pi}$$

$$L = e^{\frac{2}{\pi}}$$

Problem 2.3.9. Prove that $\lim_{x \rightarrow 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$.

Solution: Let

$$\begin{aligned}
 L &= \lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}} \quad [1^\infty \text{ form}] \quad \left[\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right] \\
 \log L &= \lim_{x \rightarrow 0} \frac{1}{x^2} \log \left(\frac{\tan x}{x} \right) \\
 &= \lim_{x \rightarrow 0} \frac{\log \left(\frac{\tan x}{x} \right)}{x^2} \quad \left[\begin{matrix} 0 \\ 0 \end{matrix} \right] \\
 &= \lim_{x \rightarrow 0} \frac{x}{\tan x} \left(\frac{x \sec^2 x - \tan x}{x^2} \right) \cdot \frac{1}{2x} \quad [\text{Applying L'Hospital's rule}] \\
 &= \lim_{x \rightarrow 0} \frac{x \sec^2 x - \tan x}{2x^3} \quad \left[\begin{matrix} 0 \\ 0 \end{matrix} \right] \quad \left[\because \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1 \right] \\
 &= \lim_{x \rightarrow 0} \frac{\sec^2 x + x \cdot 2 \sec^2 x \tan x - \sec^2 x}{6x^2} \quad [\text{Applying L'Hospital's rule}] \\
 &= \lim_{x \rightarrow 0} \frac{\sec^2 x}{3} \cdot \frac{\tan x}{x} = \frac{1}{3} \\
 \log L &= \frac{1}{3} \\
 L &= e^{\frac{1}{3}}
 \end{aligned}$$

Problem 2.3.10. Evaluate $\lim_{x \rightarrow \pi/2} (\sin x)^{\tan x}$

Solution :

$$\begin{aligned}
 K &= \lim_{x \rightarrow \pi/2} (\sin x)^{\tan x} \quad (= 1^\infty) \\
 \Rightarrow \log k &= \lim_{x \rightarrow \pi/2} \log [(\sin x)^{\tan x}] \\
 &= \lim_{x \rightarrow \pi/2} \tan x \cdot \log(\sin x) \\
 &= \lim_{x \rightarrow \pi/2} \frac{\log(\sin x)}{1/\tan x} \\
 \Rightarrow \log k &= \lim_{x \rightarrow \pi/2} \frac{\log(\sin x)}{\cot x} \quad \left(= \frac{0}{0} \right)
 \end{aligned}$$

Applying L-Hospital's rule,

$$\begin{aligned}
 \log k &= \lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sin x} \cdot (\cos x)}{-\operatorname{cosec}^2 x} \\
 &= \lim_{x \rightarrow \pi/2} -\frac{\cot x}{\operatorname{cosec}^2 x} \\
 &= -\frac{\cot(\pi/2)}{\operatorname{cosec}(\pi/2)} \\
 &= \frac{0}{2} = 0 \\
 \Rightarrow k &= e^0 = 1
 \end{aligned}$$

2.4 Partial Differentiation :

Let $z = f(x, y)$ be a function of two variables x and y . The first order partial derivative of z w.r.to x , denoted by $\frac{\partial z}{\partial x}$ or z_x is defined as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

From the above definition, we understand that $\frac{\partial z}{\partial x}$ is obtained by differentiating z only w.r.to x , treating y as a constant.

Similarly, the first order partial derivative of z w.r.to y , denoted by $\frac{\partial z}{\partial y}$ or z_y is defined as

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

From the above definition, we understand that $\frac{\partial z}{\partial y}$ is obtained by differentiating z only w.r.to y , treating x as a constant.

The second order partial derivatives are

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} \text{ or } z_{xx} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) \\ \frac{\partial^2 z}{\partial y^2} \text{ or } z_{yy} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) \\ \frac{\partial^2 z}{\partial x \partial y} \text{ or } z_{yx} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \\ \frac{\partial^2 z}{\partial y \partial x} \text{ or } z_{xy} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) \end{aligned}$$

Problem 2.4.1. If $u = \sin xy$ then find $u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{yx}$. Also verify that $u_{xy} = u_{yx}$.

Solution :

$$u = \sin xy \tag{1}$$

differentiate (1) partially w.r.to x ,

$$u_x = \frac{\partial}{\partial x} [\sin(xy)] = y \cos(xy)$$

differentiate (1) partially w.r.to y ,

$$u_y = \frac{\partial}{\partial y} [\sin(xy)] = x \cos(xy)$$

differentiate u_x partially w.r.to x ,

$$u_{xx} = \frac{\partial}{\partial x} [y \cos(xy)] = -y^2 \sin(xy)$$

differentiate u_y partially w.r.to y ,

$$u_{yy} = \frac{\partial}{\partial y} [x \cos(xy)] = -x^2 \sin(xy)$$

differentiate u_y partially w.r.to x ,

$$u_{xy} = \frac{\partial}{\partial x} [x \cos(xy)] = -xy \sin(xy) + \cos(xy)$$

differentiate u_x partially w.r.to y ,

$$u_{yx} = \frac{\partial}{\partial y} [y \cos(xy)] = -xy \sin(xy) + \cos(xy)$$

From last 2 results ,it is clear that $u_{xy} = u_{yx}$.

Problem 2.4.2. If $u = e^{(ax+by)} f(ax - by)$, then prove that $bu_x + au_y = 2abu$. (VTU Model 2022)

Solution :

$$u = e^{(ax+by)} f(ax - by) \quad (1)$$

differentiate (1) partially w.r.to x ,

$$u_x = \frac{\partial}{\partial x} [e^{(ax+by)} f(ax - by)]$$

$$u_x = a[e^{(ax+by)} f'(ax - by)] + au$$

differentiate (1) partially w.r.to y ,

$$u_y = \frac{\partial}{\partial y} [e^{(ax+by)} f(ax - by)]$$

$$u_y = -b[e^{ax+by} f'(ax - by)] + bu$$

Consider,

$$\begin{aligned} LHS &= bu_x + au_y = b [ae^{ax+by} f'(ax - by) + au] + a [-be^{ax+by} f'(ax - by) + bu] \\ &= 2abu \\ &= RHS \end{aligned}$$

Problem 2.4.3. If $u = f(x + ct) + g(x - ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ (VTU Jan 2018)

Solution :

$$u = f(x + ct) + g(x - ct) \quad (1)$$

differentiate (1) partially w.r.to x ,

$$u_x = f'(x + ct) + g'(x - ct)$$

differentiate again partially w.r.to x ,

$$u_{xx} = f''(x + ct) + g''(x - ct) \quad (2)$$

differentiate (1) partially w.r.to t ,

$$u_t = c[f'(x + ct) - g'(x - ct)]$$

differentiate again partially w.r.to t ,

$$u_{tt} = c^2[f''(x + ct) + g''(x - ct)] \quad (3)$$

Using (2) in (3), we get $u_{tt} = c^2 u_{xx}$

Problem 2.4.4. If $f = \tan^{-1}\left(\frac{y}{x}\right)$ then find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$. Also verify that $f_{xy} = f_{yx}$.

Solution: We have

$$f = \tan^{-1}\left(\frac{y}{x}\right) \quad (1)$$

Differentiating (1) partially with respect to x , we get

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{-y}{x^2}\right) = \left(\frac{-y}{x^2 + y^2}\right) \quad (2)$$

Differentiating (1) partially with respect to y , we get

$$\frac{\partial f}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{x}{x^2 + y^2} \quad (3)$$

Differentiating (2) partially with respect to y , we get

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{-y}{x^2 + y^2} \right) = \frac{(x^2 + y^2)(-1) - (-y)(2y)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned} \quad (4)$$

Differentiating (3) partially with respect to x , we get

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned} \tag{5}$$

∴ From eqns. (4) and (5), we get $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Problem 2.4.5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then prove that $u_x + u_y + u_z = \frac{3}{x+y+z}$ and hence show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{(x+y+z)^2} \tag{VTU Jan 2014}$$

Solution : Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\therefore \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \tag{... (i)}$$

$$\text{Similarly, } \frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \tag{... (ii)}$$

$$\text{and } \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \tag{... (iii)}$$

Adding eqns. (i), (ii) and (iii), we get

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} \\ &\text{or} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)} \\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3}{x + y + z} \end{aligned}$$

$$\text{As } a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\text{or } \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \tag{iv}$$

$$\begin{aligned} \text{Now, } \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) u \\ &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) \\ &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{3}{x + y + z} \right) \text{ from (iv)} \\ &= 3 \left[\frac{\partial}{\partial x} \left(\frac{1}{x + y + z} \right) + \frac{\partial}{\partial y} \left(\frac{1}{x + y + z} \right) + \frac{\partial}{\partial z} \left(\frac{1}{x + y + z} \right) \right] \\ &= 3 \left[-\frac{1}{(x + y + z)^2} - \frac{1}{(x + y + z)^2} - \frac{1}{(x + y + z)^2} \right] \\ &= \frac{-9}{(x + y + z)^2} \end{aligned}$$

Problem 2.4.6. If $u = (1 - 2xy + y^2)^{-\frac{1}{2}}$ and $x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = y^2 u^k$, then find the value of K. (VTU Jan 2016)

Solution : $u = (1 - 2xy + y^2)^{-\frac{1}{2}}$

Differentiate partially w. r. t. x and y

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{1}{2} (1 - 2xy + y^2)^{-\frac{3}{2}} (-2y) \\ &= u^3 y \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{1}{2} (1 - 2xy + y^2)^{-\frac{3}{2}} (-2x + 2y) \\ &= (x - y)u^3 \end{aligned}$$

$$\therefore x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = u^3 xy + (-xy + y^2) u^3 = u^3 y^2$$

Problem 2.4.7. If $z(x + y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Solution:

$$\begin{aligned} z(x + y) &= x^2 + y^2 \\ \Rightarrow z &= \frac{x^2 + y^2}{x + y} \end{aligned} \tag{1}$$

Differentiate (1) partially w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{(x + y)(2x) - (x^2 + y^2)(1)}{(x + y)^2} = \frac{x^2 - y^2 + 2xy}{(x + y)^2} \tag{2}$$

By Symmetry

$$\frac{\partial z}{\partial y} = \frac{y^2 - x^2 + 2xy}{(x + y)^2}$$

\therefore

$$\begin{aligned} \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 &= \left[\frac{2(x^2 - y^2)}{(x + y)^2}\right]^2 \\ &= \frac{4(x + y)^2(x - y)^2}{(x + y)^4} \\ &= \frac{4(x - y)^2}{(x + y)^2} \end{aligned} \tag{3}$$

$$\begin{aligned} 1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} &= 1 - \frac{x^2 - y^2 + 2xy}{(x + y)^2} - \frac{y^2 - x^2 + 2xy}{(x + y)^2} \\ &= 1 - \frac{4xy}{(x + y)^2} = \frac{(x + y)^2 - 4xy}{(x + y)^2} = \frac{(x - y)^2}{(x + y)^2} \end{aligned} \tag{4}$$

From (3) and (4)

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

2.5 Total Derivative

If $u = f(x, y)$ is a function of two variables, Then its total differential is denoted by du and is defined as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Similarly if $u = f(x, y, z)$ is a function of three variables then

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Problem 2.5.1. Find the total derivative for the following function : $u = x^3 + y^3 + x^2y + xy^2$

Solution :

$$\begin{aligned} du &= \frac{\partial(x^3 + y^3 + x^2y + xy^2)}{\partial x} dx + \frac{\partial(x^3 + y^3 + x^2y + xy^2)}{\partial y} dy \\ &= (3x^2 + 2xy + y^2)dx + (3y^2 + x^2 + 2xy)dy \end{aligned}$$

Problem 2.5.2. Find the total derivative of $u = xy^2z^3$

Solution :

$$\begin{aligned} du &= \frac{\partial(xy^2z^3)}{\partial x} dx + \frac{\partial(xy^2z^3)}{\partial y} dy + \frac{\partial(xy^2z^3)}{\partial z} dz \\ &= (y^2z^3)dx + (2xyz^3)dy + (3xy^2z^2)dz \end{aligned}$$

Problem 2.5.3. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t} \cos 3t$, $z = e^{2t} \sin 3t$.

Solution : Here u is a function of x, y, z and x, y, z are in turn functions of t . Thus u is a function ' t ' via the intermediate variables x, y, z . Then

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} \\ &= 2x \cdot 2e^{2t} + 2y \cdot (2e^{2t} \cos 3t - 3e^{2t} \sin 3t) \\ &\quad + 2z (2e^{2t} \sin 3t + 3e^{2t} \cos 3t) \end{aligned}$$

Rewriting in terms of $x, y =$

$$\begin{aligned} &= 2x \cdot 2 \cdot x + 2 \cdot y(2y - 3 \cdot z) + 2z(2z + 3y) \\ &= 4(x^2 + y^2 + z^2) \end{aligned}$$

or in terms of t

$$\frac{du}{dt} = 4 (e^{4t} + e^{4t} (\cos^2 3t + \sin^2 3t)) = 8e^{4t}$$

verification by direct substitution:

$$u = x^2 + y^2 + z^2 = e^{4t} + e^{4t} \cos^2 3t + e^{4t} \sin^2 3t = 2e^{4t}$$

$$\frac{du}{dt} = 8e^{4t}$$

Problem 2.5.4. Find the total differential coefficient of x^2y w.r.t. x when x, y are connected by $x^2 + xy + y^2 = 1$

Solution : Let $u = x^2y$. then the total differential is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Thus the total differential coefficient of u w.r.t. x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$$

$$\frac{du}{dx} = 2xy + x^2 \frac{dy}{dx}$$

From the implicit relation $f = x^2 + xy + y^2 = 1$, we calculate

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x + y}{x + 2y}$$

so

$$\begin{aligned} \frac{du}{dx} &= 2xy + x^2 \cdot \frac{dy}{dx} = 2xy + x^2 \left(-\frac{(2x+y)}{(x+2y)} \right) \\ \frac{du}{dx} &= 2xy - \frac{x^2(2x+y)}{(x+2y)} \end{aligned}$$

2.6 Differentiation of Composite functions :

Note 1:

If $u = f(x, y)$ and x and y are functions of single variable t , then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

Note 2 : (Chain Rule)

If $u = f(x, y)$ and x and y are functions of r, s , then u is a function of r, s . In this case,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

Note : The above formulae can be extended to functions of three or more variables.

Problem 2.6.1. If $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ find $\frac{du}{dt}$ as a function of t .

Solution : We have $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$

i.e $\frac{du}{dt} = \frac{1}{y} \cos\left(\frac{x}{y}\right) e^t - \frac{x}{y^2} \cos\left(\frac{x}{y}\right) 2t$

$$= \frac{e^t}{t^2} \cos\left(\frac{e^t}{t^2}\right) - 2 \frac{e^t}{t^3} \cos\left(\frac{e^t}{t^2}\right)$$

$$= e^t \cos\left(\frac{e^t}{t^2}\right) \left[\frac{1}{t^2} - \frac{2}{t^3}\right]$$

Problem 2.6.2. If z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$

Solution : Here z is a function of x and y , where x and y are functions of u and v .

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \dots \quad \text{(i) and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \dots \quad \text{(ii)}$$

Also given that

$$x = e^u + e^{-v} \text{ and } y = e^{-u} - e^v$$

$$\therefore \frac{\partial x}{\partial u} = e^u, \quad \frac{\partial x}{\partial v} = -e^{-v}, \quad \frac{\partial y}{\partial u} = -e^{-u}, \quad \frac{\partial y}{\partial v} = -e^v$$

\therefore From (i) we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} (e^u) + \frac{\partial z}{\partial y} (-e^{-u}) \quad \text{(iii)}$$

and from (ii) we get

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (-e^{-v}) + \frac{\partial z}{\partial y} (-e^v) \dots \quad \text{(iv)}$$

Subtracting (iv) from (iii) we get

$$\begin{aligned} \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} &= (e^u + e^{-v}) \frac{\partial z}{\partial x} - (e^{-u} - e^v) \frac{\partial z}{\partial y} \\ &= x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} \end{aligned}$$

Problem 2.6.3. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then Prove that $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$ (VTU June 2019, July 2017)

Solution : Let $r = 2x - 3y$, $s = 3y - 4z$, $t = 4z - 2x$, then

$$u \rightarrow (r, s, t) \rightarrow (x, y, z)$$

$$\therefore u \rightarrow (x, y, z)$$

$$\begin{aligned} \frac{\partial r}{\partial x} &= 2, \frac{\partial r}{\partial y} = -3, \frac{\partial r}{\partial z} = 0, \\ \frac{\partial s}{\partial x} &= 0, \frac{\partial s}{\partial y} = 4, \frac{\partial s}{\partial z} = -4, \\ \frac{\partial t}{\partial x} &= -2, \frac{\partial t}{\partial y} = 0, \frac{\partial t}{\partial z} = 4 \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} \\ &= 2 \left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \right) \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y} \\ &= 3 \left(-\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} \right) \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial z} \\ &= 4 \left(-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \right) \\ \therefore \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z &= \left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \right) + \left(-\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} \right) + \left(-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \right) \\ &= 0 \end{aligned}$$

Problem 2.6.4. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then Prove that $xu_x + yu_y + zu_z = 0$ (VTU Model 2022, June 2019, June 2018, July 2017, July 2016)

Solution : Let $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$, then $u \rightarrow (r, s, t) \rightarrow (x, y, z)$

$\therefore u \rightarrow (x, y, z)$

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} \\ &= \left(\frac{\partial u}{\partial r} \right) \frac{1}{y} + 0 - \left(\frac{\partial u}{\partial t} \right) \left(\frac{-z}{x^2} \right) \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y} \\ &= \left(\frac{\partial u}{\partial r} \right) \left(\frac{-x}{y^2} \right) + \frac{\partial u}{\partial s} \left(\frac{1}{z} \right) + 0 \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial z} \\ &= 0 - \left(\frac{\partial u}{\partial s} \right) \left(\frac{1}{z^2} \right) + \left(\frac{\partial u}{\partial t} \right) \left(\frac{1}{x} \right) \end{aligned}$$

$\therefore xu_x + yu_y + zu_z$

$$\begin{aligned}
 &= \left(\frac{\partial u}{\partial r}\right) \frac{x}{y} - \frac{\partial u}{\partial t} \left(\frac{-z}{x}\right) + \left(\frac{\partial u}{\partial r}\right) \left(\frac{-x}{y}\right) + \left(\frac{\partial u}{\partial s}\right) \left(\frac{y}{z}\right) - \left(\frac{\partial u}{\partial s}\right) \left(\frac{1}{z}\right) + \left(\frac{\partial u}{\partial t}\right) \left(\frac{z}{x}\right) \\
 &= 0
 \end{aligned}$$

Problem 2.6.5. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then prove that $x^2u_x + y^2u_y + z^2u_z = 0$ (VTU July 2014)

Solution : Let

$$v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}, \quad w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

Differentiate V and W partially w.r.to $x, y,$ and z

$$\begin{aligned}
 \frac{\partial v}{\partial x} &= -\frac{1}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial v}{\partial z} = 0, \\
 \frac{\partial w}{\partial x} &= -\frac{1}{x^2}, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = \frac{1}{z^2}
 \end{aligned}$$

Now

$$u = u(v, w)$$

∴

$$\begin{aligned}
 \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial x} = -\frac{1}{x^2} \frac{\partial u}{\partial v} - \frac{1}{x^2} \frac{\partial u}{\partial w} \\
 &= -\frac{1}{x^2} \left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \tag{1}
 \end{aligned}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial y} = \frac{1}{y^2} \frac{\partial u}{\partial v} \tag{2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial z} = \frac{1}{z^2} \frac{\partial u}{\partial w} \tag{3}$$

from (1), (2), and (3),

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} = 0$$

2.7 Jacobians (J) :

If u and v are any 2 functions of 2 independent variables x and y , then the Jacobian of (u,v) w.r. to

(x,y) is denoted by $J = \frac{\partial(u,v)}{\partial(x,y)}$ and is defined as $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ Similarly, If (u,v,w) are functions of

3 independent variables (x,y,z) then the Jacobian of (u,v,w) w.r. to (x,y,z) is denoted by $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

and is defined as $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Problem 2.7.1. If $u = x + y + z$ $v = y + z$ $w = z$, then find $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution :

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

Problem 2.7.2. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$, then Find $J \frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution :

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2x[(x+z) - (y+z)] - 2y[(y+z) - (y+x)] + 2z[(y+z) - (x+z)]$$

$$= 2(xz - xy - yz + xy + yz - xz)$$

$$= 0$$

Problem 2.7.3. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, then prove that $J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$ (VTU July 2015, July 2014).

Solution : We have

$$\begin{aligned}
 J &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \\
 J &= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix} \\
 &= \frac{1}{xyz} \begin{vmatrix} \frac{-yz}{x} & z & y \\ z & \frac{-xz}{y} & x \\ y & x & \frac{-xy}{z} \end{vmatrix} \\
 &= \frac{1}{xyz} [4xyz] \\
 &= 4
 \end{aligned}$$

Problem 2.7.4. If $u + v = e^x \cos y$ $u - v = e^x \sin y$, then find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution : We have $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

By adding given equations, we get $u = \frac{e^x}{2} (\cos y + \sin y)$

By subtracting equations, $v = \frac{e^x}{2} (\cos y - \sin y)$

$$\begin{aligned}
 \therefore J &= \begin{vmatrix} \frac{e^x}{2} (\cos y + \sin y) & \frac{e^x}{2} (\cos y - \sin y) \\ \frac{e^x}{2} (\cos y - \sin y) & -\frac{e^x}{2} (\cos y + \sin y) \end{vmatrix} \\
 J &= \frac{e^{2x}}{4} \begin{vmatrix} (\cos y + \sin y) & (\cos y - \sin y) \\ (\cos y - \sin y) & -(\cos y + \sin y) \end{vmatrix} \\
 &= \frac{-e^{2x}}{4} (2) = \frac{-e^{2x}}{2}
 \end{aligned}$$

Problem 2.7.5. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Solution :

Given $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} &= \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} \\ &= r^2 \sin \theta \cos^2 \theta + r^2 \sin \theta \sin^2 \theta \\ &= r^2 \sin \theta \end{aligned}$$

Problem 2.7.6. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - x - y$, then find $J \frac{(u,v,w)}{(x,y,z)}$ at $(1, -1, 0)$ (VTU June 2019, June 2018)

Solution :

$$u = x + 3y^2 - z^3 \quad \dots (1)$$

$$v = 4x^2yz \quad \dots (2)$$

$$w = 2z^2 - x - y \quad \dots (3)$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -1 & -1 & 4z \end{vmatrix} \end{aligned}$$

At the point $(1, -1, 0)$,

$$\begin{aligned} J &= \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ -1 & -1 & 0 \end{vmatrix} \\ &= 1(0 - 4) + 6(0 - 4) = -4 - 24 = -28 \end{aligned}$$

2.8 Maxima and Minima for a function of two variables

A function $f(x, y)$ is said to be a maximum at (a, b) when $f(a, b)$ is greater than $f(x, y)$ for all values of x and y in the neighborhood of (a, b) .

Similarly, $f(a, b)$ is said to be a minimum at (a, b) when $f(a, b)$ is less than $f(x, y)$ for all values

of x and y in the neighborhood of (a, b) .

These definitions may be stated in analytical form as follows:

If, for all values of h and k numerically less than some small positive quantity,

- If $f(a + h, b + k) - f(a, b) =$ a negative number, then $f(a, b)$ is a maximum value of $f(x, y)$.
- If $f(a + h, b + k) - f(a, b) =$ a positive number, then $f(a, b)$ is a minimum value of $f(x, y)$.

Necessary and sufficient Conditions for extreme values

We know that a necessary condition that a function of one variable to have a maximum or a minimum for a given value of the variable is that its first derivative should be zero for the given value of the variable. Similarly, for a function $f(x, y)$ of two independent variables, a necessary condition that $f(a, b)$ should be a maximum or a minimum (i.e. a turning value) is that for $x = a, y = b$, $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$.

Hence the **following steps are used** for finding maximum and minimum values of a function $f(x, y)$

Step 1. Solve the simultaneous equations $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$. and find the values of (x, y) in the form $(a, b), (c, d), (e, f)$...These points are called as stationary points.

Step 2. For each stationary point, calculate the value of $\Delta = rt - s^2$ where $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}$ and $t = \frac{\partial^2 f}{\partial y^2}$

Step 3. The function will have a

maximum if $\Delta > 0$ and $r < 0$

minimum if $\Delta > 0$ and $r > 0$

neither a maximum nor a minimum if $\Delta < 0$

The question is undecided if $\Delta = 0$

Problem 2.8.1. Examine $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ for its extreme values

Solution :

$$f_x = 3x^2 - 12, f_y = 3y^2 - 12$$

$$r = f_{xx} = 6x, \quad s = f_{xy} = 0, \quad t = f_{yy} = 6y$$

Solving $f_x = 0$ and $f_y = 0$, we get

$$\Rightarrow 3x^2 - 12 = 0, \quad \text{and} \quad 3y^2 - 3 = 0$$

$$\Rightarrow x^2 - 4 = 0 \quad \text{and} \quad y^2 - 1 = 0$$

$$\Rightarrow x^2 = 4 \quad \text{and} \quad y^2 = 1$$

$$\Rightarrow x = \pm 2 \quad \text{and} \quad y = \pm 1$$

Hence stationary points are $(2, 1), (2, -1), (-2, 1), (-2, -1)$

$$\Delta = rt - s^2 = (6x)(6y) - 0 = 36xy$$

At $(2,1)$ $\Delta = 72 > 0$ and $r = 12 > 0$.

Therefore function takes minimum value at $(2,1)$

and the Minimum Value = $f(2, 1) = 2$

At $(2,-1)$ and $(-2,1)$, $\Delta = -72 < 0$ Hence $(-2,1)$ and $(2,-1)$ are saddle points.

At $(-2,-1)$ $\Delta = 72 > 0$ and $r = -12 < 0$. Therefore function takes maximum value at $(2,1)$

Maximum Value = $f(-2, -1) = 38$

Problem 2.8.2. Find the extreme points of the function $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$.

$$\text{Solution : } f(x, y) = 2(x^2 - y^2) - x^4 + y^4 \quad \dots (1)$$

diff (1) w.r.t x & y partially

$$\frac{\partial f}{\partial x} = 4x - 4x^3 \quad \dots (2)$$

$$\frac{\partial f}{\partial y} = -4y + 4y^3 \quad \dots (3)$$

Solving $f_x = 0$ and $f_y = 0$, we get

$$\Rightarrow 4x - 4x^3 = 0 \quad \text{and} \quad -4y + 4y^3 = 0$$

$$\Rightarrow x - x^3 = 0 \quad \text{and} \quad -y + y^3 = 0$$

$$\Rightarrow x(1 - x^2) = 0 \quad \text{and} \quad y(-1 + y^2) = 0$$

$$\Rightarrow x = 0, \quad x^2 = 1 \quad \text{and} \quad y = 0, \quad y^2 - 1 = 0$$

$$\Rightarrow x = 0, \quad x = 1, \quad x = -1 \quad \text{and} \quad y = 0, \quad y = 1, \quad y = -1$$

Hence stationary points are

$$(0, 0), (0, 1), (0, -1), (1, 0), (1, 1), (1, -1), (-1, 0), (-1, 1), (-1, -1)$$

$$r = f_{xx} = 4 - 12x^2, \quad s = f_{xy} = 0, \quad t = f_{yy} = -4 + 12y^2$$

Points	r	s	t	$rt - s^2$	conclusion
(0, 0)	$4 > 0$	0	-4	$-16 < 0$	Saddle point
(0, 1)	$4 > 0$	0	8	$32 > 0$	minimum
(1, 0)	$-8 < 0$	0	-4	$32 > 0$	maximum
(1, 1)	$-8 < 0$	0	8	$-64 < 0$	saddle point
(0, -1)	4	0	8	$32 > 0$	minimum
(1, -1)	-8	0	8	$-64 < 0$	saddle point
(-1, 0)	$-8 < 0$	0	-4	$32 > 0$	maximum
(-1, 1)	-8	0	8	$-64 < 0$	saddle point
(-1, -1)	-8	0	8	$-64 < 0$	saddle point

$$\max f = f(1, 0) = f(-1, 0) = 2(1 - 0) - 1 + 0 = 2 - 1 = 1$$

$$\min f = 2(0 - 1) - 0 + 1 = -2 + 1 = -2$$

Problem 2.8.3. Find the extreme values of the function $f(x, y) = x^3 + y^3 - 3axy$

Solution : We have, $f(x, y) = x^3 + y^3 - 3axy$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3ay, \quad q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x, \quad s = \frac{\partial^2 f}{\partial x \partial y} = -3a, \quad t = \frac{\partial^2 f}{\partial y^2} = 6y$$

For maxima and minima
 $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$3x^2 - 3ay = 0 \quad 3y^2 - 3ax = 0$$

$$\Rightarrow x^2 = ay \Rightarrow y = \frac{x^2}{a} \quad \dots \tag{1}$$

$$\Rightarrow y^2 = ax \quad \dots \tag{2}$$

Putting the value of y from (1) in (2), we get

$$x^4 = a^3x \Rightarrow x(x^3 - a^3) = 0$$

$$\Rightarrow x(x - a)(x^2 + ax + a^2) = 0$$

$$\Rightarrow x = 0, a$$

Putting $x = 0$ in (1), we get $y = 0$,

Putting $x = a$ in (1), we get $y = a$,

Stationary pairs	$(0, 0)$	(a, a)
r	0	$6a$
s	$-3a$	$-3a$
t	0	$6a$
$rt - s^2$	$-9a^2 < 0$	$27a^2 > 0$

At $(0, 0)$ there is no extremum value, since $rt - s^2 < 0$.

At (a, a) , $rt - s^2 > 0, r > 0$

Therefore (a, a) is a point of minimum value.

The minimum value of $f(a, a) = a^3 + a^3 - 3a^3 = -a^3$

Problem 2.8.4. Find the extreme values of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Solution: $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For stationary point of $f(x, y)$, we solve $f_x = 4x^3 - 4x + 4y = 0$ or $x^3 - x + y = 0$ (1)

$f_y = 4y^3 + 4x - 4y = 0$ or $y^3 + x - y = 0$ (2)

Adding (1) and (2), we get $x^3 + y^3 = 0$

$\therefore y = -x$

Substituting this value in (1), we get $x^3 - x - x = 0$

$x^3 - 2x = 0 \rightarrow x(x^2 - 2) = 0 \rightarrow x = 0, x = \sqrt{2}, -\sqrt{2}$

for $x = 0, y = 0$; for $x = \sqrt{2}, y = -\sqrt{2}$ and for $x = -\sqrt{2}, y = -\sqrt{2}$

\therefore Stationary points of f are $(0, 0), (\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

$$r = f_{xx} = 12x^2 - 4, s = f_{xy} = 4, t = f_{yy} = 12y^2 - 4$$

Point	r	s	t	$rt - s^2$
$(0, 0)$	-4	4	-4	0
$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	4	20	$384 > 0$
$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	4	20	$384 > 0$

\therefore Minima at both the points $(\sqrt{2}, -\sqrt{2})$ and

$(-\sqrt{2}, \sqrt{2})$ and minimum value at both these points = $4 + 4 - 4 - 8 - 4 = -8$.

Problem 2.8.5. Examine the function $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (VTU

Model ME 2022)

Solution :

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 \quad \dots (1)$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \quad \dots (2)$$

$$\frac{\partial f}{\partial y} = 6xy - 30y \quad \dots (3)$$

For stationary point of $f(x, y)$, we solve

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$3x^2 + 3y^2 - 30x + 72 = 0 \quad \dots (4)$$

$$\text{and } 6xy - 30y = 0 \quad \dots (5)$$

$$\Rightarrow xy - 5y = 0$$

$$\Rightarrow y(x - 5) = 0$$

$$\Rightarrow y = 0, \quad x - 5 = 0$$

$$\Rightarrow y = 0 \quad x = 5$$

When $y = 0$, from (4) we get

$$x^2 - 10x + 24 = 0$$

$$\Rightarrow x^2 - 4x - 6x + 24 = 0$$

$$\Rightarrow x(x - 4) - 6(x - 4) = 0$$

$$\Rightarrow (x - 4)(x - 6) = 0$$

$$\Rightarrow x = 4, 6$$

\therefore When $y = 0$ stationary points are $(4, 0), (6, 0)$,

when $x = 5$ from (4) we get

$$\Rightarrow 25 + y^2 - 50 + 24 = 0$$

$$y^2 - 1 = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

\therefore when $x = 5$ the stationary point are $(5, 1), (5, -1)$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial x \partial y} = 6x - 30$$

Points	r	s	t	$rt - s^2$	conclusion
(4, 0)	$-6 < 0$	0	-6	$36 > 0$	max
(6, 0)	$6 > 0$	0	6	$36 > 0$	min
(5, 1)	0	6	0	$-36 < 0$	saddle point
(5, -1)	0	-6	0	$-36 < 0$	saddle point

Maximum value is $f(4, 0) = 4^3 + 0 - 15(4^2) - 0 + 72(4) = 112$

Minimum value is $f(6, 0) = 6^3 + 0 - 15(6^2) - 0 + 72(6) = 108$

Question Bank- Module 2

Maclaurin's series :

- Determine the Maclaurin series expansion of the function, $y = \sqrt{1 + \sin 2x}$ by Maclaurin's series. (VTU July 2021, June 2019, June 2018, Model 2018, Jan 2016, July 2014) **Ans :**
 $1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
- Determine the Maclaurin series expansion of the function, $f(x) = \log(1 + \cos x)$ upto the term containing x^4 . (VTU Model CV 2022, June 2019, July 2017) **Ans :**
 $\log 2 - \frac{x^2}{4} - \frac{x^4}{96} + \dots$
- Expand $\log(\sec x)$ upto 6th degree term by Maclaurin's series (VTU Jan 2018, July 2017, 2009) **Ans :** $\frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \dots$
- Expand $\tan x$ by Maclaurin's series (VTU July 2015)
- Expand $\sin(e^x - 1)$ by Maclaurin's series (VTU June 2014) **Ans :** $x + \frac{x^2}{2!} - 5\frac{x^4}{4!} + \dots$
- Expand $f(x) = e^{\sin x}$ by Maclaurin's series (VTU Model EE 2022, June 2018, 2011) **Ans :**
 $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$
- Expand $y = \log(1 + e^x)$ by Maclaurin's series (VTU Model CV 2022, July 2016) **Ans :**

$$\log 2 + \frac{x}{2} + \frac{x^2}{8} + \dots$$

8. Expand $y = \frac{x}{(e^x-1)}$ by Maclaurin's series (VTU Jan 2015) **Ans :**
 $e(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \dots)$

9. Expand $\frac{e^x}{(1+e^x)}$ using Maclaurin's series upto 3rd degree terms. (VTU Jan 2017)

10. Expand $y = \sec x$ using Maclaurin's series up to x^4 term. (VTU Jan 2016)

11. Using Maclaurin's series Prove that $\sqrt{1 + \cos 2x} = \sqrt{2} \left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots \right]$ (VTU Jan 2021)

12. Expand $\log(1 + \sin x)$ up to the term containing x^4 using Maclaurin's series (VTU Model 2022) **Ans :** $x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

13. Expand $\log \cos x$ by Maclaurin's series up to the term containing x^6 (VTU Model ME 2022)

14. Expand $f(x) = e^{\cos x}$ by Maclaurin's series up to the term containing x^4 (VTU Model ME 2022)

15. Determine the Maclaurin series expansion of the function, $\tan x$ (VTU July 2015) **Ans :**
 $x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \dots$ **Ans :** $\log 2 - \frac{1}{2} - \frac{x^2}{2!} - \frac{1}{4} + \frac{x^4}{4!} + \dots$

16. Determine the Maclaurin series expansion of the function, $f(x) = e^x \sin x$ **Ans :**
 $1 + x + \frac{x^2}{2!} - \frac{3}{4!}x^4 - \frac{8}{5!}x^5 + \dots$

17. Determine the Maclaurin series expansion of the function, $\log(1 + x)$ **Ans :**
 $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

18. Determine the Maclaurin series expansion of the function, $e^{\tan^{-1}x}$ **Ans :**
 $1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - 7\frac{x^4}{4!} + \dots$

19. Determine the Maclaurin series expansion of the function, $e^{x \cos x}$ **Ans :**
 $1 + x + \frac{x^2}{2!} - 2\frac{x^3}{3!} - 11\frac{x^4}{4!} + \dots$

20. Determine the Maclaurin series expansion of $y = e^{\tan x}$ **Ans :** $1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \dots$

Indeterminate Forms:

1. Evaluate $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$ (VTU Model 2022, Jan 2021, Jan 2020, June 2019, Jan 2018, July 2015, Model 2014) **Ans :** $(abc)^{\frac{1}{3}}$

2. Evaluate $\lim_{x \rightarrow a} \left[2 - \frac{x}{a} \right]^{\tan\left(\frac{\pi x}{2a}\right)}$ (VTU May 2010) **Ans : $e^{\frac{2}{\pi}}$**
3. Evaluate $\lim_{x \rightarrow 0} (\cos x)^{\cot^2 x}$ (VTU July 2017)
4. Evaluate $\lim_{x \rightarrow 0} \left[\frac{\tan x}{x} \right]^{\frac{1}{x^2}}$ (VTU Model EE 2022, Jan 2017, July 2016) **Ans : $e^{\frac{1}{3}}$**
5. Evaluate $\lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right]^{\frac{1}{x^2}}$ (VTU Model ME 2022, July 2017)
6. Evaluate $\lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right]^{\frac{1}{x}}$ (VTU Jan 2015)
7. Evaluate $\lim_{x \rightarrow 0} \left(\frac{2^x + 3^x + 4^x}{3} \right)^{\frac{1}{x}}$ (VTU Jan 2018, Jan 2014)
8. Evaluate $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x + d^x}{4} \right)^{\frac{1}{x}}$ (VTU Model ME 2022, July 2021, June 2019, June 2018, Model 2018, July 2017, Jan 2016, July 2014, Dec 2011)
9. Evaluate $\lim_{x \rightarrow 0} \left[\frac{1}{x} - \frac{\log(1+x)}{x^2} \right]$ (VTU Jan 2018)
10. Evaluate $\lim_{x \rightarrow 0} \left[\frac{1}{x} \right]^{2 \sin x}$ (VTU Model 2018)
11. Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x} \right)^{2 \sin x}$ (VTU Jan 2021)
12. Evaluate $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$ (VTU Model EE 2022, July 2021) $\lim_{x \rightarrow 0} [a^x + x]^{\frac{1}{x}}$ (VTU Model 2022) **Ans : ae**
13. $\lim_{x \rightarrow 0} \left[\frac{\tan x}{x} \right]^{\frac{1}{x}}$ (VTU Model 2022)
14. Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{\tan 2x}$ (VTU Model EE 2022)
15. Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{\cos x}$ **Ans : 1**
16. Evaluate $\lim_{x \rightarrow \pi/2} (\sin x)^{\tan x}$ (VTU Model CV 2022, June 2018) **Ans : 1**
17. Evaluate $\lim_{x \rightarrow 0} (\cot x)^{\tan x}$ (VTU Model CV 2022)

Partial Differentiation :

1. If $u = e^{(ax+by)} f(ax - by)$, then prove that $bu_x + av_y = 2abu$. (VTU Model 2022)
2. If $z = \sin(ax + y) + \cos(ax - y)$, prove that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$ (VTU July 2017)
3. If $u(x + y) = x^2 + y^2$, then prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)^2 = 4 \left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)$ (VTU Model ME 2022, July 2017)

4. If $x = r \cos \theta$, $y = r \sin \theta$ then show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right]$ (VTU July 2017)
5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then prove that $u_x + u_y + u_z = \frac{3}{x+y+z}$ and hence show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{(x+y+z)^2}$$
 (VTU Jan 2014)
6. If $V = (1 - 2xy + y^2)^{-\frac{1}{2}}$ and $x \frac{\partial V}{\partial x} - y \frac{\partial V}{\partial y} = y^2 V^k$, then find the value of K. (VTU Jan 2016) **Ans : 3**
7. If $u = f(x + ct) + g(x - ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ (VTU Jan 2018)
8. If $u = \log(\tan x + \tan y + \tan z)$ show that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$. (VTU Model 2022)
9. If $u = \frac{1}{\sqrt{(x^2+y^2+z^2)}}$, then prove that $v_{xx} + v_{yy} + v_{zz} = 0$
10. If $u = \log\left[\frac{x^2+y^2}{x+y}\right]$ then show that $xu_x + yu_y = 1$
11. If $u = \log(x^2 + y^2 + z^2)$, then show that $(x^2 + y^2 + z^2)(u_{xx} + u_{yy} + u_{zz}) = 2$
12. If $u = \tan^{-1}\left(\frac{y}{x}\right)$ then find the value of $u_{xx} + u_{yy}$ (VTU Model EE 2022) **Ans : 0**

Differentiation of Composite functions :

1. Find total derivative of u with respect to t where $u = \tan^{-1}\left(\frac{y}{x}\right)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ (VTU Model EE 2022, July 2017)
2. If $u = x^3 y^2 + x^2 y^3$, $x = at^2$, $y = 2at$, then find $\frac{du}{dt}$
3. If $u = xy^2 + x^2 y$, $x = at^2$, $y = 2at$, then find $\frac{du}{dt}$ (VTU Model ME 2022)
4. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then find the value of $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$ (VTU June 2019, July 2017)
5. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then Prove that $xu_x + yu_y + zu_z = 0$ (VTU Model 2022, June 2019, June 2018, July 2017, July 2016)
6. If $u = f(x - y, y - z, z - x)$ then Prove that $u_x + u_y + u_z = 0$ (VTU Model ME 2022, July 2021, Jan 2020, Model 2018, Model 2014, July 2003)

7. If $u = f(y - z, z - x, x - y)$ then Prove that $u_x + u_y + u_z = 0$ (VTU July 2019)
8. If $u = f(xz, \frac{y}{z})$ then Prove that $xu_x - yu_y - zu_z = 0$ (VTU July 2004)
9. If $u = f(\frac{y-x}{xy}, \frac{z-x}{xz})$ then prove that $x^2u_x + y^2u_y + z^2u_z = 0$ (VTU Model EE 2022, July 2014)
10. If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then prove that $(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = (\frac{\partial z}{\partial r})^2 + \frac{1}{r^2} (\frac{\partial z}{\partial \theta})^2$ (VTU Jan 2018, July 2018, Jan 2015)
11. If $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, then prove that $xz_x - yz_y = z_u - z_v$. (VTU Jan 2016)
12. If $u = \log(\tan x + \tan y + \tan z)$ show that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$. (VTU Model 2022) If $u = e^{(ax+by)} f(ax - by)$, then prove that $bu_x + au_y = 2abu$ by using the concept of composite functions. (VTU Model 2022)
13. If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, then show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$
14. If $z = f(x, y)$ where $x = u - v$, $y = uv$ then show that
 (i) $uz_u - vz_v = (u + v)z_x$
 (ii) $z_u + z_v = (u + v)z_y$

Jacobians (J) :

1. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, then prove that $J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$ (VTU Model EE 2022, Model 2018, June 2018, July 2015, July 2014).
2. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$, then Find $J \frac{\partial(u,v,w)}{\partial(x,y,z)}$ (VTU Model ME 2022, Jan 2020, Jan 2018, July 2017, Jan 2015) Ans : 0
3. If $u = x + y + z$, $vu = y + z$, $uvw = z$, then find $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ (VTU Model 2022, Jan 2016) Ans : u^2v
4. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then find the Jacobian of x, y, z w.r.to r, θ, ϕ . (VTU Model 2022, July 2016, July 2014) Ans : $r^2 \sin \theta$
5. If $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$, find the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. Determine whether u, v, w are functionally dependent. (VTU July 2017)

6. If $u = \frac{x_2 x_3}{x_1}$, $v = \frac{x_1 x_3}{x_2}$, $w = \frac{x_1 x_2}{x_3}$, then find the value of $J = \frac{\partial(u,v,w)}{\partial(x_1, x_2, x_3)}$ (VTU Jan 2017)
7. If $u = x + 3y^2$, $v = 4x^2 yz$, $w = 2z^2 - xy$, then Find $J \frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1, -1, 0)$ (VTU Jan 2018)
8. If $u = x + 3y^2 - z^3$, $v = 4x^2 yz$, $w = 2z^2 - x - y$, then find $J \frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1, -1, 0)$ (VTU June 2019, June 2018) Ans : -28
9. If $u = x + 3y^2 - z^3$, $v = 4x^2 yz$, $w = 2z^2 - xy$, then Find $J \frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1, -1, 0)$ (VTU Model CV 2022)
10. If $u = 3x + 2y - z$, $v = x - 2yz$, $w = x^2 + 2xy - xz$, then Show that $J \left(\frac{u,v,w}{x,y,z} \right) = 0$ (VTU 2021)
11. If $u = x + y + z$, $v = y + z$, $w = z$, then find $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ Ans : 1
12. If $u + v = e^x \cos y$, $u - v = e^x \sin y$, then find $\frac{\partial(u,v)}{\partial(x,y)}$. Ans : $\frac{-e^{2x}}{2}$

Maxima and minima for a function of two variables

Find the extreme values of the following functions

1. $f(x, y) = x^3 + 3xy^2 - 3y^2 - 3x + 4$ (VTU Model 2022)
2. $f(x, y) = x^2 + y^2 + 6x - 12$ (VTU Model 2022)
3. $f(x, y) = x^3 + 3x^2 + 4xy + y^2$ (VTU Model ME 2022)
4. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (VTU Model ME 2022) Ans : Maximum value is $f(4, 0) = 112$ and Minimum value is $f(6, 0) = 108$
5. $f(x, y) = x^3 + y^3 - 3axy$ (VTU Model EE 2022) Ans : Extreme value = $-a^3$
6. $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ Ans : Minimum value at two points = -8 .
7. $f(x, y) = x^3 y^2 (1 - x - y)$
8. $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$
9. $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ (VTU Jan 2020, June 2019)

10. $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ Ans : Minimum Value = $f(2, 1) = 2$, Maximum Value = $f(-2, -1) = 38$
11. $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ (VTU Jan 2021)
12. S.T. $f(x, y) = xy(a - x - y)$ is max. at $(\frac{a}{3}, \frac{a}{3})$. Find the maximum value. (VTU Model CV 2022, July 2021)
13. Find the extreme points of the function $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$ (VTU Model CV 2022)
Ans : $\max f = 1$ & $\min f = -2$
14. Find the extreme values of the function $f(x, y) = \sin x \sin y \sin(x + y)$ Where $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$
15. In a plane triangle find the maximum value of $\cos A \cos B \cos C$ (VTU June 2019) Ans : maximum value is $\frac{1}{8}$
16. In a plane triangle find the maximum value of $\sin x \sin y \sin z$ (VTU 2021)
17. Examine the function $f(x, y) = \sin x + \sin y + \sin(x + y)$ for extremum. Ans : Max value is $\frac{3\sqrt{3}}{2}$
18. Examine the function $f(x, y) = 1 + \sin(x^2 + y^2)$ for extremum
(Ans : minimum value is $f(0, 0) = 1$)
19. Find the stationary points of $z = x^2 - xy + y^2 - 2x + y$ and hence find its maximum and minimum values.
(Ans: Minimum value is $f(1, 0) = -1$)
20. Show that the function $f(x, y) = x^3 + y^3 - 3xy + 1$ is minimum at the point $(1, 1)$ (VTU Model EE 2022)



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Mathematics I for Civil Stream - Lecture Notes

Subject Code : BMATC101

Module 3 - Ordinary Differential Equations (ODEs) of First Order

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Module 3-Ordinary Differential Equations (ODE's) of first order

Syllabus

Linear and Bernoulli's differential equations. Exact and reducible to exact differential equations, Orthogonal trajectories, L-R & C-R circuits, Problems

Non-linear differential equations: Introduction to general and singular solutions, solvable for p only, Clairaut's equations, reducible to Clairaut's equations. Problems.

Self-Study: Applications of ODE, Solutions of nonlinear ODEs-Solvable for x and y

Definition of ODE: A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives. A differential equation is called an **ordinary differential equation(ODE)**, if it has ordinary derivatives in it. An ODE involves derivatives of an unknown function with respect to a single independent variable, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t).

e.g. : $\frac{dy}{dx} + 5y = \sin x$ is an ordinary differential equation.

Partial Differential Equations:

A differential equation is called a **partial differential equation(PDE)**, if it has partial derivatives of an unknown function of two or more variables.

Order of a Differential Equation: The order of a differential equation is the order of the highest order derivative occurring in it.

Degree of a Differential Equation: The degree of a differential equation is the degree of the highest order derivative occurring in it, when the derivatives are free from radicals and fractions.

Solution of a Differential Equation: A solution or Integral of a differential equation is a relation between the independent and the dependent variables which satisfies the given differential equation.

General Solution or Complete Integral: A solution containing the number of arbitrary constants equal to the order of the equation is called the general solution or complete integral.

Particular Solution : Any solution obtained from the general solution by giving specific values to one or more of the arbitrary constants is called a particular solution.

Initial Value Problem: In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an initial condition $y(x_0) = y_0$ with given values and , that is used to determine a value of the arbitrary constant c . Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy -plane. An ODE, together with an initial condition, is called an initial value problem. Thus, if the ODE is explicit, $y' = f(x, y)$ the initial value problem is of the form $y' = f(x, y), y(x_0) = y_0$

3.1 Exact Differential equation:

A differential Equation of the form $Mdx + Ndy = 0$ or $M(x, y)dx + N(x, y)dy = 0$ is said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and its solution is obtained by using the formula,

$$\int_{(y-\text{constant})} M dx + \int (\text{Terms of } N \text{ not containing } x) dy = c$$

Problem 3.1.1. Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Solution: Given equation is of the form $Mdx + Ndy = 0$

where $M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$

$$\therefore \frac{\partial M}{\partial y} = -4x - 4y, \quad \frac{\partial N}{\partial x} = -4y - 4x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\therefore Given equation is exact.

$$\therefore \text{solution is } \int_{(y-\text{constant})} M dx + \int (\text{Terms of } N \text{ not containing } x) dy = c$$

$$\text{i.e. } \int (x^2 - 4xy - 2y^2)dx + \int y^2 dy = c$$

$$\frac{x^3}{3} - 4\frac{x^2}{2}y - 2y^2x + \frac{y^3}{3} = c$$

$$\text{i.e. } x^3 - 6x^2y - 6xy^2 + y^3 = 3c = C$$

Problem 3.1.2. Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.(VTU Jan 2014, Model 2014)

Solution : Here

$$M = y^2e^{xy^2} + 4x^3, \quad N = 2xye^{xy^2} - 3y^2$$

$$\therefore \frac{\partial M}{\partial y} = y^2 \cdot e^{xy^2} 2xy + e^{xy^2} \cdot 2y = (2y + 2xy^3) e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2e^{xy^2} y + 2xy [e^{xy^2} y^2] = (2y + 2xy^3) e^{xy^2}$$

and since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

$$\int_{(y-\text{constant})} M dx + \int (\text{Terms of N not containing } x) dy = c \tag{1}$$

Integrating M w.r.t. x keeping y as constant, we get

$$\int (y^2 e^{xy^2} + 4x^3) dx = y^2 \frac{e^{xy^2}}{y^2} + 4 \frac{x^4}{4} = e^{xy^2} + x^4$$

Integrating the terms of N which are free from x , w.r.t. y , we have

$$\int (-3y^2) dy = -3 \frac{y^3}{3} = -y^3$$

Substituting in (1) we get the solution in the form :

$$x^4 + e^{xy^2} - y^3 = c$$

Problem 3.1.3. Solve $(1 + e^{\frac{x}{y}}) dx + (1 - \frac{x}{y}) e^{\frac{x}{y}} dy = 0$. (VTU June 2014)

Solution : Here

$$M = 1 + e^{\frac{x}{y}}, N = e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) = e^{\frac{x}{y}} - \frac{x}{y} e^{\frac{x}{y}}$$

$$\therefore \frac{\partial M}{\partial y} = e^{\frac{x}{y}} \left(-\frac{x}{y^2}\right)$$

$$\text{and } \frac{\partial N}{\partial x} = e^{x/y} \cdot \frac{1}{y} - \frac{1}{y} \left(e^{\frac{x}{y}} + x e^{\frac{x}{y}} \frac{1}{y}\right) = -\frac{x}{y^2} e^{\frac{x}{y}}$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$\text{Solution is given by } \int_{(y-\text{constant})} M dx + \int (\text{Terms of N not containing } x) dy = c$$

(1)

Integrating M w.r.t. x keeping y as constant, we have

$$\int_{(y-\text{constant})} \left(1 + e^{\frac{x}{y}}\right) dx = x + \frac{e^{\frac{x}{y}}}{(1/y)} = x + y e^{\frac{x}{y}}$$

since in N there is no term free from x , hence the solution (1) becomes

$$x + ye^{x/y} = c$$

Problem 3.1.4. Solve $(y \cos x + \sin y + y)dx + (\sin x + x \cos y + x)dy = 0$

(VTU Jan 2016)

(or)

Solve $\frac{dy}{dx} + \frac{(y \cos x + \sin y + y)}{(\sin x + x \cos y + x)} = 0$ (VTU Jan 2020, Jan 2019, June 2018, Jan 2018)

Solution : Given equation is of the form $Mdx + Ndy = 0$

Here $M = y \cos x + \sin y + y$, $N = \sin x + x \cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1, \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Solution is given by

$$\int_{(y-\text{constant})} M dx + \int (\text{Terms of } N \text{ not containing } x) dy = c$$

$$(y \cos x + \sin y + y)dx = c$$

(y-constant)

$$i.e. y \sin x + (\sin y + y)x = c. \quad (\because \int k dx = kx, \text{ where } k \text{ is a constant})$$

Problem 3.1.5. Solve $[y(1 + \frac{1}{x}) + \cos y] dx + [x + \log x - x \sin y] dy = 0$

(VTU Jan 2021)

Solution : Given equation is of the form $Mdx + Ndy = 0$

Here

$$M = y \left(1 + \frac{1}{x} \right) + \cos y$$

$$N = x + \log x - x \sin y$$

$$\therefore \frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y = \frac{dN}{dx}$$

Equation is exact.

Solution is given by $\int_{(y-\text{constant})} M dx + \int (\text{Terms of N not containing } x) dy = c$

$$\int_{(y-\text{constant})} \left[y \left(1 + \frac{1}{x} \right) + \cos y \right] dx + 0 = C$$

i.e. $y(x + \log x) + x \cos y = C$

Problem 3.1.6. Solve $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$ (VTU Jan 2018)

Solution :Given equation is of the form $Mdx + Ndy = 0$

Here $M = ye^{xy}, N = xe^{xy} + 2y$

$$\begin{aligned} \frac{\partial M}{\partial y} &= y(e^{xy} \cdot x) + e^{xy} = e^{xy}(xy + 1) \\ \frac{\partial N}{\partial x} &= x(e^{xy} \cdot y) + e^{xy} = e^{xy}(xy + 1) \\ \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x}, \text{ the equation is exact.} \end{aligned}$$

Solution is given by $\int_{(y-\text{constant})} M dx + \int (\text{Terms of N not containing } x) dy = c$

$$\begin{aligned} \int_{(y-\text{constant})} ye^{xy} dx + \int 2y dy &= c \\ y \cdot \frac{e^{xy}}{y} + 2 \cdot \frac{y^2}{2} &= c \quad \text{or} \quad e^{xy} + y^2 = c. \end{aligned}$$

Problem 3.1.7. Solve $(e^y + 1) \cos x dx + e^y \sin x dy = 0$.

Solution : Here $M = (e^y + 1) \cos x, N = e^y \sin x$

$$\Rightarrow \frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Solution is given by

$$\int_{(y-\text{constant})} M dx + \int (\text{Terms of N not containing } x) dy = c$$

Integrating M w.r.t. x keeping y as constant, we have

$$\int (e^y + 1) \cos x dx = (e^y + 1) \sin x$$

Now since in N there is no term free from x , the solution is

$$(e^y + 1) \sin x = c$$

Problem 3.1.8. Solve:

$$(5x^4 + 3x^2y^2 - 2xy^3) dx + (2x^3y - 3x^2y^2 - 5y^4) dy = 0$$

Solution : Here, $M = 5x^4 + 3x^2y^2 - 2xy^3$, $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now Solution is given by $\int M dx + \int (\text{terms of } N \text{ is not containing } x) dy = C$

(y constant)

$$\begin{aligned} \int (5x^4 + 3x^2y^2 - 2xy^3) dx + \int -5y^4 dy &= C \\ \Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 &= C \end{aligned}$$

3.2 Equations Reducible to exact form

Sometimes a differential equation which is not exact may become exact on multiplication by a suitable function known as an **Integrating Factor**.(I.F.)

Consider the general differential equation $M dx + N dy = 0$

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then compute $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ and check whether it is near to M or N .

Case 1 : If it is near to M then compute $\frac{(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})}{M}$ and if this gives a function of y

only, say $f(y)$, then

$$I.F. = e^{\int f(y) dy}$$

Case 2 : If it is near to N then compute $\frac{(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})}{N}$ and if this gives a function of x only, say $g(x)$, then

$$I.F. = e^{\int g(x)dx}$$

Finally multiply the given differential equation by the integrating factor which reduces to the exact form. and the Solution is given by

$$\int \underset{(y\text{-constant})}{M} dx + \int (\text{Terms of N not containing } x) dy = c$$

Problem 3.2.1. Solve $(x^2 + y^2 + x)dx + xydy = 0$. (VTU July 2017, Aug 2001, Mar 2000)

Solution : Here $M = x^2 + y^2 + x$ and $N = xy$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Consider $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = 2y - y = y$ which is near to N.

$$\therefore \text{compute } \frac{(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})}{N} = \frac{1}{xy} y = \frac{1}{x} = g(x)$$

$$I.F. = e^{\int g(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Multiplying the given differential equation by the integrating factor (i.e. x), we get

$$(x^3 + xy^2 + x^2)dx + x^2ydy = 0.$$

This is in exact form.

\therefore its solution is

$$\int \underset{(y\text{-constant})}{M} dx + \int (\text{Terms of N not containing } x) dy = c$$

$$\text{i.e. } \int (x^3 + xy^2 + x^2)dx + 0 = c$$

$$\text{i.e. } \frac{x^4}{4} + \frac{(x^2y^2)}{2} + \frac{x^3}{3} = c$$

Problem 3.2.2. Solve $(xy^3 + y) dx + 2(x^2y^2 + x + y^4) dy = 0$. (VTU Jan 2015)

Solution : Here $M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$.

$$\frac{\partial M}{\partial y} = 3xy^2 + 1, \quad \frac{\partial N}{\partial x} = 4xy^2 + 2$$

Now
$$\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{(4xy^2 + 2) - (3xy^2 + 1)}{y(1 + xy^2)}$$

$$= \frac{xy^2 + 1}{y(1 + xy^2)}$$

$$= \frac{1}{y}$$

Hence I.F. = $e^{\int \frac{1}{y} dy}$

$$= e^{\log y}$$

$$= y$$

Multiplying the equation by y , we have

$$(xy^4 + y^2) dx + 2(x^2y^3 + xy + y^5) dy = 0$$

Above equation is exact.

Solution is given by

$$\int \frac{M}{(y-\text{constant})} dx + \int (\text{Terms of N not containing } x) dy = c$$

i.e.
$$\int (xy^4 + y^2) dx + 2 \int y^5 dy = cc$$

or

$$\frac{x^2y^4}{2} + y^2x + 2\frac{y^6}{6} = c$$

or

$$3x^2y^4 + 6y^2x + 2y^6 = c_1$$

Problem 3.2.3. Solve $(y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0$

Solution :

$$\text{Here } M = y^4 + 2y, \quad N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2,$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (y^3 - 4) - (4y^3 + 2)$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{(y^3 - 4) - (4y^3 + 2)}{y^4 + 2y}$$

$$= \frac{-3(y^3 + 2)}{y(y^3 + 2)}$$

$$= -\frac{3}{y} = f(y)$$

$$\therefore \text{ I.F.} = e^{\int f(y) dy}$$

$$= e^{\int -\frac{3}{y} dy}$$

$$= e^{-3 \log y}$$

$$= e^{\log y^{-3}}$$

$$= y^{-3} = \frac{1}{y^3}$$

Multiplying the given equation by $\frac{1}{y^3}$, it becomes

$$\left(y + \frac{2}{y^2} \right) dx + \left(x + 2y - \frac{4x}{y^3} \right) dy = 0$$

Which is exact.

Hence the solution is

$$\int_{(y-\text{constant})} M dx + \int (\text{Terms of } N \text{ not containing } x) dy = c$$

$$\text{i.e. } \int_{(y-\text{constant})} \left(y + \frac{2}{y^2} \right) dx + \int 2y dy = c$$

$$\text{or } \left(y + \frac{2}{y^2} \right) x + y^2 = c.$$

Problem 3.2.4. Solve $(x^2 + y^2 + 2x) dx + 2y dy = 0$

Solution : Here $M = (x^2 + y^2 + 2x)$, $N = 2y$

$$\Rightarrow \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 0$$

$$\text{Now } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{2y}{2y} = 1$$

This can be treated as a function of x only.

Hence, the I.F. = $e^{\int dx} = e^x$

Multiplying the equation by e^x , we have

$$e^x (x^2 + y^2 + 2x) dx + 2ye^x dy = 0$$

Above equation is exact.

Solution is given by

$$\int_{(y-\text{constant})} M dx + \int (\text{Terms of } N \text{ not containing } x) dy = c$$

$$\text{Hence the solution is } \int e^x (x^2 + y^2 + 2x) dx + 0 = c$$

($\because 2ye^x$ is not free from x)

$$\text{or } \int (x^2 + 2x) e^x dx + y^2 \int e^x dx = c$$

Integrating by parts

$$(x^2 + 2x) e^x - \int (2x + 2) e^x dx + y^2 \int e^x dx = c$$

$$(x^2 + 2x) e^x - [(2x + 2)e^x - \int 2e^x dx] + y^2 e^x = c$$

$$(x^2 + 2x) e^x - (2x + 2)e^x + 2e^x = c$$

$$[x^2 + 2x - 2x - 2 + 2]e^x = c$$

$$\text{or } (x^2 + y^2) e^x = c$$

Problem 3.2.5. Solve $x^2 y dx - (x^3 + y^3) dy = 0$ (VTU Jan 2010, Dec 2009)

Solution : The given equation is $x^2 y dx - (x^3 + y^3) dy = 0$

Comparing it with $Mdx + Ndy = 0$

$$M = x^2y; \quad N = -x^3 - y^3$$

$$\frac{\partial M}{\partial y} = x^2; \quad \frac{\partial N}{\partial x} = -3x^2$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3x^2 - x^2}{x^2y} = \frac{-4}{y}$$

Hence I.F. is

$$I.F. = e^{\int \frac{-4}{y} dy} = e^{-4 \log y} = e^{\log y^{-4}} = y^{-4} = \frac{1}{y^4}$$

Multiplying the given equation by $\frac{1}{y^4}$ it becomes

$$\frac{x^2}{y^3} dx - \left(\frac{x^3}{y^4} + \frac{1}{y} \right) dy = 0$$

Solution is given by

$$\int \frac{M}{(y-\text{constant})} dx + \int (\text{Terms of N not containing } x) dy = c$$

Hence the solution is

$$\int \frac{x^2}{y^3} dx - \int \frac{1}{y} dy = c$$

or

$$\frac{x^3}{3y^3} - \log y = c.$$

Problem 3.2.6. Solve $(xy^2 - e^{\frac{1}{x^3}})dx - x^2ydy = 0$.

(VTU Jan 2015)

Solution : Here

$$\begin{aligned}
 M &= xy^2 - e^{\frac{1}{x^3}}, \quad N = -x^2y \\
 \frac{\partial M}{\partial y} &= 2xy, \quad \frac{\partial N}{\partial x} = -2xy \\
 \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} &= \frac{2xy - (-2xy)}{-x^2y} \\
 &= \frac{4xy}{-x^2y} \\
 &= -\frac{4}{x} = f(x) \\
 \text{I.F.} &= e^{\int f(x)dx} \\
 &= e^{\int -\frac{4}{x}dx} \\
 &= e^{-4 \log x} \\
 &= e^{\log x^{-4}} \\
 &= x^{-4} \\
 &= \frac{1}{x^4}
 \end{aligned}$$

Multiplying the given equation by $\frac{1}{x^4}$, it becomes

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}} \right) dx - \frac{y}{x^2} dy = 0$$

which is exact.

Hence the solution is

$$\int_{(y-\text{constant})} M dx + \int (\text{Terms of } N \text{ not containing } x) dy = c$$

$$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}} \right) dx + 0 = c$$

$$\text{or } y^2 \int \frac{1}{x^3} dx - \int \frac{1}{x^4} e^{\frac{1}{x^3}} dx = c$$

$$y^2 \cdot \frac{x^{-2}}{-2} - \int \frac{1}{x^4} e^{\frac{1}{x^3}} dx + c \quad (*)$$

To evaluate $\int \frac{1}{x^4} e^{\frac{1}{x^3}} dx$, put $\frac{1}{x^3} = t$, i.e. $x^{-3} = t$

$$\therefore -\frac{3}{x^4} dx = dt \text{ or } \frac{1}{x^4} dx = -\frac{1}{3} dt$$

$$\begin{aligned} \therefore \int \frac{1}{x^4} e^{\frac{1}{x^3}} dx &= \int e^t \left(-\frac{1}{3}\right) dt \\ &= -\frac{1}{3} e^t \\ &= -\frac{1}{3} e^{\frac{1}{x^3}} \end{aligned}$$

Hence, from (*), the solution is

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = c.$$

Problem 3.2.7. Solve $(x^2 + y^3 + 6x)dx + y^2x dy = 0$ (VTU Model 2021, July 2016)

Solution : Here $M = x^2 + y^3 + 6x$ and $N = xy^2$

$$\frac{\partial M}{\partial y} = 3y^2, \quad \frac{\partial N}{\partial x} = y^2$$

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{xy^2} (3y^2 - y^2) = \frac{2}{x}$$

which is a function of x alone.

Hence

$$I.F. = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2$$

Multiplying the given equation by x^2 we get

$$(x^4 + x^2y^3 + 6x^3)dx + x^3y^2dy = 0$$

This is exact. Hence the solution is

$$\int (x^4 + x^2y^3 + 6x^3)dx + 0 = c$$

y-constant

i.e. $\frac{x^5}{5} + y^3 \frac{x^3}{3} + 6 \frac{x^4}{4} = c$

3.3 Linear Differential Equation

A differential equation is said to be linear if the dependant variable and its differential co-efficients occur only in the first degree, and are not multiplied together

The first order linear differential equation is of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x only or constants. Its solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c \text{ where } I.F. = e^{\int Pdx}$$

Note : A first-order ODE is said to be linear if it can be brought into the form

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only or constants (linear in y)

The equation linear in x can be written as

$$\frac{dx}{dy} + Px = Q$$

where P and Q are functions of y only or constants.

In this case solution is given by

$$x(I.F.) = \int Q(I.F.)dy + c \text{ where } I.F. = e^{\int Pdy}$$

3.4 Equations reducible to linear

General equation reducible to linear form is $f'(y) \frac{dy}{dx} + P f(y) = Q$ (1)

where P and Q are functions of x only or constants.

Putting $f(y) = z$ so that $f'(y) \frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes $\frac{dz}{dx} + Pz = Q$, which is linear in z.

Its solution is given by

$$z(I.F.) = \int Q(I.F.)dx + c \text{ where } I.F. = e^{\int Pdx}$$

3.5 Bernoulli's Linear differential Equation :

General Form :

$$\frac{dy}{dx} + Py = Qy^n \tag{1}$$

where P and Q are functions of x only.

To solve this, we use the following method.

Divide by y^n

$$\text{then } \frac{1}{y^n} \frac{dy}{dx} + \frac{1}{y^n} Py = Q \tag{2}$$

$$\text{i.e. } y^{-n} \frac{dy}{dx} + y^{1-n} P = Q$$

$$\text{Put } y^{1-n} = t \text{ then } (1-n)y^{-n} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\implies y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dt}{dx}$$

$$\therefore (2) \text{ becomes } \frac{1}{1-n} \frac{dt}{dx} + Pt = Q$$

$$\text{or } \frac{dt}{dx} + (1-n)Pt = Q(1-n) \text{ which is linear in } t.$$

Its integrating factor is given by

$$I.F. = e^{\int (1-n)Pdt}$$

And its solution is given by

$$t(I.F.) = \int (1-n)Q(I.F.)dt + c$$

Note : Another form of Bernoulli's equation is

$$\frac{dx}{dy} + Px = Qx^n$$

where P and Q are functions of y .

Problem 3.5.1. Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Solution : Dividing by $\cos^2 y$, we get $\frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{\sin 2y}{\cos^2 y} x = x^3$

$$\text{i.e. } \frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{2 \sin y \cos y}{\cos^2 y} x = x^3$$

$$\text{or } \sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

Putting $\tan y = z$, we get $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$

Therefore equation reduces to $\frac{dz}{dx} + 2xz = x^3$

This is in the form $\frac{dz}{dx} + Pz = Q$, where $p = 2x$ and $Q = x^3$

$$\text{I.F.} = e^{(\int 2x dx)} = e^{\frac{2x^2}{2}} = e^{x^2}$$

Hence the solution is

$$\begin{aligned} z(\text{I.F.}) &= \int Q(\text{I.F.}) dx + c \\ ze^{x^2} &= \int x^3 e^{x^2} dx + c \\ &= \int x \cdot x^2 e^{x^2} dx + c \quad \dots (*) \end{aligned}$$

Put $t = x^2$

$$dt = 2x dx \Rightarrow x dx = \frac{dt}{2}$$

So (*) becomes

$$\begin{aligned} ze^{x^2} &= \frac{1}{2} \int t e^t dt + c \\ &= \frac{1}{2} \left[t e^t - \int e^t dt \right] + c \quad (\text{by using integration by parts}) \\ &= \frac{1}{2} [t e^t - e^t] + c \\ &= \frac{1}{2} [x^2 e^{x^2} - e^{x^2}] + c \end{aligned}$$

or

$$z = \frac{x^2}{2} - \frac{1}{2} + c e^{-x^2}$$

or

$$\tan y = \frac{1}{2} (x^2 - 1) + ce^{-x^2}$$

Problem 3.5.2. Solve $\frac{dy}{dx} + y \tan x = y^3 \sec x$ (VTU June 2019, Jan 2018, Jan 2015)

Solution : Dividing by y^3 we get

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{y^2} \tan x = \sec x \quad \dots (1)$$

Let $\frac{1}{y^2} = z$

Then $\left(\frac{-2}{y^3}\right) (dy/dx) = (dz/dx) \Rightarrow \frac{1}{y^3} \frac{dy}{dx} = \frac{-1}{2} \frac{dz}{dx}$

So given equation can be written as

$$\frac{-1}{2} \frac{dz}{dx} + \tan x z = \sec x$$

$$\Rightarrow \frac{dz}{dx} - 2 \tan x z = -2 \sec x$$

This is linear equation in variable z .

Here $P = -2 \tan x$ and $Q = -2 \sec x$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int -2 \tan x dx} = e^{-2 \log \sec x} = \frac{1}{\sec^2 x}$$

Hence solution is

$$z(\text{I.F.}) = \int Q (\text{I.F.}) dx + c$$

$$\Rightarrow (1/y^2) \cdot \frac{1}{\sec^2 x} = \int (-2 \sec x) \frac{1}{\sec^2 x} dx + c \quad [\because z = 1/y^2]$$

$$\Rightarrow (1/y^2) \cdot \frac{1}{\sec^2 x} = \int -2 \cos x + c$$

$$\text{i.e. } (1/y^2) \cdot \frac{1}{\sec^2 x} = -2 \sin x + x + c$$

Problem 3.5.3. Solve $x \frac{dy}{dx} + y = x^3 y^6$. (VTU Model 2014)

Solution : Let $x \frac{dy}{dx} + y = x^3 y^6$. (1)

Divide both sides by xy^6

$$y^{-6} \frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \quad (2)$$

Put $y^{-5} = t$

Then $-5y^{-6} \frac{dy}{dx} = \frac{dt}{dx}$

$$\therefore y^{-6} \frac{dy}{dx} = \frac{1}{-5} \frac{dt}{dx} \quad (3)$$

Using (3) in (2), we get

$$-\frac{1}{5} \frac{dt}{dx} + \frac{t}{x} = x^2$$

$$i.e. \frac{dt}{dx} - 5 \frac{t}{x} = -5x^2$$

This is in the form $\frac{dt}{dx} + Pt = Q$,

which is linear in t, where $P = \frac{-5}{x}$, $Q = -5x^2$

$$\begin{aligned} \therefore \text{I.F.} &= e^{\int P dx} \\ &= e^{-5 \int \frac{1}{x} dx} \\ &= e^{-5 \log x} \\ &= e^{\log x^{-5}} \\ &= \frac{1}{x^5} \end{aligned}$$

Solution is given by

$$\begin{aligned} t(\text{I.F.}) &= \int Q(\text{I.F.}) dx + c \\ t \frac{1}{x^5} &= -5 \int x^{-3} dx + c \\ &= -5 \frac{x^{-2}}{-2} + c \\ i.e. \frac{1}{x^5 y^5} &= \frac{5}{2x^2} + c \end{aligned}$$

Problem 3.5.4. Solve: $(1 + y^2) dx = (\tan^{-1} y - x) dy$.

Solution : The given equation is $(1 + y^2) dx = (\tan^{-1} y - x) dy$

$$(1 + y^2) \frac{dx}{dy} = \tan^{-1} y - x$$

$$(1 + y^2) \frac{dx}{dy} + x = \tan^{-1} y$$

$$\frac{dx}{dy} + \frac{1}{1 + y^2} x = \frac{\tan^{-1} y}{1 + y^2}$$

It is of the form $\frac{dx}{dy} + Px = Q$

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

The solution is

$$\begin{aligned} x(\text{I.F.}) &= \int Q (\text{I.F.}) dx + c \\ x e^{\tan^{-1} y} &= \int \frac{\tan^{-1} y}{1 + y^2} e^{\tan^{-1} y} dy + c \\ &= \int t e^t dt + c, \text{ where } t = \tan^{-1} y, dt = \frac{1}{1 + y^2} dy \\ &= t e^t - \int e^t dt + c \text{ (Integrating by parts)} \\ &= t e^t - e^t + c \\ &= e^t (t - 1) + c \\ \text{i.e. } x e^{\tan^{-1} y} &= e^{\tan^{-1} y} (\tan^{-1} y - 1) + c \\ \text{or } x &= \tan^{-1} y - 1 + c e^{-\tan^{-1} y}. \end{aligned}$$

Problem 3.5.5. Solve: $xy (1 + xy^2) \frac{dy}{dx} = 1$.

VTU July 2017

Solution: The given equation can be written as $\frac{dx}{dy} = xy (1 + xy^2)$

$$\frac{dx}{dy} = xy - x^2 y^3$$

$$\frac{dx}{dy} - yx = y^3 x^2$$

Dividing by x^2 , we have $x^{-2} \frac{dx}{dy} - yx^{-1} = y^3$ (1)

Putting $x^{-1} = z$ so that $-x^{-2} \frac{dx}{dy} = \frac{dz}{dy}$

$$\text{or } x^{-2} \frac{dx}{dy} = -\frac{dz}{dy}$$

$$\text{Equation (1) becomes } -\frac{dz}{dy} - yz = y^3$$

$$\text{or } \frac{dz}{dy} + yz = -y^3,$$

$$\text{This is in the form } \frac{dz}{dy} + Pz = Q$$

which is linear in z , with $P = y$ and $Q = -y^3$

$$\text{I.F.} = e^{\int P dy} = e^{\int y dy} = e^{\frac{1}{2}y^2}$$

\therefore The solution is

$$z (I.F.) = \int Q (I.F.) dy + c$$

$$\begin{aligned} z \cdot e^{\frac{1}{2}y^2} &= \int [-y^3 e^{\frac{1}{2}y^2}] dy + c \\ &= - \int y^2 e^{\frac{1}{2}y^2} y dy + c \quad \dots (*) \end{aligned}$$

$$\text{Put } t = \frac{1}{2}y^2 \text{ then } dt = \frac{1}{2}(2y dy) = y dy \text{ and } y^2 = 2t$$

Now (*) becomes

$$\begin{aligned} z \cdot e^{\frac{1}{2}y^2} &= - \int 2te^t dt + c, \\ &= -2 \left[te^t - \int 1 \cdot e^t dt \right] + c \quad (\text{integrating by parts}) \\ &= -2 (te^t - e^t) + c \\ &= -2e^t(t - 1) + c \\ \text{i.e. } z \cdot e^{\frac{1}{2}y^2} &= -2e^{\frac{1}{2}y^2} \left(\frac{1}{2}y^2 - 1 \right) + c \\ \Rightarrow z &= -2 \left(\frac{1}{2}y^2 - 1 \right) + ce^{-\frac{1}{2}y^2} \\ \Rightarrow \frac{1}{x} &= 2 - y^2 + ce^{-\frac{1}{2}y^2} \end{aligned}$$

Problem 3.5.6. Solve $(x + 2y^3) \frac{dy}{dx} = y$

(VTU July 2015)

Sol: The given equation is $(x + 2y^3) \frac{dy}{dx} = y$

$$y \cdot \frac{dx}{dy} = x + 2y^3$$

Divide by y , we get

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{1}{y} \cdot x = 2y^2$$

It is of the form $\frac{dx}{dy} + Px = Q$

Here $P = -\frac{1}{y}$, $Q = 2y^2$

$$\therefore \text{I.F.} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$$

\therefore The solution is

$$x (\text{I.F.}) = \int Q (\text{I.F.}) dy + c$$

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} dy + c$$

$$= \int 2y dy + c$$

$$= y^2 + c$$

$$\text{or } x = y^3 + cy$$

Problem 3.5.7. Solve $r \sin \theta - \cos \theta \frac{dr}{d\theta} = r^2$ (VTU Jan 2020, June 2018, Model 2018)

Solution : The given equation can be written as

$$-\frac{dr}{d\theta} \cos \theta + r \sin \theta = r^2$$

Dividing by $r^2 \cos \theta$, we get

$$-r^{-2} \frac{dr}{d\theta} + r^{-1} \tan \theta = \sec \theta$$

Putting $r^{-1} = z$ so that $-r^{-2} \frac{dr}{d\theta} = \frac{dz}{d\theta}$ in the above equation, we get

$$\frac{dz}{d\theta} + z \tan \theta = \sec \theta$$

$$\text{I.F.} = e^{\int \tan \theta d\theta} = e^{\log \sec \theta} = \sec \theta$$

solution is given by

$$z \sec \theta = \int \sec \theta, \sec \theta + C \Rightarrow v \sec \theta = \int \sec^2 \theta d\theta + C$$

$$\text{i.e. } \frac{\sec \theta}{r} = \tan \theta + C$$

$$\Rightarrow r^{-1} = (\sin \theta + C \cos \theta)$$

$$\therefore r = \frac{1}{\sin \theta + C \cos \theta}$$

3.6 Nonlinear differential equations: Differential Equations of First order and higher Degree

We are already familiar with differential equations of the first order and first degree, Now we shall study differential equations of first order and degree higher than the first. We are familiar with the solution of differential equations (d.e.) of first order and first degree Now, we shall study the methods of solving differential equations of first order and higher degree. In addition to the general solution and particular solution associated with the D.E., we also introduce singular solution. The differential equations of first order but not of first degree are also branded as $p - y - x$ equations.

If $y = f(x)$, we use the notation $\boxed{\frac{dy}{dx} = p}$ throughout this section. A differential equation of first order and nth degree is the form

$$A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_n = 0 \quad (1)$$

Where $A_0, A_1, A_2, \dots, A_n$ are functions of x and y . This being a differential equation of first order, the associated general solution will contain only one arbitrary constant.

3.7 Differential Equations solvable for p

Supposing that the LHS of (1) is expressed as a product of n linear factors, then the equivalent form of (1) is

$$(p - f_1(x, y))(p - f_2(x, y)) \cdots (p - f_n(x, y)) = 0$$

Equating each of the factors to Zero, we have

$$p = f_1(x, y), p = f_2(x, y), p = f_3(x, y), \cdots p = f_n(x, y)$$

All these are differential equations of first order and first degree. They can be solved by the known methods.

Solving each of these equations of the first order of first degree, we get the solutions

$$F_1(x, y, c) = 0, F_2(x, y, c) = 0, \cdots F_n(x, y, c) = 0$$

respectively.

General solution of (1) may be written as the product of all these solutions. i.e.

$$F_1(x, y, c) \cdot F_2(x, y, c) \cdot F_3(x, y, c) \cdots F_n(x, y, c) = 0$$

Note : We need to present the general solution with the same arbitrary constant in each factor.

Problem 3.7.1. Solve the equation $p^2 + p(x + y) + xy = 0$ (VTU Jan 2013, Dec 2014)

Solution: The equation can be written as $p^2 + px + py + xy = 0$

$$\text{i.e. } p(p + x) + y(p + x) = 0$$

$$\text{i.e. } (p + y)(p + x) = 0$$

$$\text{i.e. } p = -y \text{ and } p = -x$$

$$\text{Solving } p = -y \implies \frac{dy}{dx} = -y$$

$$\text{i.e. } \frac{dy}{y} = -dx$$

Integrating on both sides

$$\log y = -x + c$$

$$\text{i.e. } \log y + x - c = 0 \quad (1)$$

$$\text{Similarly Consider } p = -x \implies \frac{dy}{dx} = -x$$

$$\text{i.e. } dy = -x dx$$

Integrating,

$$y = -\frac{x^2}{2} + c$$

$$\text{i.e. } y + \frac{x^2}{2} - c = 0 \quad (2)$$

General Solution is obtained by multiplying the solutions (1) and (2).

$$\text{Hence the General Solution is, } (\log y + x - c)(y + \frac{x^2}{2} - c) = 0$$

Problem 3.7.2. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ (VTU Model ME 2022, June 2018, July 2017, Jan 2016, June 2015, Jan 2016)

$$\text{Solution : } \frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

$$p - \frac{1}{p} = \frac{x^2 - y^2}{xy}, \text{ where } p = \frac{dy}{dx}$$

$$xyp^2 - (x^2 - y^2)p - xy = 0$$

$$p = \frac{(x^2 - y^2) \pm \sqrt{(x^2 - y^2)^2 + 4x^2y^2}}{2xy}$$

$$= \frac{(x^2 - y^2) \pm (x^2 + y^2)}{2xy}$$

or

$$p = \frac{x}{y} ; p = -\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y} ; \frac{dy}{dx} = -\frac{y}{x}$$

$$ydy = xdx ; \frac{1}{y}dy = -\frac{1}{x}dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c ; \log y = -\log x + c$$

Integrating both sides

$$\text{or } y^2 - x^2 - 2c = 0 \text{ or } \log xy = c \text{ or } xy = e^c$$

∴ The general solution of given equation is $(y^2 - x^2 - 2c)(xy - e^c) = 0$.

Problem 3.7.3. $xy \left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\frac{dy}{dx} + xy = 0$ (VTU Model 2022, Jan 2019, Jan 2018)

or

$$xy \left\{ \left(\frac{dy}{dx}\right)^2 + 1 \right\} = (x^2 + y^2)\frac{dy}{dx} \quad \text{(VTU June 2018)}$$

Solution :

$$xy(dy/dx)^2 - (x^2 + y^2) dy/dx + xy = 0$$

$$\text{Put } dy/dx = p$$

$$xy \cdot p^2 - (x^2 + y^2)p + xy = 0$$

This is a quadratic equation in powers of p .

$$\therefore p = \frac{(x^2 + y^2) \pm \sqrt{(x^2 + y^2)^2 - 4(xy)(xy)}}{2xy}$$

$$\begin{aligned}
 p &= \frac{(x^2 + y^2) \pm \sqrt{x^4 + y^4 + 2x^2y^2 - 4x^2y^2}}{2xy} \\
 &= \frac{(x^2 + y^2) \pm \sqrt{x^4 + y^4 - 2x^2y^2}}{2xy} \\
 &= \frac{(x^2 + y^2) \pm (x^2 - y^2)}{2xy} \\
 \Rightarrow p &= \frac{x^2 + y^2 + x^2 - y^2}{2xy} = \frac{2x^2}{2xy} = \frac{x}{y} \\
 \&p &= \frac{x^2 + y^2 - x^2 + y^2}{2xy} = \frac{2y^2}{2xy} = \frac{y}{x}
 \end{aligned}$$

i.e. values of p are

$$\frac{dy}{dx} = \frac{x}{y} \quad (1)$$

$$\text{and } \frac{dy}{dx} = \frac{y}{x} \quad (2)$$

Solution of (1) :

$$\frac{dy}{dx} = \frac{x}{y}$$

$$ydy = xdx$$

$$\int ydy = \int xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

$$y^2 = x^2 + 2c_1$$

$$y^2 - x^2 - 2c_1 = 0 \quad \dots (3)$$

Solution of (2) :

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + \log c$$

$$= \log(cx)$$

$$y = cx$$

$$y - cx = 0 \quad \dots (4)$$

∴ The general solution of given equation is

$$(y^2 - x^2 - 2c_1)(y - cx) = 0$$

Problem 3.7.4. Solve $p^2 + 2pycotx = y^2$. (VTU Model 2022, Jan 2020, July 2016, June 2012)

Solution : Given $p^2 + 2pycotx = y^2$

Adding y^2cot^2x on both sides, The above equation can be written as

$$(p + y \cot x)^2 = y^2 (1 + \cot^2 x)$$

$$\text{or } p + y \cot x = \pm y \operatorname{cosec} x$$

$$\therefore \text{The component equations are } p = y(-\cot x + \operatorname{cosec} x) \quad (1)$$

$$\text{and } p = y(-\cot x - \operatorname{cosec} x) \quad (2)$$

From (1),

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$$

$$\text{or } \frac{dy}{y} = (-\cot x + \operatorname{cosec} x)dx$$

Integrating

$$\begin{aligned}\log y &= -\log \sin x + \log \tan \frac{x}{2} + \log c \\ &= \log \frac{c \tan \frac{x}{2}}{\sin x} \\ y &= \frac{c \tan \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \frac{c}{2 \cos^2 \frac{x}{2}}\end{aligned}$$

$$\text{or } y \cos^2 \frac{x}{2} = c_1, \text{ where } c_1 = \frac{c}{2}$$

From (2), $\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$

$$\text{or } \frac{dy}{y} = (-\cot x - \operatorname{cosec} x) dx$$

Integrating

$$\begin{aligned}\log y &= -\log \sin x - \log \tan \frac{x}{2} + \log c \\ &= \log \frac{c}{\sin x \tan \frac{x}{2}}\end{aligned}$$

or

$$y = \frac{c}{2 \sin^2 \frac{x}{2}} \text{ or } y \sin^2 \frac{x}{2} = C$$

\therefore The general solution of the given equation is $(y \cos^2 \frac{x}{2} - C)(y \sin^2 \frac{x}{2} - C) = 0$

Problem 3.7.5. Solve $p(p + y) = x(x + y)$ (VTU Dec 2014, July 2011, Dec 2011)

Solution: Given equation is

$$p^2 + py = x^2 + xy$$

or

$$p^2 + py - (x^2 + xy) = 0, \text{ which is quadratic in } p$$

$$\begin{aligned} \therefore p &= \frac{-y \pm \sqrt{y^2 + 4(x^2 + xy)}}{2} \\ &= \frac{-y \pm \sqrt{(y+2x)^2}}{2} \\ \therefore p &= \frac{-y + y + 2x}{2} \text{ or } p = \frac{-y - y - 2x}{2} \end{aligned}$$

Thus $p = x$ (1)

or $p = -y - x$ (2)

From (1), $\frac{dy}{dx} = x$ or $dy = x dx$

Integrating both sides,

$$\begin{aligned} y &= \frac{x^2}{2} + c \\ \text{or } y - \frac{x^2}{2} - c &= 0 \quad \dots (3) \end{aligned}$$

and From (2)

$$\begin{aligned} \frac{dy}{dx} &= -y - x \\ \text{or } \frac{dy}{dx} + y &= -x \end{aligned}$$

which is linear equation in y with $P = 1$ and $Q = -x$

$$\text{I.F.} = e^{\int P dx} = \int e^x dx = e^x$$

∴ Its solution is

$$\begin{aligned}
 y(I.F.) &= \int Q(I.F.)dx + c \\
 \Rightarrow ye^x &= \int (-x)e^x + c \\
 &= (-x)e^x - \int (-1)e^x dx + c \quad [\text{Integrating by parts}] \\
 &= -xe^x + e^x + c \\
 &= (-x + 1)e^x + c \\
 y &= -x + 1 + ce^{-x}
 \end{aligned}$$

$$\text{or } y + x - 1 - ce^{-x} = 0 \quad \dots (4)$$

Combining (3) and (4), general solution is

$$\left(y - \frac{x^2}{2} - c\right) (y + x - 1 - ce^{-x}) = 0.$$

3.8 Clairaut's equation

The equation of the form

$$y = px + f(p) \quad (1)$$

is known as Clairaut's equation.

Using $p = c$ in (1) we obtain the general solution of Clairaut's equation in the form

$$y = cx + f(c)$$

Differentiating the general solution partially w.r.to c and eliminating the constant c , we get the singular solution.

Problem 3.8.1. Solve $p = \log(px - y)$

Solution :

or $e^p = px - y$ or $y = px - e^p$, which is Clairaut's equation where $f(p) = -e^p$

\therefore Its solution is obtained by putting $p = c$

\therefore solution is $y = cx - e^c$.

Problem 3.8.2. Solve $\sin px \cos y = \cos px \sin y + p$. Also find the singular solution. (Jan 2014)

Solution :

$$\text{Given } \sin px \cos y - \cos px \sin y = p$$

$$\text{or } \sin(px - y) = p$$

$$\text{or } px - y = \sin^{-1} p$$

$$\text{or } y = px - \sin^{-1} p,$$

this is in the form $y = px + f(p)$ which is Clairaut's form

Solution is obtained by replacing p by c .

Hence the General solution is

$$y = cx + \sin^{-1} c \quad (*)$$

Differentiating equation (*) partially w.r.to c

$$0 = x + \frac{1}{\sqrt{1-c^2}}$$

$$\frac{1}{\sqrt{1-c^2}} = -x$$

$$1 = -x\sqrt{1-c^2}$$

$$1 = x^2(1-c^2)$$

$$1 - c^2 = \frac{1}{x^2}$$

$$c^2 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

$$c = \frac{\sqrt{x^2-1}}{x}$$

Substitute for c in (*) we get, $y = \frac{\sqrt{x^2-1}}{x}x + \sin^{-1} \frac{\sqrt{x^2-1}}{x}$

Problem 3.8.3. Obtain the general solution and singular solution of the Clairaut's equation $xp^3 - yp^2 + 1 = 0$. (VTU July 2017, Jan 2015, June 2014, Dec 2011)

Solution: The given equation can be written as

$$y = \frac{xp^3 + 1}{p^2}$$

$$\Rightarrow y = px + \frac{1}{p^2} \text{ which is in the Clairaut's form } y = px + f(p)$$

Thus general solution is given by $y = cx + \frac{1}{c^2}$ (*)

Differentiating partially w.r.to c we get

$$0 = x + \frac{-2}{c^3}$$

$$\Rightarrow \frac{2}{c^3} = x$$

$$\Rightarrow c^3 = \frac{2}{x}$$

$$\Rightarrow c = \left(\frac{2}{x}\right)^{1/3}$$

Thus general solution (*) becomes

$$y = \left(\frac{2}{x}\right)^{1/3} x + \left(\frac{x}{2}\right)^{2/3}$$

$$= 2^{1/3} x^{1-1/3} + \frac{x^{2/3}}{2^{2/3}}$$

$$= x^{2/3} \left(2^{1/3} + \frac{1}{2^{2/3}}\right)$$

$$= x^{2/3} \left(\frac{2 + 1}{2^{2/3}}\right)$$

$$2^{2/3} y = 3x^{2/3}$$

$$4^{1/3} y = 3(x^2)^{1/3}$$

Cubing both sides, we get

$$4y^3 = 27x^2$$

This is the singular solution of the given Clairaut's equation.

Problem 3.8.4. Find the general and singular solution for $xp^2 + xp - yp + 1 - y = 0$ (VTU June 2018)

Solution :

$$xp^2 + px - py + 1 - y = 0$$

$$xp^2 + px + 1 = y(p + 1)$$

$$y = \frac{xp^2 + px + 1}{(p + 1)}$$

$$\Rightarrow y = \frac{xp(p + 1) + 1}{(p + 1)}$$

$$y = px + \frac{1}{p+1}$$

G.S. is given by putting $p = c$

$$\text{Hence G.S. is } y = cx + \frac{1}{c+1} \quad (1)$$

Diff. partially w.r.t. c ,

$$0 = x - \frac{1}{(c+1)^2}$$

$$\text{or } \frac{1}{(c+1)^2} = x$$

$$\text{or } (c + 1)^2 = \frac{1}{x}$$

$$(c + 1) = \frac{1}{\sqrt{x}}$$

$$\text{or } c = \frac{1}{\sqrt{x}} - 1$$

Hence (1) becomes

$$y = \left(\frac{1}{\sqrt{x}} - 1 \right) x + \sqrt{x}$$

$$\Rightarrow y = \left(\frac{1-\sqrt{x}}{\sqrt{x}} \right) x + \sqrt{x}$$

$$y = (1 - \sqrt{x})\sqrt{x} + \sqrt{x}$$

$$\Rightarrow y = \sqrt{x} - x + \sqrt{x}$$

$$y = 2\sqrt{x} - x$$

$$\Rightarrow x + y = 2\sqrt{x}$$

squaring

$$(x + y)^2 = 4x$$

Reducible to Clairaut's equations

Problem 3.8.5. Solve: $(px - y)(py + x) = 2p$ by reducing into Clairaut's form, taking the substitution $X = x^2, Y = y^2$.

Solution : Put $X = x^2$ and $Y = y^2$

so that $dX = 2xdx$ and $dY = 2ydy$

$$i.e. dx = \frac{dX}{2x}, \quad dy = \frac{dY}{2y}$$

$$\therefore p = \frac{dy}{dx} = \frac{x dY}{y dX} = \frac{\sqrt{X} P}{\sqrt{Y}}$$

where $P = \frac{dY}{dX}$

The given equation becomes

$$\left(\frac{\sqrt{X} P}{\sqrt{Y}} \cdot \sqrt{X} - \sqrt{Y} \right) \left(\frac{\sqrt{X} P}{\sqrt{Y}} \cdot \sqrt{Y} + \sqrt{X} \right) = 2 \frac{\sqrt{X}}{\sqrt{Y}} P$$

$$\text{or } (PX - Y)(P + 1) = 2P$$

$$\text{or } PX - Y = \frac{2P}{P+1}$$

or $Y = PX - \frac{2P}{P+1}$, which is of Clairaut's form.

\therefore

Its solution is $Y = cX - \frac{2c}{c+1}$

and hence $y^2 = cx^2 - \frac{2c}{c+1}$.

Problem 3.8.6. Solve $(px - y)(py + x) = a^2p$, use the substitution $X = x^2, Y = y^2$. (VTU Jan 2018, July 2017, June 2014, June 2015, Dec 2011)

Solution:

Solution: Given $(px - y)(py + x) = a^2p$ (1)

$$X = x^2 \Rightarrow \frac{dX}{dx} = 2x \Rightarrow dX = 2x dx$$

$$Y = y^2 \Rightarrow \frac{dY}{dy} = 2y \Rightarrow dY = 2y dy$$

$$\frac{dY}{dX} = \frac{y dy}{x dx} = \frac{y}{x} \frac{dy}{dx}$$

$$P = \frac{\sqrt{Y}}{\sqrt{X}} p$$

$$\text{or } p = \frac{\sqrt{X}}{\sqrt{Y}} P$$

where $P = \frac{dY}{dX}$, $p = \frac{dy}{dx}$

Substitute for P , x and y in the given equation, i.e. $(px - y)(py + x) = a^2p$ we get

$$\left(\frac{\sqrt{X}}{\sqrt{Y}} P \sqrt{X} - \sqrt{Y} \right) \left(\frac{\sqrt{X}}{\sqrt{Y}} P \sqrt{Y} + \sqrt{X} \right) = a^2 \frac{\sqrt{X}}{\sqrt{Y}} P$$

$$\left(\frac{XP - Y}{\sqrt{Y}} \right) (\sqrt{X}P + \sqrt{X}) = a^2 \frac{\sqrt{X}}{\sqrt{Y}} P$$

Multiplying both sides by \sqrt{Y} we get

$$(XP - Y) (\sqrt{X}P + \sqrt{X}) = a^2P\sqrt{X}$$

$$(XP - Y) (P + 1)\sqrt{X} = a^2P\sqrt{X}$$

$$(XP - Y) (P + 1) = a^2P$$

$$(XP - Y) = \frac{a^2P}{P + 1}$$

$$\text{or } Y = XP - \frac{a^2P}{P + 1}$$

This is in the Clairaut's form and hence the associated genertal solution is $Y =$

$$Xc - \frac{a^2c}{c+1}$$

Resubstituting for X and Y we get

$$y^2 = cx^2 - \frac{a^2c}{c+1}$$

Problem 3.8.7. Find the general and singular solution of the equation $x^2(y - px) = p^2y$ by reducing into Clairaut's equation using the substitution $X = x^2, Y = y^2$ (VTU July 2016)

Solution :

$$X = x^2 \Rightarrow \frac{dX}{dx} = 2x \Rightarrow dX = 2x dx$$

$$Y = y^2 \Rightarrow \frac{dY}{dy} = 2y \Rightarrow dY = 2y dy$$

$$\frac{dY}{dX} = \frac{y dy}{x dx} = \frac{y}{x} \frac{dy}{dx}$$

$$P = \frac{\sqrt{Y}}{\sqrt{X}} p$$

$$\text{or } p = \frac{\sqrt{X}}{\sqrt{Y}} P$$

where $P = \frac{dY}{dX}, p = \frac{dy}{dx}$

Substitute for P , x and y in the given equation, i.e. $x^2(y - px) = p^2y$

$$X \left[\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} \cdot P \cdot \sqrt{X} \right] = \frac{X}{Y} P^2 \sqrt{Y}$$

$$X \left[\frac{Y - PX}{\sqrt{Y}} \right] = \frac{X}{\sqrt{Y}} \cdot P^2$$

$$\Rightarrow Y - PX = P^2$$

$\Rightarrow Y = PX + P^2$ Which is Clairaut's form.

Its solution is

$$Y = cX + c^2$$

$$y^2 = cx^2 + c^2 \quad \dots (*)$$

Diff. partially w.r.to c

$$0 = x^2 + 2c$$

$$c = -\frac{x^2}{2}$$

Hence (*) becomes

$$y^2 = -\frac{x^2}{2}x^2 + \left(-\frac{x^2}{2}\right)^2$$

$$y^2 = -\frac{x^4}{2} + \frac{x^4}{4}$$

$$= -\frac{x^4}{4}$$

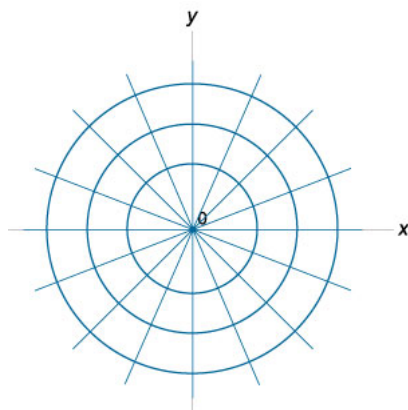
$$4y^2 = -x^4$$

$$x^4 + 4y^2 = 0$$

3.9 Orthogonal trajectories

Definition: If two family of curves are such that every member of one family intersect every member of the other family at right angles then they are said to be orthogonal trajectories of each other.

Example : the orthogonal trajectory of the family of straight lines defined by the equation $y = kx$, where k is a parameter (the slope of the straight line), is the family of circle having centre at the origin (Figure 1)



Note :

- If $y = f(x)$ be a curve then $\frac{dy}{dx} = m =$ slope of the tangent.
- Two curves intersect each other orthogonally, if the tangents at point of intersection are at right angles.
- Condition for orthogonality of two curves is given by $m_1 m_2 = -1$ where m_1 and m_2 are slopes of tangents.
- If each member of a family cuts every other member of the same family orthogonally, then we say that family of curves is self-orthogonal.

Rules to find the equation of orthogonal trajectories of a family of cartesian curves: Given a relation in the form $f(x, y, c) = 0$ (1)

Step 1. Diff (1) and eliminate c , which gives the differential equation of the given family (1).

Step 2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ which gives the differential equation of orthogonal tra-

jectory(OT).

Step 3. Solve this new equation to get the required orthogonal trajectory.

Rules to find the equation of orthogonal trajectories of a family of Polar curves

:

Given a polar family of curves in the form $f(r, \theta, c) = 0$ (1)

(i) Differentiate (1) and eliminate c which gives the differential equation of the given family (1).

(ii) Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ which gives the differential equation of orthogonal trajectory(OT).

(iii) Solve this new equation to get the required orthogonal trajectory.

Problem 3.9.1. Find the Orthogonal trajectories of the family of parabola $y^2 = 4ax$ (VTU Jan 2020)

Solution :

$$y^2 = 4ax$$

$$\frac{y^2}{x} = 4a$$

Differentiate w.r.to x

$$\frac{x \cdot 2y \frac{dy}{dx} - y^2 \cdot 1}{x^2} = 0$$

$$2xy \frac{dy}{dx} - y^2 = 0$$

$$\frac{dy}{dx} = \frac{y}{2x}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to get the D.E. of Orthogonal trajectory in the form

$$\begin{aligned}\frac{-dx}{dy} &= \frac{y}{2x} \\ \frac{dy}{dx} &= -\frac{2x}{y} \quad (\text{by rearranging the terms})\end{aligned}$$

$$-ydy = 2xdx$$

$$2xdx + ydy = 0$$

integrating both sides

$$\int 2xdx + \int ydy = \int 0$$

$$x^2 + \frac{y^2}{2} = C$$

$$2x^2 + y^2 = 2C$$

$$2x^2 + y^2 = c_1, \text{ where } c_1 = 2c$$

Problem 3.9.2. Find the Orthogonal trajectories of the family of confocal ellipses

$\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1$, where λ is a parameter . (VTU July 2021, Model 2018, July

2015, Dec 2011, July 2011, June 2009, Aug 2000, Mar 2000, Aug 1999)

Solution : Let $\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1$ (1)

Differentiate (1) w.r.to x, we get

$$\begin{aligned}\frac{2x}{a^2} + \frac{2y}{(b^2 + \lambda)} \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{x}{a^2} + \frac{y}{(b^2 + \lambda)} \frac{dy}{dx} &= 0\end{aligned}$$

$$\Rightarrow \frac{x}{a^2} = \frac{-y \frac{dy}{dx}}{(b^2 + \lambda)} \quad \dots (2)$$

From (1) we have $\frac{x^2}{a^2} - 1 = \frac{-y^2}{(b^2 + \lambda)}$

$$\Rightarrow \frac{x^2 - a^2}{a^2} = \frac{-y^2}{(b^2 + \lambda)} \quad \dots (3)$$

Dividing (2) from (3) we obtain,

$$\frac{x}{x^2 - a^2} = \frac{1}{y} \frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$,

$$\frac{x}{x^2 - a^2} = \frac{-1}{y} \frac{dx}{dy}$$

$$i.e \ ydy = - \left(\frac{x^2 - a^2}{x} \right) dx$$

$$i.e \ ydy = \left(\frac{a^2}{x} - x \right) dx$$

Integrating on both sides

$$\int ydy = \int \frac{a^2}{x} - \int xdx$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$$

$$\frac{y^2}{2} + \frac{x^2}{2} = a^2 \log x + c$$

Problem 3.9.3. Show that the family of confocal parabolas having x -axis as their axis, is of the form $y^2 = 4a(x + a)$ is self orthogonal (VTU Jan 2021, June 2019, June 2018, Jun 2013)

Solution : The given equation is $y^2 = 4a(x + a)$ (1)

Differentiating w.r. to x ,

$$y \frac{dy}{dx} = 2a$$
 (2)

Substituting the value of a from (2) in (1), we get

$$y^2 = 2y \frac{dy}{dx} \left(x + \frac{1}{2} y \frac{dy}{dx} \right)$$

$$y \left(\frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} - y = 0$$
 (3)

This is the differential equation of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we obtain

$$y \left(\frac{dx}{dy} \right)^2 - 2x \frac{dx}{dy} - y = 0$$

Multiplying both sides by $\left(\frac{dy}{dx} \right)^2$, we get

$$y - 2x \frac{dy}{dx} - y \left(\frac{dy}{dx} \right)^2 = 0$$

Multiplying by -1 and rearranging the terms we get

$$y \left(\frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} - y = 0 \text{ which is the same as (3).}$$

Thus we see that a system of confocal and coaxial parabolas is self-orthogonal, i.e., each member of the family cuts every other member of the same family orthogonally.

Problem 3.9.4. Find the orthogonal trajectories of the confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, λ being the parameter. (VTU July 2017, Jan 2016, Jan 2015, Jan 2013)

Solution: Solution: The given family is $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$. (1)

Differentiating (1) with respect to x ,

$$\frac{2x}{a^2+\lambda} + \frac{2yy_1}{b^2+\lambda} = 0$$

$$x(b^2 + \lambda) + yy_1(a^2 + \lambda) = 0$$

$$\lambda(x + yy_1) = -(b^2x + a^2yy_1) \Rightarrow \lambda = -\frac{(b^2x + a^2yy_1)}{(x + yy_1)} \tag{2}$$

$$\text{Now, } (a^2 + \lambda) = a^2 - \frac{(b^2x + a^2yy_1)}{(x + yy_1)} = \frac{(a^2 - b^2)x}{(x + yy_1)} \tag{3}$$

$$\text{Similarly } (b^2 + \lambda) = b^2 - \frac{(b^2x + a^2yy_1)}{(x + yy_1)} = -\frac{(a^2 - b^2)yy_1}{(x + yy_1)} \tag{4}$$

On using results (3) and (4) in (1), we get

$$\frac{x(x + yy_1)}{(a^2 - b^2)} + \frac{y(x + yy_1)}{-(a^2 - b^2)y_1} = 1$$

$$\frac{(x + yy_1)}{(a^2 - b^2)} \left[x - \frac{y}{y_1} \right] = 1 \tag{5}$$

Equation (5) is the differential equation of the given family.

In order to find the differential equation of the desired orthogonal trajectory, replace

y_1 by $-\frac{1}{y_1}$ in (5) i.e.

$$\frac{\left(x - \frac{y}{y_1}\right)}{(a^2 - b^2)} [x + yy_1] = 1 \quad (6)$$

Clearly equations (5) and (6) are the same.

Hence the given system is self orthogonal.

Problem 3.9.5. Find the O.T of the family of astroids $x^{2/3} + y^{2/3} = a^{2/3}$

Solution : Consider $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w.r.to x , we have

$$\frac{2}{3} \cdot x^{-1/3} + \frac{2}{3} \cdot y^{-1/3} \frac{dy}{dx} = 0$$

ie.,

$$x^{-1/3} + y^{-1/3} \frac{dy}{dx} = 0$$

. This is the D.E of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have

$$x^{-1/3} + y^{-1/3} \left(-\frac{dx}{dy}\right) = 0$$

$$\text{ie., } x^{-1/3} dy = y^{-1/3} dx$$

$$y^{1/3} dy = x^{1/3} dx$$

by separating the variables.

$$\Rightarrow \int y^{3/3} dy - \int x^{1/3} dx = c$$

$$\Rightarrow \frac{y^{4/3}}{(4/3)} - \frac{x^{4/3}}{(4/3)} = c$$

or

$$x^{4/3} - y^{4/3} = -\frac{4c}{3} = k \text{ (say)}$$

Thus $x^{4/3} - y^{4/3} = k$ is the required O.T.

Problem 3.9.6. Find the Orthogonal trajectories of the cardioid $r = a(1 - \cos\theta)$

(VTU Model 2022, Jan 2015, Jan 2014, Mar 2001, Jul 2007, Jan 2011)

Solution : Let $r = a(1 - \cos\theta)$ (1)

Apply log on both sides

$\log r = \log a + \log(1 - \cos\theta)$ (2) differentiate (1) w.r.to θ , we get

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{1}{1 - \cos\theta} (\sin\theta)$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ we get,

$$-r \frac{d\theta}{dr} = \frac{1}{1 - \cos\theta} (\sin\theta)$$

$$i.e. \frac{1 - \cos\theta}{(\sin\theta)} d\theta = \frac{-dr}{r}$$

$$\tan \frac{\theta}{2} d\theta = \frac{-dr}{r} \text{ (This is the differential equation of the orthogonal trajectory)}$$

Integrating on both sides

$$\frac{-\log(\cos \frac{\theta}{2})}{\frac{1}{2}} = -\log r - \log c$$

$$i.e. 2\log(\cos \frac{\theta}{2}) = \log r + \log c = \log(rc)$$

$$i.e. \cos^2 \frac{\theta}{2} = rc \text{ or } r = \frac{1}{c} \cos^2 \frac{\theta}{2} = \frac{1}{2c} (1 + \cos\theta)$$

$$i.e. r = b(1 + \cos\theta) \text{ where } b = \frac{1}{2c} \text{ and}$$

Problem 3.9.7. Find the Orthogonal trajectories of $r^2 = a^2 \cos 2\theta$ (VTU Jun

2014, Jan 2009, Aug 2003)

Solution: Given $r^2 = a^2 \cos 2\theta$ (1)

On differentiating (1), we have

$$r \frac{dr}{d\theta} = -a^2 \sin 2\theta \quad (2)$$

Eliminate 'a' from (2), using (1)

$$r \frac{dr}{d\theta} = -r^2 \tan 2\theta$$

$$\text{i.e. } \frac{dr}{d\theta} = -r \tan 2\theta \quad (3)$$

which is the differential equation of the given family.

To obtain the differential of the trajectories, replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in equation (3).

$$\text{i.e. } -r^2 \frac{d\theta}{dr} = -r \tan 2\theta$$

or $\cot 2\theta d\theta = \frac{dr}{r}$ (4) Integration of (4), gives the equation of the desired system.

$$\therefore \frac{\log \sin 2\theta}{2} + \log c = \log r$$

$$\log(\sin 2\theta) + 2 \log(c) = 2 \log r$$

$$\text{i.e. } \log(\sin 2\theta) + \log(c^2) = \log r^2$$

$$\text{i.e. } \log(c^2 \sin 2\theta) = \log(r^2)$$

$$\text{i.e. } c^2 \sin 2\theta = r^2$$

Problem 3.9.8. Find the Orthogonal trajectories of $r^n = a^n \cos n\theta$ (VTU July 2016, Feb 2003)

Solution :

$$r^n = a^n \cos n\theta$$

Diff. w.r. to θ

$$nr^{n-1} \frac{dr}{d\theta} = -a^n \sin n\theta (n)$$

$$\frac{r^n dr}{r d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\frac{1 dr}{r d\theta} = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\frac{1 dr}{r d\theta} = -\tan n\theta$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = -\tan n\theta$$

$$\frac{1}{r} dr = \frac{1}{\tan n\theta} d\theta$$

Integrating on both sides

$$\int \frac{1}{r} dr = \int \frac{1}{\tan n\theta} d\theta$$

$$\log r = \int \cot n\theta d\theta$$

$$\log r = \frac{\log(\sin n\theta)}{n} + \log c$$

$$n \log r = \log \sin n\theta + n \log c$$

$$\log r^n = \log(\sin n\theta) + \log c^n$$

$$\log r^n = \log [\sin n\theta c^n]$$

$$r^n = c^n \sin n\theta$$

Problem 3.9.9. Find the Orthogonal trajectories of $r^n = a^n \sin n\theta$ (VTU July 2017)

SOLution :

$$\frac{dr}{d\theta} n r^{n-1} = a^n (\cos n\theta) n$$

$$\frac{r^n dr}{r d\theta} = a^n \cos n\theta$$

$$\frac{1 dr}{r d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\frac{1 dr}{r d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\frac{1 dr}{r d\theta} = \cot n\theta$$

$$\text{Replace } \left(\frac{dr}{d\theta} \right) = -\frac{r^2 d\theta}{dr}$$

$$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) = \cot n\theta$$

$$\frac{-1}{r dr} = \frac{1}{\cot n\theta} d\theta$$

Integrating on both sides

$$\int \frac{-1}{r} dr = \int \tan n\theta d\theta$$

$$\Rightarrow -\log r = \frac{\log(\sec n\theta)}{n} - \log c$$

$$\Rightarrow -n \log r = \log \sec n\theta - n \log c$$

$$\Rightarrow -\log r^n = \log \sec n\theta - \log c^n$$

$$\Rightarrow \log c^n - \log r^n = \log \sec n\theta$$

$$\Rightarrow \log \left(\frac{c^n}{r^n} \right) = \log(\sec n\theta)$$

$$\Rightarrow r^n = \frac{c^n}{\sec n\theta}$$

$$\Rightarrow r^n = c^n \cos n\theta$$

L-R and C-R circuits

Generally, an electrical circuit is made up of a combination of resistors, inductors and capacitors. If a circuit is made up of resistors and capacitors, then it is known as an **RC circuit**. Similarly, when the circuit is containing capacitors and inductors,

then it is known as an LC circuit. An **LR circuit** is a circuit which is made up of resistors and inductors.

Let $i = I(t)$ denote the current in the circuit and $q = q(t)$ denote the charge on the capacitor. Furthermore, let L denote inductance in henrys (H), R denote resistance in ohms (Ω), and C denote capacitance in farads (F). Last, let $E(t)$ denote electric potential in volts (V).

The formation of differential equation for an electric circuit depends upon the following laws.

(i) $i = \frac{dq}{dt}$,

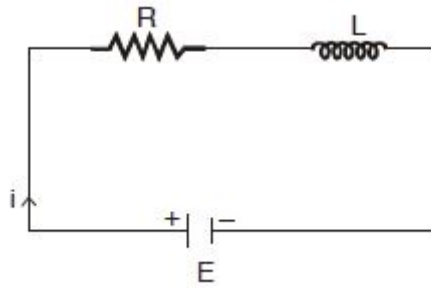
(ii) Voltage drop across resistance, R is $V_R = Ri$ (Ohm's law)

(iii) Voltage drop across inductance L is $V_L = L \cdot \frac{di}{dt}$ (Faraday's law and Lenza's law)

(iv) Voltage drop across capacitance C is, $V_C = \frac{q}{C}$

Kirchhoff's Voltage law :

The algebraic sum of the voltage drop around any closed circuit is equal to the resultant electromotive force in the circuit.

L - R series circuit :

Let i be the current flowing in the circuit containing resistance R and inductance L in series, with voltage source E , at any time t .

By voltage law

$$Ri + L \frac{di}{dt} = E$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \dots$$

This is the linear differential equation of the form $\frac{di}{dt} + P i = Q$

$$\text{I.F.} = e^{\int P dt} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Its solution is

$$i \times (IF) = \int Q \times (IF) dt + c$$

$$\Rightarrow i \times e^{\frac{R}{L}t} = \int \frac{E}{L} \times e^{\frac{R}{L}t} dt + c$$

$$= \frac{E}{L} \times \frac{L}{R} e^{\frac{R}{L}t} + c$$

Dividing by $e^{\frac{R}{L}t}$

$$\Rightarrow i = \frac{E}{R} + ce^{-\frac{Rt}{L}}$$

This represents the current in the circuit, at any time t

$$\text{At } t = 0, \quad i = 0 \Rightarrow c = -\frac{E}{R}$$

Thus, above current i becomes

$$i = \frac{E}{R} \left[1 - e^{-\frac{Rt}{L}} \right]$$

Growth of Current

An LR Circuit is analysed in three ways. The first one is the initial state, which is present at the instant of closing the switch or opening the switch in the circuit. The second one is the transient state, which appears at any instant after closing or opening the switch. The third one is **steady-state**, which appears after a long time after closing and opening the switch.

Let's start with the initial and steady states of an LR circuit.

Initial State : Let us assume a circuit of EMF E has the inductance L and the resistance R , as shown in the figure.

The voltage drop across the inductor is V_L and the voltage drop across the resistor is V_R .

At $t = 0$, the inductor offers an infinite opposition to the current flow and hence

there is no current flow in the circuit at the time of closing the switch. Due to high opposition to the current flow, the voltage is dropped entirely at the inductor and there is no voltage drop across the resistor.

i.e., at $t = 0$, $V_R = 0$ and $V_L = E$.

Steady State: At a certain point of time, say $t = \infty$, the current in the inductor does not vary with time after closing or opening the switch for a long period of time. We can see that the current has reached its maximum value and therefore the inductor does not offer any position to the current flow. So, the voltage drop across the inductor becomes zero and the entire voltage drops across a resistor.

At $t = \infty$, $V_R = E$ and $V_L = 0$

Transient State : In this state, the voltage is dropped both across the resistor and the inductor. At any instant $t = 0$ and $t = \infty$ is taken for this state. We know that the voltage drop across the inductor is equal to the inductance multiplied by the rate of change in current across the inductor.

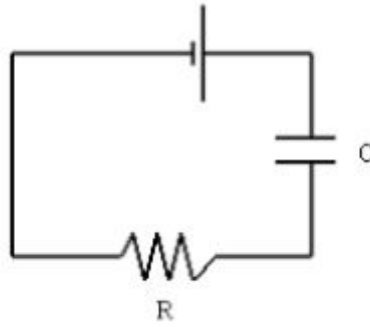
i.e., $V_L = L \frac{dI}{dt}$

And the voltage across the resistor is given by $V_R = IR$.

In the transient state, when the switch is closed gradually, the current starts increasing across the inductor. Due to the increase in the current, there will be a self-induced EMF in the inductor which opposes the change of the current in the circuit.

C – R series circuit :

Let i be current in the circuit containing resistance R , L , and capacitance C in series with voltage source E , at any time t .



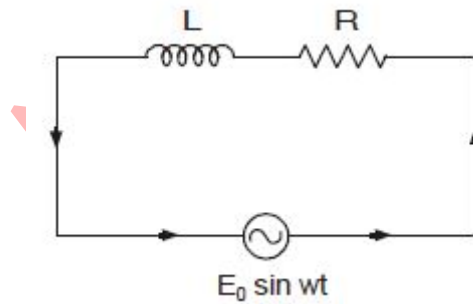
By voltage law

$$Ri + \frac{q}{C} = E$$

$$\Rightarrow R \frac{dq}{dt} + \frac{q}{C} = E \quad \left[\because i = \frac{dq}{dt} \right]$$

Problem 3.9.10. Solve the equation $L \frac{di}{dt} + Ri = E_0 \sin wt$ where L, R and E_0 are constants and discuss the case when t increases indefinitely. (VTU July 2021, Jan 2021)

Solution :



$$L \frac{di}{dt} + Ri = E_0 \sin wt$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L} \sin wt$$

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

$$i \cdot e^{\frac{R}{L}t} = \frac{E_0}{L} \int e^{\frac{R}{L}t} \sin wt dt + c$$

$$\left[\int e^{ax} \sin bx dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) \right]$$

$$\Rightarrow i e^{\frac{R}{L}t} = \frac{E_0}{L} \frac{e^{\frac{R}{L}t}}{\sqrt{\frac{R^2}{L^2} + \omega^2}} \sin \left(\omega t - \tan^{-1} \frac{L\omega}{R} \right) + c$$

$$i = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} \sin \left(\omega t - \tan^{-1} \frac{L\omega}{R} \right) + ce^{-\frac{R}{L}t}$$

As t increases indefinitely, then $ce^{-\frac{Rt}{L}}$ tends to zero. SO

$$i = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} \sin \left(\omega t - \tan^{-1} \frac{L\omega}{R} \right)$$

Problem 3.9.11. Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force $E \sin \omega t$ satisfies the equation

$$iR + L \frac{di}{dt} = E \sin \omega t$$

Find the value of the current at any time t , if initially there is no current in the circuit. (VTU Model EE 2022)

Solution : By Kirchoff's first law, we have sum of voltage drops across R and $L = E \sin \omega t$ i.e., $Ri + L \frac{di}{dt} = E \sin \omega t$ This is the required differential equation which can be written as $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \sin \omega t$

This is a linear equation.

Its I.F. = $e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

\therefore the solution is i (I.F.) = $\int \frac{E}{L} \sin \omega t \cdot (\text{I.F.}) dt + c$

or $ie^{\frac{Rt}{L}} = \frac{E}{L} \int e^{\frac{Rt}{L}} \sin \omega t dt + c = \frac{E}{L} \frac{e^{\frac{Rt}{L}}}{\sqrt{[(R/L)^2 + \omega^2]}} \sin \left(\omega t - \tan^{-1} \frac{L\omega}{R} \right) + c$

or $i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \phi) + ce^{-Rt/L}$ where $\tan \phi = L\omega/R$

Initially when $t = 0; i = 0$.

$$\therefore 0 = \frac{E \sin(-\phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + c,$$

$$\text{i.e., } c = \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}}$$

$$\text{Thus (i) takes the form } i = \frac{E \sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}} \cdot e^{-Rt/L}$$

$$\text{or } i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} [\sin(\omega t - \phi) + \sin \phi \cdot e^{-Rt/L}]$$

which gives the current at any time t .

Problem 3.9.12. A resistance of 100Ω s, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5 \text{ sec}$, if $i = 0$ at $t = 0$

Solution :

$$L \frac{di}{dt} + Ri = E$$

$$\therefore \frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

Solution is given by,

$$i, e^{\int (\frac{R}{L}) dt} = \int e^{\int (\frac{R}{L}) dt} \frac{E}{L} dt + c$$

$$\therefore i \cdot e^{\frac{Rt}{L}} = \frac{E e^{\frac{Rt}{L}}}{R} + c$$

$$\text{At } t = 0, i = 0 \therefore c = \frac{E}{R}$$

$$\therefore i \cdot e^{\frac{Rt}{L}} = \frac{E e^{\frac{Rt}{L}}}{R} + \frac{-E}{R}$$

$$\therefore i = \frac{E}{R} (1 - e^{-\frac{Rt}{L}})$$

For given condition $R = 100, L = 0.5, E = 20$

$$\therefore i = 0.2 (1 - e^{-200t})$$

Problem 3.9.13. When a resistance R Ohms is connected in series with an inductance L henries with an emf of E volts, the current i amperes at time t is given by

$L \frac{di}{dt} + Ri = E$. If $E = 10 \sin t$ volts, and $i = 0$ when $t = 0$, find i as a function

of t .

(VTU Model EE 2022)

Solution : We have,

$$\begin{aligned} L \frac{di}{dt} + Ri &= E \\ \Rightarrow \frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \dots \\ \therefore I.F. &= e^{\int \frac{R}{L} dt} \\ &= e^{\frac{R}{L}t} \end{aligned}$$

Multiplying both sides of (1) by $I.F. = e^{\frac{R}{L}t}$, we get

$$\begin{aligned} e^{\frac{R}{L}t} \left(\frac{di}{dt} + \frac{R}{L}i \right) &= e^{\frac{R}{L}t} \times \frac{E}{L} \\ \Rightarrow e^{\frac{R}{L}t} \frac{di}{dt} + e^{\frac{R}{L}t} \frac{R}{L}i &= e^{\frac{R}{L}t} \times \frac{E}{L} \end{aligned}$$

Integrating both sides with respect to t , we get

$$\begin{aligned} e^{\frac{R}{L}t}i &= \frac{E}{L} \int e^{\frac{R}{L}t} dt + C \\ \Rightarrow e^{\frac{R}{L}t}i &= \frac{E}{L} \times \frac{L}{R} e^{\frac{R}{L}t} + C \\ \Rightarrow e^{\frac{R}{L}t}i &= \frac{E}{R} e^{\frac{R}{L}t} + C \dots \end{aligned}$$

Now,

$$\begin{aligned} i &= 0 \text{ at } t = 0 \\ \therefore e^0 \times 0 &= \frac{E}{R} e^0 + C \\ \Rightarrow C &= -\frac{E}{R} \end{aligned}$$

Putting the value of C in (2), we get

$$\begin{aligned} e^{\frac{R}{L}t}i &= \frac{E}{R} e^{\frac{R}{L}t} - \frac{E}{R} \\ \Rightarrow i &= \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L}t} \\ \Rightarrow i &= \frac{E}{R} \left(1 - e^{-\frac{R}{L}t} \right) \end{aligned}$$

Question Bank

Exact, Reducible to Exact Differential Equations

1. Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$. (VTU July 2017, Jan 2014, Model 2014) Ans : $x^4 + e^{xy^2} - y^3 = c$
2. Solve $(1 + e^{\frac{x}{y}})dx + (1 - \frac{x}{y})e^{\frac{x}{y}}dy = 0$. (VTU June 2019, June 2014)
Ans : $x + ye^{x/y} = c$
3. Solve $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$ (VTU Jan 2016)
(or)
Solve $\frac{dy}{dx} + \frac{(y\cos x + \sin y + y)}{(\sin x + x\cos y + x)} = 0$ (VTU Model ME 2022, Jan 2020, Jan 2019, June 2018, Jan 2018, 2016) Ans : $y \sin x + (\sin y + y)x = c$
4. Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$ (VTU June 2013)
Ans : $x^3 - 6x^2y - 6xy^2 + y^3 = 3c = C$
5. Solve $(x^2 + y^2 + x)dx + xydy = 0$. (VTU Model 2022, July 2021, June 2018, July 2017, Aug 2001, Mar 2000) Ans : $\frac{x^4}{4} + \frac{(x^2y^2)}{2} + \frac{x^3}{3} = c$
6. Solve $x^2ydx - (x^3 + y^3)dy = 0$ (VTU Jan 2010, Dec 2009) Ans :
 $\frac{x^3}{3y^3} - \log y = c$
7. Solve $(x^2 + y^3 + 6x)dx + y^2xdy = 0$ (VTU Model ME 2022, July 2016)
Ans : $\frac{x^5}{5} + y^3\frac{x^3}{3} + 6\frac{x^4}{4} = c$
8. Solve $(xy^2 - e^{\frac{1}{x^3}})dx - x^2ydy = 0$. (VTU Jan 2015) Ans :
 $-\frac{y^2}{2x^2} + \frac{1}{3}e^{\frac{1}{x^3}} = c$.

9. Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$. (VTU Jan 2015) **Ans :**
 $3x^2y^4 + 6y^2x + 2y^6 = c_1$
10. Solve $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$ (VTU Jan 2020)
11. Solve $[y(1 + \frac{1}{x}) + \cos y] dx + [x + \log x - x \sin y] dy = 0$ (VTU Jan 2021) **Ans :** $y(x + \log x) + x \cos y = C$
12. Solve $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$ (VTU Jan 2018) **Ans :** $e^{xy} + y^2 = c$
13. $ylogydx + (x - logy)dy = 0$ (VTU Model ME 2022) **Ans :**
 $xlogy - \frac{1}{2}(logy)^2 = c$
14. Solve $(e^y + 1) \cos x dx + e^y \sin x dy = 0$. **Ans :** $(e^y + 1) \sin x = c$
15. $(1 + 2xy \cos x^2 - 2xy)dx + (\sin x^2 - x^2)dy = 0$ (July 2015) **Ans**
 $:x + y \sin x^2 - x^2y = c$
16. Solve $\frac{2x}{y^3}dx + \frac{(y^2 - 3x^2)}{y^4}dy = 0$. **Ans :** $x^2 - y^2 = cy^3$
17. Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$.
 $x^5 + x^3y^2 - x^2y^3 - y^5 = c$
18. Solve $y \sin 2x dx - (1 + y^2 + \cos^2 x)dy = 0$. **Ans :**
 $3y \cos 2x + 6y + 2y^3 = c$
19. Solve $(y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0$ **Ans :**
 $(y + \frac{2}{y^2}) x + y^2 = c$
20. Solve $(x^2 + y^2 + 2x) dx + 2y dy = 0$ **Ans :** $(x^2 + y^2) e^x = c$
21. $(x - ylogy + ylogx)dx + x(logy - logx)dy = 0$ **Ans :**
 $logx + \frac{y}{x}logy - \frac{y}{x}(logx + 1) = c$

22. $y(x + y + 1)dx + x(x + 3y + 2)dy = 0$ Ans : $\frac{x^2y^2}{2} + xy^3 + xy^2 = c$

23. $(xy + y^2)dx + (x + 2y - 1)dy = 0$ (VTU Feb 2004) Ans :
 $xye^x - ye^x + y^2e^x = c$

24. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$ (VTU Feb 2002) Ans :
 $x^3y^2 + \frac{x^2}{y} = c$

25. $y(x + y)dx + (x + 2y - 1)dy = 0$ Ans : $y(xe^x - e^x) + e^xy^2 = c$

26. $(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy = 0$ (VTU Model CV 2022) Ans :
 $x^4y + x^3y^2 - x^4 = c$

27. $y(2x - y + 1)dx + x(3x - 4y + 3)dy = 0$ Ans :
 $x^2y^3 - xy^4 + xy^3 = c$

28. $y(2xy + 1)dx - xdy = 0$ Ans : $x^2 + \frac{x}{y} = c$

Linear and Bernoulli Differential Equations:

1. Solve $x\frac{dy}{dx} + y = x^3y^6$. (VTU Model 2022, Model 2014) Ans :
 $\frac{1}{x^5y^5} = \frac{5}{2x^2} + c$

2. Solve $xy(1 + xy^2)\frac{dy}{dx} = 1$ VTU June 2018, July 2017 Ans :
 $\frac{1}{x} = 2 - y^2 + ce^{-\frac{1}{2}y^2}$

3. Solve $\frac{dy}{dx} + y\tan x = y^3\sec x$ (VTU June 2019, Jan 2018, Jan 2015) Ans :
 $(1/y^2) \cdot \frac{1}{\sec^2 x} = -2\sin x + x + c$

4. Solve $(x + 2y^3)\frac{dy}{dx} = y$ (VTU July 2015) Ans : $x = y^3 + cy$

5. Solve $(1 + y^2)dx + (x - e^{\tan^{-1}y})dy = 0$ (VTU Jan 2014, June 2013)

Ans : $x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}$

6. Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ (VTU July 2017, Jan 2015, June 2013, July 2014, July 2011)

Ans : $\tan y = \frac{1}{2}x^2 - 1 + ce^{(-x^2)}$

7. Solve $\frac{dy}{dx} = xy^3 - xy$ (VTU Jan 2017)

Ans : $\frac{e^{-x^2}}{y^2} = e^{-x^2} + c$

8. Solve $\frac{dy}{dx} - \frac{2}{x}y = \frac{y^2}{x^3}$ (VTU Jan 2018)

9. Solve $r \sin \theta - \cos \theta \frac{dr}{d\theta} = r^2$ (VTU Jan 2020, June 2018, Model 2018)

10. Solve $y(2xy + 1)dx - xdy = 0$ (VTU Model 2022, Model 2018)

11. Solve $\frac{dy}{dx} + y \tan x = y^2 \sec x$ (VTU Model ME 2022, Jan 2019)

Ans : $\sec x = (\tan x + c)y$

12. Solve $(1 + y^2)dx + (x - e^{\tan^{-1}y})dy = 0$ (VTU Jan 2014, June 2013)

Ans : $xe^{\tan^{-1}y} = \tan^{-1}y + c$

13. Solve $e^y \frac{dy}{dx} + 1 = e^x$. (VTU Jan 2016)

Ans : $\frac{1}{y} = x^2 - 2 + ce^{\frac{-x^2}{2}}$

14. Solve $(2x \log x - xy)dy + 2ydx = 0$ (VTU Jan 2017)

Ans : $2y \log x - \frac{1}{2}y^2 = c$

15. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2x$. (VTU Model 2022)

Differential Equations solvable for p and Clairaut's equations

1) $xy \left(\frac{dy}{dx}\right)^2 - (x^2 + y^2) \frac{dy}{dx} + xy = 0$ (VTU Model 2022, Jan 2019, Jan 2018)

or

$$xy \left\{ \left(\frac{dy}{dx} \right)^2 + 1 \right\} = (x^2 + y^2) \frac{dy}{dx} \quad (\text{VTU June 2018}) \quad \text{Ans :}$$

$$(y^2 - x^2 - 2c_1)(y - cx) = 0$$

2) $x^2p^2 + 3xyp + 2y^2 = 0$ (VTU Jan 2017) Ans: $(x^2y - c)(xy - c) = 0$

3) $x^2p^2 + xyp - 6y^2 = 0$ (VTU July 2017)

4) Solve the equation $p^2 + p(x + y) + xy = 0$ (VTU Jan 2013, Dec 2014) Ans
: $(\log y + x - c)(y + \frac{x^2}{2} - c) = 0$

5) Solve $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$ (VTU July 2017, July 2013) Ans
: $(y - cx^2)(y^2 + 3x^2 - c) = 0$

6) Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ (VTU Model ME 2022, June 2018, July 2017, Jan 2016, June 2015, Jan 2016) Ans : $(y^2 - x^2 - 2c)(xy - e^c) = 0$.

7) Solve $p^2 + 2p \cot x = y^2$. (VTU Model 2022, Jan 2020, July 2016, June 2012) Ans : $(y \cos^2 \frac{x}{2} - C)(y \sin^2 \frac{x}{2} - C) = 0$ Ans: $y(1 \pm \cos x) = c$

8) $xp^2 - (2x + 3y)p + 6y = 0$ (VTU Jan 2015)
Ans: $(y - 2x - c)(y - cx^3) = 0$

9) Solve $p(p + y) = x(x + y)$ (VTU Dec 2014, July 2011, Dec 2011)
Ans : $(y - \frac{x^2}{2} - c)(y + x - 1 - ce^{-x}) = 0$ or
 $(2y - x^2 - c)(e^x(y + x + 1) - c) = 0$

10) Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$ (VTU Model EE 2022)

11) Solve $p^2 + 2p \cosh x + 1 = 0$ (VTU Jan 2014)

12) Solve $p^2 - 2p \sinh x - 1 = 0$ (VTU Jan 2015)

Ans: $(y - \cosh x - \sinh x - c)(y - \cosh x + \sinh x - c) = 0$

13) Solve $p^2 + p(x + y) + xy = 0$ (VTU Jan 2013)

14) Solve $y \left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ (VTU July 2016)

Ans: $(y - x - c)(y^2 + x^2 - c) = 0$

15) Solve $\frac{dy^2}{dx} - 5dy/(dx) + 6 = 0$ Ans: $(y - 3x - c)(y - 2x - c) = 0$

16) Solve $x^2 \left(\frac{dy}{dx}\right)^2 - xy\frac{dy}{dx} - 6y^2 = 0$ Ans: $(y - cx^2)(x^3y - c) = 0$

17) Obtain the general solution and singular solution of the equation, $y = px + p^3$
(VTU Jan 2016) Ans : $y = cx + c^3$

18) Obtain the general solution and singular solution of the Clairaut's equation $xp^3 - yp^2 + 1 = 0$.

(VTU July 2017, Jan 2015, June 2014, Dec 2011) Ans : $cx + \frac{1}{c^2}$ and

$4y^3 = 27x^2$

19) Obtain the general solution and singular solution of the Clairaut's equation $(y - px)(p - 1) = p$ (VTU Jan 2015)

20) Solve the equation $y^2(y - xp) = x^4p^2$ using the substitution, $X = \frac{1}{x}$ and $Y = \frac{1}{y}$ (VTU June 2014)

21) Solve $(px - y)(py + x) = 2p$, by reducing into Clairaut's form, taking the substitution $X = x^2, Y = y^2$. (VTU Model 2022, June 2018, July 2017, July 2016, June 2012, Dec 2011).

22) Solve $(px - y)(py + x) = a^2p$, use the substitution $X = x^2, Y = y^2$.
(VTU Model 2022, June 2019, Jan 2018, July 2017, June 2014, June 2015, Dec 2011)

Ans : $y^2 = cx^2 - \frac{a^2c}{c+1}$

- 23) Solve $x^2(y - px) = yp^2$ using the substitution $X = x^2, Y = y^2$ (VTU Model ME 2022, July 2016) **Ans :** $y^2 = cx^2 + c^2, x^4 + 4y^2 = 0$
- 24) Find the general and singular solution of the equation $p = \log(px - y)$ (VTU Model 2015). **Ans:** $y = x \log x - x$
- 25) Solve $y = 2px - y^2p^3$ Take $X = 2x, Y = y^2$
- 26) Obtain the general solution and singular solution of the equation $\sin px \cos y = \cos px \sin y + p$ (VTU Jan 2013) **Ans :** $y = \frac{\sqrt{x^2-1}}{x}x + \sin^{-1} \frac{\sqrt{x^2-1}}{x}$
- 27) Solve $p + \cos y \sin px = \sin y \cos px$ **Ans:** $y = cx + \sin^{-1}c$
- 28) Find the general and singular solution of the equation $xp^2 - py = a = 0$ (VTU Model 2018)
- 29) Find the general and singular solution for $xp^2 + xp - yp + 1 - y = 0$ (VTU Model ME 2022, June 2018) **Ans :** $y = cx + \frac{1}{c+1}$ and $(x + y)^2 = 4x$

Orthogonal trajectories

- Find the Orthogonal trajectories of the family of parabola $y^2 = 4ax$ (VTU Jan 2020) **Ans :** $2x^2 + y^2 = c_1$
- Find the Orthogonal trajectories of the family of confocal ellipses $\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1$, where λ is a parameter . (VTU July 2021, Model 2018, July 2015, Dec 2011, July 2011, June 2009, Aug 2000, Mar 2000, Aug 1999) **Ans :** $\frac{y^2}{2} + \frac{x^2}{2} = a^2 \log x + c$
- Show that the orthogonal trajectories of a family of circles passing through the origin having centres on x-axis is a family of circles passing through the origin

having their centres on y-axis.

(VTU Model 2014)

4. P.T. the system of confocal and coaxial parabolas $y^2 = 4a(x + a)$ is self orthogonal.

(VTU Model EE 2022, Jan 2021, June 2019, June 2018, Jun 2013)

5. Prove that the family of curves $\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1$, where λ is the parameter is self - orthogonal.

(VTU Model 2022, July 2017, Jan 2016, Jan 2015, Jan 2013)

6. Given $y = ke^{-2x} + 3x$ find member of its orthogonal trajectory passing through the point (0, 3) (VTU Feb 2006)

7. Find the Orthogonal trajectories of the family of curves $y^2 = cx^3$ (VTU Jan 2018)

8. Find the O.T of the family of astroids $x^{2/3} + y^{2/3} = a^{2/3}$ **Ans :**
 $x^{4/3} - y^{4/3} = k$

9. Find the Orthogonal trajectories of the family $y^2 = cx^3$ **Ans :**
 $3y^2 + 2x^2 = k$

10. Find the Orthogonal trajectories of the family $y = ax^2$ **Ans :** $y^2 + \frac{x^2}{2} = c$

11. Find the Orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\lambda x + c = 0$, λ is a parameter . (VTU Feb 2002) **Ans :** $x^2 + y^2 - ky = c$

12. Show that the orthogonal trajectories of a family of circles passing through the origin having centers on x-axis is a family of circles passing through the origin having their centers on y-axis. (VTU Model 2014)

13. Find the Orthogonal trajectories of the family of curves $x^3 - 3xy^2 = c$ Ans
: $y^3 - 3x^2y = c$
14. Find the Orthogonal trajectories of the cardioid $r = a(1 + \cos\theta)$ (VTU
Model 2022, Aug 2001) Ans : $r = b(1 - \cos\theta)$
15. Find the Orthogonal trajectories of $r^n = a^n \sin n\theta$ Ans : $r^n = b^n \cos n\theta$
16. Find the Orthogonal trajectories of $r = 2a \cos\theta$, where 'a' is a parameter.
VTU July 2017
17. Find the Orthogonal trajectories of $r^2 = a^2 \cos 2\theta$ (VTU Jun 2014, Jan 2009,
Aug 2003) Ans : $r^2 = c^2 \sin 2\theta$
18. Find the Orthogonal trajectories of $(r + \frac{k^2}{r}) \cos\theta = a$ where 'a' is the param-
eter. (VTU Feb 2005) Ans :
 $(r - \frac{k^2}{r}) \sin\theta = c$
19. Find the Orthogonal trajectories of $\frac{2a}{r} = 1 - \cos\theta$ Ans :
 $r(1 + \cos\theta) = 2b$
20. Find the Orthogonal trajectories of $r = \frac{2a}{1 + \cos\theta}$ Ans : $r = \frac{2b}{1 - \cos\theta}$
21. Find the Orthogonal trajectories of $r = 4a(\sec\theta \tan\theta)$. Ans :
 $r(1 + \sin^2\theta)^{\frac{1}{2}} = c$
22. Find the Orthogonal trajectories of $r^n \sin n\theta = a^n$ (VTU Model 2022) Ans
: $r^n \cos n\theta = b^n$
23. Find the Orthogonal trajectories of $r^n \cos n\theta = a^n$ (VTU Jan 2017, June
2013, Jun 2011, Jan 2010,, Dec 2009) Ans : $r^n \sin n\theta = b^n$

24. Find the Orthogonal trajectories of $r^n = a \sin n\theta$ (VTU Jun 2012, July 2006, Aug 2002 , Mar 1999) **Ans : $r^n = b \cos n\theta$**
25. Find the Orthogonal trajectories of $r^n = a^n \cos n\theta$ (VTU July 2016, Feb 2003) **Ans : $r^n = b^n \sin n\theta$**
26. Find the Orthogonal trajectories of $r^n = a^n \sin n\theta$ (VTU July 2017) **Ans : $r^n = c^n \cos n\theta$**
27. $r = c \sin^2 \theta$ **Ans : $r^2 = b \cos \theta$**
28. Show that orthogonal trajectories of the family of cardioids $r = a \cos^2 \left(\frac{\theta}{2}\right)$ is another family of cardioids $r = b \sin^2 \left(\frac{\theta}{2}\right)$ (VTU Jan 2015)
29. Find the orthogonal trajectories of the family curve $r = 2a(\cos \theta + \sin \theta)$ (VTU Model ME 2022)

L-R and C-R circuits

- An RL circuit has an emf of 5 V, a resistance of 50ω , an inductance of $1H$, and no initial current. Find the current in the circuit at any time t. Distinguish between the transient and steady-state current.
- If we close a switch in a circuit that contains a battery E , an inductance L, and a resistance R, the current i builds up at a rate given by $L \frac{di}{dt} + Ri = E$. Find i as a function of t. How long does it will be before the current has reached one-half of its final value if E=6 Volts , R=100 ohms, and L = 0.1 henry? (VTU Model 2022, Jan 2020) **ANS : 0.0006931 sec**

3. When a resistance R Ohms is connected in series with an inductance L henries with an emf of E volts, the current i amperes at time t is given by $L \frac{di}{dt} + Ri = E$. If $E = 10 \sin t$ volts, and $i = 0$ when $t = 0$, find i as a function of t .

(VTU Model EE 2022) ANS : $\frac{10}{L^2 + R^2} (R \sin t - L \cos t + L e^{-\frac{Rt}{L}})$

4. A resistance of 100Ω s, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5 \text{ sec}$, if $i = 0$ at $t = 0$

5. Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force $E \sin \omega t$ satisfies the equation

$$iR + L \frac{di}{dt} = E \sin \omega t$$

Find the value of the current at any time t , if initially there is no current in the circuit. (VTU Model EE 2022)

6. Solve $L \frac{di}{dt} + Ri = E_0 \sin \omega t$, where L, R & E_0 are constants and discuss the case when t increases indefinitely. (VTU July 2021, Jan 2021)



Lecture Notes

VTU - BMATC101 - Mathematics-I for Civil Stream- Module 4 : Differential Equations of Higher order

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Module 4

Differential Equations of higher order

Module 4 Syllabus

Module 4: Differential Equations of higher order: -Second order linear ODE's with constant coefficients - Inverse differential operators, method of variation of parameters; Cauchy's and Legendre homogeneous equations. (RBT Levels: L1 ,L2 and L3)

Linear Differential Equations of Higher Order

The general form of n th order linear differential equation is

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y = R(x) \quad \dots \quad (1)$$

where a_0, a_1, \dots, a_n , and $R(x)$ are functions of only x .

The general form of n th order linear differential equation with constant coefficients is

$$k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} \frac{dy}{dx} + k_n y = R(x) \quad \dots \quad (2)$$

where k_0, k_1, \dots, k_n are constants and $R(x)$ is a function of only x .

Operator D : Denoting $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, \dots , (2) can be written in symbolic form as

$$(k_0 D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n) y = R(x)$$

This takes the form

$$\boxed{f(D)y = R(x)}$$

where $f(D) = (k_0 D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)$

4.1 Method of Solving a linear differential equation

To solve the differential equation $f(D)y = R(x)$, we use the following steps :

Step 1 :

Find the Complementary function:

Solve the Auxiliary Equation(A.E.), $f(D) = 0$. (or $f(m) = 0$).

Solving the A.E. we get n roots $m_1, m_2, m_3, \dots, m_n$.

Then the solution of the homogeneous linear differential equation part $f(D)y = 0$ is called as the **Complementary Function (C.F.)** or y_c and is given by using the following rules.

Sl. No.	Roots	Complementary Function (C.F.)
1.	One real root m_1	$C.F. = c_1 e^{m_1 x}$
	Two real and different roots m_1 and m_2	$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
	Three real and different roots m_1, m_2 and m_3	$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2.	Two real and equal roots $m_1 = m_2$	$C.F. = (c_1 + c_2 x) e^{m_1 x}$
	Three real and equal roots $m_1 = m_2 = m_3$	$C.F. = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$
3	One pair of complex roots $a \pm ib$	$C.F. = e^{ax} (c_1 \cos bx + c_2 \sin bx)$
4	Two pair of complex and equal roots (say $m = a \pm ib, a \pm ib$)	$C.F. = e^{ax} [(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx]$

Some Examples :

Roots of A.E.	Complementary Function (C.F.)
$m = 1, 2$	$y_c = c_1 e^x + c_2 e^{2x}$
$m = 2, -2, 5, 6$	$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{5x} + c_4 e^{6x}$
$m = 3, 3$	$y = (c_1 + c_2 x) e^{3x}$
$m = 2, 2, 2$	$y = (c_1 + c_2 x + c_3 x^2) e^{2x}$
$m = \pm i = 0 \pm i$	$y = e^{0x} (c_1 \cos x + c_2 \sin x)$
$m = 2 \pm 3i$	$y = e^{2x} (c_1 \cos 3x + c_2 \sin 3x)$
$m = 2, -1, 6, 6$	$y = c_1 e^{2x} + c_2 e^{-x} + (c_3 + c_4 x) e^{6x}$
$m = -1 \pm 2i, -1 \pm 2i$	$y = e^{-x} ((c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x)$

Step 2 :

Find the Particular Integral: For the Differential Equation $f(D)y = R(x)$, by considering the Right Hand Side function $R(x)$, we find a particular solution which is also called as **particular Integral (P.I.)**.

The methods used for this are:

- (i) Inverse Operator method
- (ii) Method of Variation of Parameters

4.2 Inverse Operator method :

For a differential equation of the type $f(D)y = R(x)$, where $R(x)$ is a function of the form e^{ax} , $\sin(ax + b)$, $\cos(ax + b)$, x^n , The particular integral, *P.I* or y_p is given by

$$y_p = \frac{1}{f(D)}R(x)$$

and the general solution is

$$y = C.F. + P.I.$$

4.3 Type 1 : $R(x) = e^{ax}$:

If $R(x) = e^{ax}$ then

$$\begin{aligned} y_p &= \frac{1}{f(D)} \Big|_{D=a} e^{ax}, \text{ provided } f(D)|_{D=a} = f(a) \neq 0 \\ &= x \frac{1}{f'(D)} \Big|_{D=a} e^{ax}, \text{ provided } f(D)|_{D=a} = f(a) = 0, \text{ and } f'(D)|_{D=a} = f'(a) \neq 0 \\ &= x^2 \frac{1}{f''(D)} \Big|_{D=a} e^{ax}, \text{ provided } f'(D)|_{D=a} = f'(a) = 0, \text{ and } f''(D)|_{D=a} = f''(a) \neq 0 \\ &\vdots \end{aligned}$$

and so on

4.4 Type 2: $R(x) = \sin(ax + b)$ or $\cos(ax + b)$

If $R(x) = \sin(ax + b)$ or $\cos(ax + b)$ or $k_1 \sin(ax + b) + k_2 \cos(ax + b)$ then

$$\begin{aligned} y_p &= \frac{1}{f(D)} \Big|_{D^2=-a^2} \sin ax \text{ provided } f(D)|_{D^2=-a^2} \neq 0 \\ y_p &= x \frac{1}{f'(D)} \Big|_{D^2=-a^2} \sin ax, \text{ provided } f(D)|_{D^2=-a^2} = 0 \text{ and } f'(D)|_{D^2=-a^2} \neq 0 \\ y_p &= x^2 \frac{1}{f''(D)} \Big|_{D^2=-a^2} \sin ax \text{ provided } f'(D)|_{D^2=-a^2} = 0 \text{ and } f''(D)|_{D^2=-a^2} \neq 0 \\ &\vdots \end{aligned}$$

and so on.

4.5 Type 3 : If $R(x)$ is a polynomial

If $R(x)$ is a polynomial of degree n ,

(say, $R(x) = k_0 + k_1x + k_2x^2 + k_3x^3 + \dots + k_nx^n$) then

$y_p = \frac{1}{f(D)}R(x)$ Take out the lowest degree term(or constant term) from the Denominator and express the Denominator in the form $(1 + \psi(D))$.

Then y_p reduces to any one of the following forms.

- $[1 + \psi(D)]^{-1} R(x)$
- $[1 - \psi(D)]^{-1} R(x)$
- $[1 + \psi(D)]^{-2} R(x)$
- $[1 - \psi(D)]^{-2} R(x)$

These are expanded as far as the term containing D^n by using following binomial expansions, and then operated on $R(x)$ term by term.

- $(1 + z)^{-1} = 1 - z + z^2 - z^3 + \dots$
- $(1 - z)^{-1} = 1 + z + z^2 + z^3 + \dots$
- $(1 + z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \dots$
- $(1 - z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \dots$

Remember : $Df(x) = \frac{d}{dx}(f(x))$ and $\frac{1}{D}f(x) = \int f(x)dx$

Depending on the type of the function $R(x)$ we use the above three **rules** for finding the particular integral.

Note : The number of roots of A.E. is equal to the order of the D.E.

Example 4.5.1. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Solution :

The given Differential Equation in symbolic form is $(D^2 + D - 2)y = 0$

Auxiliary equation (A.E.) is $m^2 + m - 2 = 0$

Solving this we get $(m - 1)(m + 2) = 0 \implies m = 1, -2$ (real and different roots)

Hence the Solution is, $y = c_1e^x + c_2e^{-2x}$

Problem 1. Solve $(D^2 - 3D - 4)y = 0$

Solution : Auxiliary equation is $m^2 - 3m - 4 = 0$

or $(m - 4)(m + 1) = 0$

$m = 4, -1$ (real and distinct roots)

\therefore The solution is $y = c_1e^{4x} + c_2e^{-x}$.

Problem 2. Solve : $(D^2 + 2D + 2)y = 0$

Solution: A.E. is $m^2 + 2m + 2 = 0$

$$\begin{aligned} m &= \frac{-2 \pm \sqrt{4 - 8}}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= -1 \pm i = -1 + i, -1 - i \text{ (Complex roots in the form } a + ib, a - ib) \\ \text{C.F} &= e^{-x} [c_1 \cos x + c_2 \sin x] \end{aligned}$$

Problem 3. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

Solution : Given equation in the symbolic form is $(D^2 - 4D + 1)y = 0$

Auxiliary equation is $m^2 - 4m + 1 = 0$

$$\begin{aligned} m &= \frac{4 \pm \sqrt{16 - 4}}{2} \\ &= \frac{4 \pm 2\sqrt{3}}{2} \\ &= 2 \pm \sqrt{3} \\ &= 2 + \sqrt{3}, 2 - \sqrt{3} \end{aligned}$$

Note that The roots are real and distinct. (and not complex!)

Hence the general solution is $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

Problem 4. Solve: $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

Solution : Using the operator $D = \frac{d}{dx}$ the given DE can be written as $(D^3 + 6D^2 + 11D + 6)y = 0$

Auxiliary Equation(A.E.) is $m^3 + 6m^2 + 11m + 6 = 0$ (This cannot be factorized)

We shall find one root by inspection by taking values for $m = 1, -1, 2, -2$ etc. Since all the terms are positive we have to try only negative values for m .

Putting $m = -1$ we have

$$(-1)^3 + 6(-1)^2 + 11(-1) + 6 = -1 + 6 - 11 + 6 = 0$$

$\therefore m = -1$ is a root.

The other roots can be found through the process of synthetic division.

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

Remaining factor is, $m^2 + 5m + 6 = 0$.

Solving: $(m + 2)(m + 3) = 0 \Rightarrow m = -2, -3$

Hence, $m = -1, -2, -3$ are the roots of the A.E. which are all real and distinct.

Thus $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ is the general solution.

Problem 5. Solve : $(D^3 - 3D^2 + 3D - 1)y = 0$

Solution : Given : $(D^3 - 3D^2 + 3D - 1) y = 0$

The auxiliary equation is $m^3 - 3m^2 + 3m - 1 = 0$.

$$(m - 1)^3 = 0$$

$m = 1, 1, 1$ (Three roots are real and equal.)

The general solution is given by $y = [c_1 + c_2x + c_3x^2] e^x$

Problem 6. Solve : $(D^3 - 3D + 2) y = 0$

A.E. is $m^3 - 3m + 2 = 0$, i.e. $m^3 + 0m^2 - 3m + 2 = 0$

By inspection we can see that $m = 1$ is a root.

The other two roots can be found through synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & & & & \\ \hline & 0 & 1 & 1 & -2 \\ & & & & \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

Now, $m^2 + m - 2 = 0$ or $(m - 1)(m + 2) = 0 \Rightarrow m = 1, -2$

Hence, $m = 1, 1, -2$ are the roots of the A.E.

Thus $y = (c_1 + c_2x) e^x + c_3e^{-2x}$ is the general solution.

Problem 7. Solve : $\frac{d^3y}{dx^3} + y = 0$

Solution : We have $(D^3 + 1) y = 0$

AE is $m^3 + 1 = 0$,

$m = -1$ is a root by inspection.

The other two roots are found by synthetic division.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & & & \\ \hline & 0 & -1 & 1 & -1 \\ & & & & \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

Now, let us solve $m^2 - m + 1 = 0$

$$\therefore m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i^2}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Hence $m = -1, \frac{1}{2} \pm i \left(\frac{\sqrt{3}}{2} \right)$ are the roots of the AE.

Thus $y = c_1 e^{-x} + e^{\frac{x}{2}} \left[c_2 \cos \left(\frac{\sqrt{3}}{2} x \right) + c_3 \sin \left(\frac{\sqrt{3}}{2} x \right) \right]$

Problem 8. Solve: $(4D^4 - 4D^3 - 23D^2 + 12D + 36) y = 0$

Solution : A.E. is $4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0$

By inspection, if $m = 2$ then $64 - 32 - 92 + 24 + 36 = 124 - 124 = 0$

$\Rightarrow m = 2$ is a root by inspection.

Now by synthetic division,

$$2 \begin{array}{r|rrrrr} 4 & -4 & -23 & 12 & 36 & \\ 0 & 8 & 8 & -30 & -36 & \\ \hline 4 & 4 & -15 & -18 & 0 & \end{array}$$

Remaining factor is $4m^3 + 4m^2 - 15m - 18 = 0$

If $m = 2$, $32 + 16 - 30 - 18 = 48 - 48 = 0$

Hence $m = 2$ is a root.

Again by Synthetic division,

$$2 \begin{array}{r|rrrr} 4 & 4 & -15 & -18 & \\ 0 & 8 & 24 & 18 & \\ \hline 4 & 12 & 9 & 0 & \end{array}$$

Remaining factor is $4m^2 + 12m + 9 = 0$

or $(2m + 3)^2 = 0 \Rightarrow m = -\frac{3}{2}, -\frac{3}{2}$

Hence, roots of the A.E are $2, 2, -\frac{3}{2}, -\frac{3}{2}$

Thus $y = (c_1 + c_2x) e^{2x} + (c_3 + c_4x) e^{-\frac{3}{2}x}$ is the general solution.

Problem 9. Solve $(D^3 + D^2 + 4D + 4) = 0$.

Solution: Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$

$(m^2 + 4)(m + 1) = 0 \therefore m = -1, \pm 2i$.

i.e. $m = -1, 0 + 2i, 0 - 2i$

Hence the G.S. is $y = c_1 e^{-x} + e^{0x} (c_2 \cos 2x + c_3 \sin 2x)$

$y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$

Problem 10. Solve $:\frac{d^4 y}{dx^4} - y = 0$

Solution: Given equation in symbolic form is $(D^4 - 1) y = 0$

A.E. is $D^4 - 1 = 0$ or $m^4 - 1 = 0$

i.e. $(m^2 - 1)(m^2 + 1) = 0$

$\therefore m = \pm 1, \pm i$

i.e. $m = 1, -1, 0 + i, 0 - i$

Solution is

$$\begin{aligned} \text{C.F.} &= c_1 e^x + c_2 e^{-x} + e^{0x} (c_3 \cos x + c_4 \sin x) \\ &= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x \end{aligned}$$

Problem 11. Solve $:(D^3 - D^2 - D - 2) y = 0$

Solution : The given equation is $(D^3 - D^2 - D - 2) y = 0$

Auxiliary equation is $m^3 - m^2 - m - 2 = 0$

$m = 2$ satisfies this equation.

$\therefore m = 2$ is a root.

By synthetic division

$$2 \left| \begin{array}{cccc} 1 & -1 & -1 & -2 \\ 0 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 0 \end{array} \right.$$

∴ remaining factor is $(m^2 + m + 1) = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Hence roots of AE are $\Rightarrow m = 2, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

The general solution is

$$y = c_1 e^{2x} + e^{-\frac{1}{2}x} \left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x \right)$$

Example 4.5.2. Solve $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$ (VTU Jan 2017, July 2016, June 2012, June 2015, Jan 2008)

Solution :

A.E. is $4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0$

By inspection, we get $m = 2$ as one of the roots.

Using synthetic division, we obtain,

$$4m^3 - 7m - 3 = 0$$

$m = -1$ is a root of this equation.

2	4	-8	-7	11	6
	0	8	0	-14	-6
	4	0	-7	-3	0
-1	4	0	-7	-3	
	0	-4	4	3	
	4	-4	-3	0	

Using syn-

thetic division again, we get

$$4m^2 - 4m - 3 = 0$$

Solving this equation, we get the remaining two roots as $m = \frac{-1}{2}$ and $m = \frac{3}{2}$.

∴ roots of the A.E. are

$$m = 2, m = -1, m = \frac{-1}{2}, m = \frac{3}{2}$$

Hence the solution is

$$C.F. \text{ or } y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-\frac{1}{2}x} + c_4 e^{\frac{3x}{2}}$$

Problem 12. Solve : $\frac{d^4 y}{dx^4} + 8\frac{d^2 y}{dx^2} + 16y = 0$

Solution : Given equation in symbolic form is $(D^4 + 8D^2 + 16)y = 0$

Auxiliary equation is $m^4 + 8m^2 + 16 = 0$

We can't use inspection method as all m terms are with even power.

Let us denote $m^2 = p$ then the auxiliary equation can be written as

Hence the Complementary function is $y_c = c_1 e^{2x} + c_2 e^{-6x}$

The particular integral is

$$\begin{aligned} y_p &= \frac{1}{D^2 + 4D - 12} e^{3x} \\ &= \left[\frac{1}{D^2 + 4D - 12}_{D=3} \right] e^{3x} \\ &= \frac{1}{9} e^{3x} \end{aligned}$$

Hence the general Solution is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-6x} + \frac{1}{9} e^{3x}$$

Problem 14. Solve : $(D^3 - 6D^2 + 11D - 6)y = e^{2x}$

(VTU Jan 2017)

Solution: The given equation can be written as $(D^3 - 6D^2 + 11D - 6)y = 0$

The A.E is $m^3 - 6m^2 + 11m - 6 = 0$

Here, $m = 1$ is one of the roots.

$$m = 1 \left| \begin{array}{cccc} 1 & -6 & 11 & -6 \\ 0 & 1 & -5 & 6 \\ \hline 1 & -5 & 6 & 0 \end{array} \right.$$

Remaining equation is, $m^2 - 5m + 6 = 0$

$$\Rightarrow m^2 - 2m - 3m + 6 = 0$$

$$\Rightarrow m(m - 2) - 3(m - 2) = 0$$

$$\Rightarrow (m - 3)(m - 2) = 0$$

$$\Rightarrow m = 3, 2$$

$\Rightarrow m = 1, 2, 3$ are the roots of AE

Hence, C.F. $y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

$$\begin{aligned} y_p &= \frac{1}{(D^3 - 6D^2 + 11D - 6) |_{D=2}} e^{2x} \\ &= x \frac{1}{(3D^2 - 12D + 11) |_{D=2}} e^{2x} \quad (\text{Since the Denominator } = 0) \\ &= x \frac{1}{-1} e^{2x} = -e^{2x} \end{aligned}$$

$$\begin{aligned} \text{G.S., } y &= y_c + y_p \\ &= c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - e^{2x} \end{aligned}$$

Problem 15. Solve : $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x$

Solution : A.E. is $(m + 2)(m - 1)^2 = 0$

Roots of AE are $m = -2, 1, 1$ (Two equal and one distinct roots)

Hence C.F. is $y_c = c_1 e^{-2x} + (c_2 + c_3 x) e^x$

Particular integral is

$$\begin{aligned} y_p &= \frac{1}{(D+2)(D-1)^2} [e^{-2x} + 2 \sinh x] \\ &= \frac{1}{(D+2)(D-1)^2} [e^{-2x} + e^x - e^{-x}] \\ &= \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^x - \frac{1}{(D+2)(D-1)^2} e^{-x} \end{aligned}$$

Let us evaluate each of these terms separately.

$$\begin{aligned} y_{p1} &= \frac{1}{(D+2)(D-1)^2} \Big|_{D=-2} e^{-2x} \\ &= \frac{1}{D+2} \cdot \left[\frac{1}{(D-1)^2} \Big|_{D=-2} e^{-2x} \right] \\ &= \frac{1}{D+2} \cdot \frac{1}{(-2-1)^2} e^{-2x} \\ &= \frac{1}{9} \cdot \frac{1}{(D+2)} \Big|_{D=-2} e^{-2x} \\ &= \frac{1}{9} \cdot x \cdot \frac{1}{1} e^{-2x} \\ &= \frac{x}{9} e^{-2x} \end{aligned}$$

$$\begin{aligned} y_{p2} &= \frac{1}{(D+2)(D-1)^2} \Big|_{D=1} e^x \\ &= \frac{1}{1+2} \cdot \frac{1}{(D-1)^2} \Big|_{D=1} e^x \\ &= \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^x \\ &= \frac{x^2}{6} e^x \end{aligned}$$

$$\begin{aligned} y_{p3} &= \frac{1}{(D+2)(D-1)^2} \Big|_{D=-1} e^{-x} \\ &= \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{e^{-x}}{4} \end{aligned}$$

Hence, G.S. C.F.+ P.I. = $c_1 e^{-2x} + (c_2 + c_3 x) e^x + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$

Problem 16. Solve : $\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$ (VTU July/August 2005)

Solution : We have $(D^2 - 6D + 9) y = 6e^{3x} + 7e^{-2x} - \log 2$ AE is given by $m^2 - 6m + 9 = 0$ or $(m - 3)^2 = 0 \Rightarrow m = 3, 3 \therefore y_c = (c_1 + c_2 x) e^{3x}$

$$\begin{aligned} y_p &= \frac{1}{D^2 - 6D + 9} 6e^{3x} + \frac{1}{D^2 - 6D + 9} 7e^{-2x} - \frac{\log 2}{D^2 - 6D + 9} e^{0x} \\ &= p_1 + p_2 + p_3 \text{ (say)} \end{aligned}$$

$$\begin{aligned}
 p_1 &= \left[\frac{1}{(D^2 - 6D + 9) |_{D=3}} \right] 6e^{3x} \\
 &= \frac{1}{3^2 - 6 \times 3 + 9} 6e^{3x} \quad (\text{since denominator} = 0) \\
 &= 6x \left[\frac{1}{(2D - 6) |_{D=3}} \right] e^{3x} \\
 &= 6x \frac{1}{2 \cdot 3 - 6} e^{3x} \quad (\text{since denominator} = 0) \\
 p_1 &= 6x^2 \frac{e^{3x}}{2} \\
 &= 3x^2 e^{3x}
 \end{aligned}$$

$$\begin{aligned}
 p_2 &= \frac{1}{(D^2 - 6D + 9) |_{D=-2}} 7e^{-2x} \\
 &= \frac{1}{(-2)^2 - 6(-2) + 9} 7e^{-2x} \\
 &= \frac{7e^{-2x}}{25} p_3 = -\log 2 \left(\frac{1}{(D^2 - 6D + 9) |_{D=0}} \right) e^{0x} \\
 &= -\frac{\log 2}{0 - 0 + 9} \\
 &= -\frac{\log 2}{9}
 \end{aligned}$$

General solution : $y = y_c + p_1 + p_2 + p_3$

$$= (c_1 + c_2x) e^{3x} + 3x^2 e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$$

Problem 17. Solve : $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = e^x + 1$ (VTU June 2012, July 2014, June 2015, 2015 Model)

Solution : In symbolic form we can write the given equation as

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

$$\text{A.E. is } m^3 + 6m^2 + 11m + 6 = 0$$

By inspection we can see that $m = -1$ is a root.

By synthetic division, we get the remaining equation as $m^2 + 5m + 6 = 0$

Solving this, we get $m = -2, -3$

Hence the roots of the auxiliary equation are $m = -1, -2, -3$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

$$\begin{aligned}
 y_p &= \left[\frac{1}{D^3 + 6D^2 + 11D + 6} \right] (e^x + 1) \\
 &= \left[\frac{1}{(D^3 + 6D^2 + 11D + 6) |_{D=1}} \right] e^x + \left[\frac{1}{(D^3 + 6D^2 + 11D + 6) |_{D=0}} \right] e^{0x} \\
 &= \left[\frac{1}{1^3 + 6 + 11 + 6} \right] e^x + \left[\frac{1}{0 + 0 + 0 + 6} \right] e^{0x} \\
 &= \frac{1}{24} e^x + \frac{1}{6}
 \end{aligned}$$

$$\text{Hence } Y = C.F. + y_p = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} + \frac{e^x}{24} + \frac{1}{6}$$

Problem 18. $(D^2 + 7D + 12)y = \cosh x$

(VTU Jan 2016, Dec 2015)

Solution : The AE is $m^2 + 7m + 12 = 0$

$$m^2 + 3m + 4m + 12 = 0$$

$$m(m + 3) + 4(m + 3) = 0$$

$$(m + 4)(m + 3) = 0$$

Therefore, $m = -4, -3$ are the roots of AE

$$\text{C.F.} = c_1 e^{-4x} + c_2 e^{-3x}$$

$$\begin{aligned}
 \text{P.I.} &= \frac{\cosh x}{D^2 + 7D + 12} = \frac{1}{D^2 + 7D + 12} \left[\frac{1}{2} (e^x + e^{-x}) \right] \\
 &= \frac{1}{2} \left[\frac{e^x}{D^2 + 7D + 12} + \frac{e^{-x}}{D^2 + 7D + 12} \right] = \frac{1}{2} \left[\frac{e^x}{20} + \frac{e^{-x}}{6} \right]
 \end{aligned}$$

$$\text{P.I.} = \frac{1}{40} e^x + \frac{1}{12} e^{-x}$$

General solution is

$$y = y_c + y_p = c_1 e^{-4x} + c_2 e^{-3x} + \frac{1}{40} e^x + \frac{1}{12} e^{-x}$$

4.8 Particular Integral -Type 1 : $R(x) = e^{ax}$:

If $R(x) = e^{ax}$ then $y_p = \left[\frac{1}{f(D)} \right] e^{ax}$ is given by

$$\begin{aligned} y_p &= \left[\frac{1}{f(a)} \right] e^{ax}, \text{ provided } f(a) \neq 0 \\ &= x \left[\frac{1}{f'(a)} \right] e^{ax}, \text{ provided } f'(a) \neq 0 \\ &= x^2 \left[\frac{1}{f''(a)} \right] e^{ax} \text{ provided } f''(a) \neq 0 \end{aligned}$$

⋮

and so on

General Solution is $y = y_c + y_p$

Problem 19. Solve : $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = \sinh(2x + 3)$

(VTU Jan 2018)

Solution : AE is $m^3 - m^2 + 4m - 4 = 0$

i.e. $m^2(m - 1) + 4(m - 1) = 0$

$(m - 1)(m^2 + 4) = 0 \Rightarrow m = 1, \pm 2i$

Hence $y_c = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x$

$$\begin{aligned} y_p &= \frac{1}{(D^3 - D^2 + 4D - 4)} \sinh(2x + 3) \\ &= \frac{1}{2} \left[\frac{1}{(D^3 - D^2 + 4D - 4) |_{D=2}} e^{2x+3} \right. \\ &\quad \left. - \frac{1}{(D^3 - D^2 + 4D - 4) |_{D=-2}} e^{-(2x+3)} \right] \\ &= \frac{1}{2} \left[\frac{e^{2x+3}}{8 - 4 + 8 - 4} - \frac{e^{-(2x+3)}}{-8 - 4 - 8 - 4} \right] \\ &= \frac{1}{2} \left[\frac{e^{2x+3}}{8} - \frac{e^{-(2x+3)}}{-24} \right] \\ y_p &= \frac{1}{48} [3e^{2x+3} + e^{-(2x+3)}] \end{aligned}$$

General solution : $y = y_c + y_p$

Thus $y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{48} [3e^{2x+3} + e^{-(2x+3)}]$

Problem 20. Solve : $\frac{d^2y}{dx^2} + 4y = 2^{-x}$

(VTU June 2015)

Solution : AE is given by $m^2 + 4 = 0$ and hence $m = \pm 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

$$\begin{aligned}
 y_p &= \frac{1}{D^2 + 4} 2^{-x} \\
 &= \frac{1}{D^2 + 4} \left(e^{\log 2^{-x}} \right) \quad (\because e^{\log f(x)} = f(x)) \\
 &= \frac{1}{(D^2 + 4) |_{D=-\log 2}} e^{(-x \log 2)} \quad (\because \log(a^n) = n \log a) \\
 &= \frac{1}{(-\log 2)^2 + 4} e^{(-\log 2)x} = \frac{2^{-x}}{(\log 2)^2 + 4}
 \end{aligned}$$

General solution: $y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{2^{-x}}{(\log 2)^2 + 4}$

Problem 21. Solve : $\frac{d^2 y}{dx^2} - 4y = \cosh(2x - 1) + 3^x$ (VTU July 2017, July 2016, July 2014, Jan 2009, VTU 2015 Model, 2000)

Solution : We have $(D^2 - 4)y = \cosh(2x - 1) + 3^x$

AE is $m^2 - 4 = 0$ or $(m - 2)(m + 2) = 0 \Rightarrow m = 2, -2$

$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}$

(1)

$$\begin{aligned}
 y_p &= \left[\frac{1}{D^2 - 4} \right] (\cosh(2x - 1) + 3^x) \\
 &= \frac{1}{2} \left[\frac{1}{D^2 - 4} \right] e^{(2x-1)} + \frac{1}{2} \left[\frac{1}{D^2 - 4} \right] e^{-(2x-1)} + \left[\frac{1}{D^2 - 4} \right] 3^x \\
 &= p_1 + p_2 + p_3 \quad (\text{say})
 \end{aligned}$$

$$\begin{aligned}
 p_1 &= \frac{1}{2} \frac{1}{(D^2 - 4) |_{D=2}} e^{2x-1} \\
 &= \frac{1}{2} \frac{1}{2^2 - 4} e^{2x-1} \quad (\text{But, Denominator} = 0) \\
 &= \frac{1}{2} \frac{1}{2D} |_{D=2} e^{2x-1} \\
 &= \frac{1}{2} \frac{e^{2x-1}}{4} \\
 &= \frac{x}{8} e^{2x-1}
 \end{aligned}$$

$$\begin{aligned}
 p_2 &= \frac{1}{2} \frac{1}{(D^2 - 4) |_{D=-2}} e^{-(2x-1)} \\
 &= \frac{1}{2} x \frac{1}{2D} |_{D=-2} e^{-(2x-1)} \quad (\text{Since Denominator} = 0) \\
 &= \frac{1}{2} x \frac{e^{-(2x-1)}}{-4} = \frac{-x}{8} e^{-(2x-1)} \\
 p_3 &= \frac{1}{D^2 - 4} 3^x \\
 &= \frac{1}{D^2 - 4} (e^{\log 3^x}) \\
 &= \frac{1}{(D^2 - 4) |_{D=\log 3}} e^{(\log 3) x} \\
 &= \frac{e^{(\log 3)x}}{(\log 3)^2 - 4} = \frac{3^x}{(\log 3)^2 - 4} \\
 y_p &= p_1 + p_2 + p_3 \\
 &= \frac{x}{8} e^{2x-1} + \frac{-x}{8} e^{-(2x-1)} + \frac{3^x}{(\log 3)^2 - 4} \quad (2)
 \end{aligned}$$

General solution : $y = y_c + y_p$

where y_c and y_p are given by (1) and (2) respectively.

Problem 22. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2$

Solution : (i) To find C.F.

Its A.E. is $D^2 + D + 1 = 0, \Rightarrow D = \frac{1}{2}(-1 \pm \sqrt{3}i)$

Thus C.F. = $e^{\frac{-x}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x \right)$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 + D + 1} (1 - 2e^x + e^{2x}) \\
 &= \frac{1}{D^2 + D + 1} (e^{0x} - 2e^x + e^{2x}) \\
 &= \frac{1}{0^2 + 0 + 1} e^{0x} - 2 \times \frac{1}{1^2 + 1 + 1} e^x \\
 &\quad + \frac{1}{2^2 + 2 + 1} e^{2x} \\
 &= 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}
 \end{aligned}$$

General Solution is

$$y = e^{\frac{-x}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x \right) + 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}$$

4.9 Practice Problems:

1. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = e^{2x}$ (VTU July 2006)
 Ans : $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} + \frac{1}{60}e^{2x}$
2. $(D^3 + D^2 - 4D - 4)y = 3e^{-x} - 4x - 6$ (VTU July 2016)
3. $(D^3 - 6D^2 + 11D - 6)y = e^{2x}$ (VTU Jan 2017)
4. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x}$ (VTU Jan 2007)
 Ans : $y = (c_1 + c_2x + c_3x^2)e^{-x} + \frac{x^3}{6}e^{-x}$
5. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$ (VTU July/August 2005)
 Ans : $(c_1 + c_2x)e^{3x} + 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{\log 2}{9}$
6. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = e^x + 1$ (VTU June 2012, July 2014, June 2015, 2015 Model)
 Ans : $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} + \frac{e^x}{24} + \frac{1}{6}$
7. $\frac{d^3y}{dx^3} + y = e^x$ (VTU 2006)
 Ans : $y = c_1e^{-x} + e^{\frac{1}{2}x} \left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x \right) + \frac{1}{2}e^x$
8. $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 9x = 5e^{-2t}$ (VTU July 2013)
 Ans : $x = (c_1 + c_2t)e^{3t} + \frac{1}{5}e^{-2t}$
9. $(D^4 - 18D^2 + 81)y = 36e^{3x}$
 Ans : $(c_1 + c_2x)e^{3x} + (c_3 + c_4x)e^{-3x} + \frac{x^2e^{3x}}{2}$
10. $(D^2 + 7D + 12)y = \cosh x$ (VTU Jan 2016, Dec 2015)
 Ans : $y = c_1e^{-3x} + c_2e^{-4x} + \frac{1}{2} \left(\frac{e^x}{20} + \frac{e^{-x}}{6} \right)$
11. $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-x}$ (VTU Dec 2010)
 Ans : $c_1e^{-\frac{4}{3}x} + c_2e^{-\frac{3}{2}x} + e^{-x}$
12. $\frac{d^2y}{dx^2} + 4y = 2^{-x}$ (VTU June 2015)
 Ans : $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{(\log 2)^2 + 4} 2^{-x}$
13. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 4x = 2 \cosh 2t$ (VTU Jan 2006)
 Ans : $x = c_1e^{-t} + (c_2 + c_3t)e^{2t} + \frac{t^2e^{2t}}{6} - \frac{e^{-2t}}{16}$
14. $\frac{d^2y}{dx^2} - 4y = \cosh(2x - 1) + 3^x$ (VTU July 2017, July 2016, July 2014, Jan 2009, VTU 2015 Model, 2000)
 Ans : $y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4} \sinh(2x - 1) + \frac{3^x}{(\log 3)^2 - 4}$
15. $y'' + 4y' + 3y = e^{-x}$ with $y(0) = y'(0) = 1$, (VTU July 2013)
 Ans : $y = \frac{7}{4}e^{-x} - \frac{3}{4}e^{-3x} + \frac{x}{2}e^{-x}$
16. $y'' + 4y' + 5y + 2 \cosh x = 0$ with $y = 0, y' = 0$ at $x = 0$, (VTU July 2017, June 2012)
 Ans : $y = e^{2x} \left(\frac{3}{5} \cos x - \frac{8}{5} \sin x \right) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$

$$17. (D + 2)(D - 1)^2 y = e^{-2x} + 2\sinh x$$

$$\text{Ans : } y = (c_1 + c_2 x)e^x + c_3 e^{-2x} + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x - \frac{1}{4}e^{-x}$$

$$18. (D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$$

$$\text{Ans : } y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{120}(2e^{-2x} + e^{-3x})$$

$$19. \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2$$

$$\text{Ans : } y = e^{-\frac{x}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) + 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$$

$$20. y'' - 4y = \sinh^2 x$$

$$\text{Ans : } y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{16}(x e^{2x} - x e^{-2x} + 2)$$

$$21. \frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 4y = \sinh(2x + 3)$$

(VTU Jan 2018)

$$22. (D^3 - 6D^2 + 11D - 6)y = e^{2x}$$

(VTU Jan 2017)

$$23. (D^2 + 4D + 5)y = -2\cosh x$$

(VTU July 2017)

$$\text{Ans : } y = \frac{3}{5}e^{-2x}(\cos x + 3\sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}$$

$$24. \text{Solve : } (D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$$

$$\text{Ans : } y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{120}(2e^{-2x} + e^{-3x})$$

4.10 Particular Integral : Type 2 - $R(x) = \sin(ax+b)$ or $\cos(ax+b)$

If $R(x) = \sin(ax + b)$

$$y_p = \frac{1}{f(D)} \Big|_{D^2 = -a^2} \sin(ax + b) \text{ provided } f(-a^2) \neq 0$$

$$y_p = x \frac{1}{f'(D)} \Big|_{D^2 = -a^2} \sin(ax + b), \text{ provided } f'(-a^2) \neq 0$$

$$y_p = x^2 \frac{1}{f''(D)} \Big|_{D^2 = -a^2} \sin(ax + b) \text{ } f''(-a^2) \neq 0$$

∴ and so on.

We use the above rule for $R(x) = \cos(ax + b)$ or $R(x) = k_1 \sin(ax + b) + k_2 \cos(ax + b)$

Problem 23. Solve $(D^2 - 4D + 4)y = \sin 2x$

(VTU July 2011)

Solution: The A.E. is $m^2 - 4m + 4 = 0$

Solving this we get $(m - 2)(m - 2) = 0$

$\implies m = 2, 2$ (two real and equal roots)

Hence the Complementary function is

$$y_c = (c_1 + c_2 x)e^{2x}$$

The Particular integral is given by

$$y_p = \frac{1}{D^2 - 4D + 4} \sin 2x$$

$$= \left[\frac{1}{D^2 - 4D + 4} \right]_{D^2 = -(2^2) = -4} \sin 2x$$

Here we replace only D^2 term by $-(a^2)$ and keep D term as it is.

$$= \frac{1}{-4 - 4D + 4} \sin 2x$$

$$= \frac{1}{-4D} \sin 2x$$

$$= \frac{1}{-4} \frac{1}{D} \sin 2x$$

$$y_p = \frac{1}{-4} \int \sin 2x dx \quad (\because \frac{1}{D} \text{ stands for } \textit{integration})$$

$$= \frac{1}{-4} \left(\frac{-\cos 2x}{2} \right)$$

$$= \frac{1}{8} \cos 2x$$

Hence the General Solution is

$$y = y_c + y_p = (c_1 + c_2 x) e^{2x} + \frac{1}{8} \cos 2x$$

Problem 24. Solve $(D^2 + D + 1) y = \sin 2x$

Solution: Here the auxiliary equation is $m^2 + m + 1 = 0$ which gives $m = -\frac{1}{2} \pm i \left(\frac{1}{2} \sqrt{3} \right)$

$\therefore C.F. = e^{-\frac{x}{2}} \left\{ C_1 \cos \left(\frac{\sqrt{3}}{2} x \right) + C_2 \sin \left(\frac{\sqrt{3}}{2} x \right) \right\}$
 where C_1 and C_2 are arbitrary constants, and

$$P.I. = \frac{1}{D^2 + D + 1} \sin 2x$$

$$= \frac{1}{-2^2 + D + 1} \sin 2x \quad \text{replacing } D^2 \text{ by } -2^2$$

$$= \frac{1}{D - 3} \sin 2x$$

$$= \frac{1}{(D - 3)(D + 3)} (D + 3) \sin 2x$$

$$= \frac{1}{D^2 - 9} (D + 3) \sin 2x$$

$$= \frac{1}{-2^2 - 9} (D + 3) \sin 2x$$

$$= -\frac{1}{13} (D + 3) \sin 2x$$

$$\begin{aligned}
 y_p &= \frac{-1}{13} [D(\sin 2x) + 3 \sin 2x] \\
 &= -\frac{1}{13} [2 \cos 2x + 3 \sin 2x] \\
 &\quad (\text{Since } D \text{ means differentiation with respect to } x)
 \end{aligned}$$

∴ The complete solution is

$$\begin{aligned}
 y &= y_c + y_p \\
 &= e^{-\frac{x}{2}} \left[C_1 \cos \left(\frac{\sqrt{3}}{2} x \right) + C_2 \sin \left(\frac{\sqrt{3}}{2} x \right) \right] - \frac{1}{13} (2 \cos 2x + 3 \sin 2x)
 \end{aligned}$$

Problem 25. Solve $(D^2 - 5D + 6)y = \sin 3x$

Solution: Its auxiliary equation is $m^2 - 5m + 6 = 0$ which gives $m = 2, 3$
 ∴ C.F. = $C_1 e^{2x} + C_2 e^{3x}$, where C_1 and C_2 are arbitrary constants and

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 5D + 6} \sin 3x \\
 &= \frac{1}{-3^2 - 5D + 6} \sin 3x, \text{ replacing } D^2 \text{ by } -(3^2) \\
 &= \frac{1}{-(5D + 3)} \sin 3x \\
 &= \frac{-1}{(5D + 3)(5D - 3)} (5D - 3) \sin 3x \\
 &= \frac{-1}{(25D^2 - 9)} (5D - 3) \sin 3x \\
 &= \frac{-1}{\{25(-3^2) - 9\}} (5D - 3) \sin 3x \\
 &= \frac{1}{234} [5D(\sin 3x) - 3 \sin 3x] \\
 &= \frac{1}{234} [5 \times 3 \cos 3x - 3 \sin 3x] \\
 &= \frac{1}{78} (5 \cos 3x - \sin 3x)
 \end{aligned}$$

Hence the required solution is

$$y = y_c + y_p = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{78} (5 \cos 3x - \sin 3x)$$

Problem 26. Solve $(D^3 + D^2 - D - 1)y = \cos 2x$

Solution: Its auxiliary equation is

$$m^3 + m^2 - m - 1 = 0 \text{ or } (m^2 - 1)(m + 1) = 0$$

$$\Rightarrow m = 1, -1, -1$$

∴ C.F. = $C_1 e^x + (C_2 x + C_3) e^{-x}$, where C_1, C_2 and C_3 are arbitrary constants

and

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^3 + D^2 - D - 1} \cos 2x \\
 &= \frac{1}{D(D^2) + D^2 - D - 1} \cos 2x \\
 &= \frac{1}{D(-2^2) + (-2^2) - D - 1} \cos 2x \\
 &= -\frac{1}{5} \frac{1}{D + 1} \cos 2x \\
 &= -\frac{1}{5} \frac{D - 1}{(D - 1)(D + 1)} \cos 2x \\
 &= -\frac{1}{5} \frac{1}{D^2 - 1} (D - 1) \cos 2x \\
 &= -\frac{1}{5} \frac{1}{(-2^2 - 1)} [D(\cos 2x) - \cos 2x] \\
 \text{P.I.} &= \frac{1}{25} (-2 \sin 2x - \cos 2x)
 \end{aligned}$$

Therefore the general solution is

$$y = \text{C.F.} + \text{P.I} = C_1 e^x + (C_2 x + C_3) e^{-x} - \frac{1}{25} (2 \sin 2x + \cos 2x)$$

Problem 27. Solve $\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 2y = e^x + \cos x$

Solution: Using $D \equiv \frac{d}{dx}$, the given equation can be rewritten as $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$, Here the auxiliary equation is

$$\begin{aligned}
 m^3 - 3m^2 + 4m - 2 &= 0 \\
 (m - 1)(m^2 - 2m + 2) &= 0 \\
 \Rightarrow m &= 1, 1 \pm i
 \end{aligned}$$

$\therefore \text{C.F.} = C_1 e^x + e^x (C_2 \cos x + C_3 \sin x)$, where C_1, C_2 and C_3 are arbitrary constants.

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^3 - 3D^2 + 4D - 2} (e^x + \cos x) \\
 &= \frac{1}{(D^3 - 3D^2 + 4D - 2) |_{D=1}} e^x \\
 &\quad + \frac{1}{(D^3 - 3D^2 + 4D - 2) |_{D^2=-(1^2)}} \cos x \\
 &= x \frac{1}{(3D^2 - 6D + 4) |_{D=1}} e^x + \frac{1}{(- (1^2)D - 3(-1^2) + 4D - 2)} \cos x \\
 &= x \frac{1}{1} e^x + \frac{1}{3D + 1} \cos x \\
 &= x e^x + \frac{(3D - 1)}{9D^2 - 1} \cos x
 \end{aligned}$$

$$\begin{aligned}
&= xe^x + \frac{3D - 1}{9(-1^2) - 1} \cos x \\
&= xe^x - \frac{1}{10}(3D - 1) \cos x \\
&= xe^x - \frac{1}{10}(3D \cos x - \cos x) \\
&= xe^x - \frac{1}{10}(-3 \sin x - \cos x) \\
&= xe^x + \frac{1}{10}(3 \sin x + \cos x)
\end{aligned}$$

Hence, the required solution is

$$y = \text{C.F} + \text{P.I} = C_1 e^x + e^x (C_2 \cos x + C_3 \sin x) + xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Problem 28. Solve $(D^2 + 5D + 6)y = \cos x + e^{-2x}$.

Solution: The A.E. is $m^2 + 5m + 6 = 0$

i.e, $(m + 2)(m + 3) = 0 \Rightarrow m = -2, -3$

C.F. = $C_1 e^{-2x} + C_2 e^{-3x}$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 5D + 6} [\cos x + e^{-2x}] \\
&= \frac{1}{D^2 + 5D + 6} \cos x + \frac{1}{D^2 + 5D + 6} e^{-2x} \\
&= P_1 + P_2 \text{ (say)}
\end{aligned}$$

$$\begin{aligned}
P_1 &= \frac{1}{(D^2 + 5D + 6) |_{D^2 = -(1^2)}} \cos x \\
&= \frac{1}{-1 + 5D + 6} \cos x \\
&= \frac{1}{5D + 5} \cos x \\
&= \frac{1}{5} \frac{(D - 1)}{(D + 1)(D - 1)} \cos x \\
&= \frac{1}{5} \frac{(D - 1) \cos x}{D^2 - 1} \\
&= \frac{1 - \sin x - \cos x}{5(-1^2 - 1)} \\
&= \frac{-1 \sin x + \cos x}{5(-2)} = \frac{1}{10}(\sin x + \cos x)
\end{aligned}$$

$$\begin{aligned}
 P_2 &= \frac{1}{(D^2 + 5D + 6) |_{D=-2}} e^{-2x} \\
 &= \frac{1}{(-2)^2 + 5 \times -2 + 6} e^{-2x} \quad (\text{Denominator} = 0) \\
 &= x \frac{1}{(2D + 5) |_{D=-2}} e^{-2x} \\
 &= x \frac{1}{2(-2) + 5} e^{-2x} \\
 &= \frac{1}{1} x e^{-2x} \\
 &= x e^{-2x}
 \end{aligned}$$

$$P.I. = p_1 + p_2 = \frac{1}{10}(\sin x + \cos x) + x e^{-2x}$$

∴ The general solution is

$$y = C.F. + P.I = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{10}(\sin x + \cos x) + x e^{-2x}$$

Problem 29. Solve: $y'' - 4y' + 13y = \cos 2x$

Solution : We have $(D^2 - 4D + 13)y = \cos 2x$

AE is $m^2 - 4m + 13 = 0$ and by solving,

$$\begin{aligned}
 m &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} \\
 &= \frac{4 \pm \sqrt{-36}}{2} \\
 &= \frac{4 \pm \sqrt{36i^2}}{2} \\
 &= \frac{4 \pm 6i}{2} = 2 \pm 3i
 \end{aligned}$$

$$\therefore y_c = e^{2x} (c_1 \cos 3x + c_2 \sin 3x)$$

Here $a = 2$ and hence replace D^2 by $-a^2 = -4$

$$y_p = \frac{1}{(-4 - 4D + 13)} \cos 2x = \frac{\cos 2x}{9 - 4D}$$

Now multiply and divide by $(9 + 4D)$

$$\begin{aligned}
 &= \frac{(9 + 4D)}{(9 + 4D)(9 - 4D)} \cos 2x \\
 &= \frac{9 \cos 2x + 4D(\cos 2x)}{81 - 16D^2} \\
 &= \frac{9 \cos 2x - 8 \sin 2x}{81 - 16(-4)} \\
 &= \frac{9 \cos 2x - 8 \sin 2x}{145}
 \end{aligned}$$

General solution: $y = y_c + y_p$

$$\text{Thus } y = e^{2x} (c_1 \cos 3x + c_2 \sin 3x) + \frac{9 \cos 2x - 8 \sin 2x}{145}$$

Problem 30. $(D^3 - 1)y = 3 \cos 2x$

(VTU Jan 2020, June 2012, 2010)

Solution : AE is $(m^3 - 1) = 0$ or $(m - 1)(m^2 + m + 1) = 0$
 $m = 1$; $m^2 + m + 1 = 0$ and by solving this equation,

$$m = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

$$= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$y_c = c_1 e^x + e^{-x/2} \left\{ c_2 \cos(\sqrt{3}x/2) + c_3 \sin(\sqrt{3}x/2) \right\}$$

$$y_p = \frac{1}{D^3 - 1} 3 \cos 2x$$

$$= \frac{1}{(D^2 D) - 1} 3 \cos 2x \text{ Now } D^2 \rightarrow -2^2 = -4$$

$$= \frac{1}{-4D - 1} 3 \cos 2x$$

$$= \frac{3(4D - 1) \cos 2x}{-(4D + 1)(4D - 1)}$$

$$= \frac{3(-8 \sin 2x - \cos 2x)}{-(16D^2 - 1)}$$

$$= \frac{-3(8 \sin 2x + \cos 2x)}{65}$$

General solution is

$$y = y_c + y_p$$

$$= c_1 e^x + e^{-x/2} \left\{ c_2 \cos(\sqrt{3}x/2) + c_3 \sin(\sqrt{3}x/2) \right\}$$

$$+ \frac{-3(8 \sin 2x + \cos 2x)}{65}$$

Problem 31. $(D - 2)^2 y = 8(e^{2x} + \sin 2x)$ (VTU Jan 2017, Jun 2012, Dec 2011, Dec 2014, Jun 2014)

Solution: AE is $(m - 2)^2 = 0$ $m = 2, 2$

Therefore, $y_c = (c_1 + c_2 x) e^{2x}$

$$\text{P.I.} = \frac{1}{(D - 2)^2} 8e^{2x} + \frac{1}{(D - 2)^2} 8 \sin 2x = p_1 + P_2 \text{ (say)}$$

$$\begin{aligned}
 p_1 &= \frac{1}{(D-2)^2} 8e^{2x} \\
 &= x \frac{1}{2(D-2)} 8e^{2x} \\
 &= x^2 \frac{8e^{2x}}{2} \\
 &= 4x^2 e^{2x}
 \end{aligned}$$

$$\begin{aligned}
 p_2 &= \frac{1}{(D-2)^2} 8 \sin 2x \\
 &= \frac{1}{D^2 - 4D + 4} 8 \sin 2x \\
 &= \frac{1}{-2^2 - 4D + 4} 8 \sin 2x \\
 &= \frac{1}{-4D} 8 \sin 2x \\
 &= -2 \int \sin 2x dx \\
 &= -2 \left(\frac{-\cos 2x}{2} \right) \\
 &= \cos 2x
 \end{aligned}$$

$$\text{P.I.} = p_1 + p_2 = 4x^2 e^{2x} + \cos 2x$$

Hence, the general solution is

$$y = (c_1 + c_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x$$

Problem 32. $y'' + 4y' - 12y = e^{2x} - 3 \sin 2x$

(VTU Jan 2016, Dec.2015)

Solution: We have $(D^2 + 4D - 12)y = e^{2x} - 3 \sin 2x$

AE is $m^2 + 4m - 12 = 0$ or $(m+6)(m-2) = 0 \Rightarrow m = 2, -6$

$\therefore y_c = c_1 e^{2x} + c_2 e^{-6x}$

$$\begin{aligned}
 y_p &= \frac{1}{D^2 + 4D - 12} e^{2x} - \frac{1}{D^2 + 4D - 12} 3 \sin 2x \\
 &= p_1 - p_2 \text{ (say)} \\
 p_1 &= \frac{1}{D^2 + 4D - 12} e^{2x} \\
 &= \frac{1}{4 + 8 - 12} e^{2x} \quad (\text{Denominator} = 0) \\
 &= x \frac{1}{2D + 4} e^{2x} \\
 &= x \frac{e^{2x}}{8} \\
 &= \frac{x e^{2x}}{8}
 \end{aligned}$$

$$\begin{aligned}
 p_2 &= \frac{1}{D^2 + 4D - 12} \Big|_{D^2 \rightarrow -4} 3 \sin 2x; \\
 &= \frac{3 \sin 2x}{-4 + 4D - 12} \\
 &= \frac{1}{4(D - 4)} 3 \sin 2x \\
 &= \frac{3(D + 4) \sin 2x}{4(D^2 - 16)} \\
 &= \frac{3(2 \cos 2x + 4 \sin 2x)}{-80}
 \end{aligned}$$

General solution: $y = y_c + y_p$ where $y_p = p_1 - p_2$

$$y = c_1 e^{2x} + c_2 e^{-6x} + \frac{x e^{2x}}{8} + \frac{3}{40} (\cos 2x + 2 \sin 2x)$$

Problem 33. Solve : $\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = 4 \cos^2 x$

Solution : We have $(D^2 + 3D + 2) y = 4 \cos^2 x$

AE is $m^2 + 3m + 2 = 0$

or $(m + 1)(m + 2) = 0 \Rightarrow m = -1, -2$

$\therefore y_c = c_1 e^{-x} + c_2 e^{-2x}$

We have $4 \cos^2 x = 2(1 + \cos 2x)$

$$\begin{aligned}
 y_p &= \frac{2}{D^2 + 3D + 2} + \frac{1}{D^2 + 3D + 2} 2 \cos 2x \\
 &= p_1 + p_2 \text{ (say)} \\
 p_1 &= \frac{1}{D^2 + 3D + 2} 2e^{0x} \\
 &= \frac{2e^{0x}}{0^2 + 3 \cdot 0 + 2} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 p_2 &= \frac{1}{D^2 + 3D + 2} 2 \cos 2x \\
 &= 2 \left[\frac{1}{-4 + 3D + 2} \right] \cos 2x \\
 &= 2 \left[\frac{1}{3D - 2} \right] \cos 2x \\
 &= \frac{2(3D + 2)(\cos 2x)}{(3D + 2)(3D - 2)} \\
 &= \frac{2(-6 \sin 2x + 2 \cos 2x)}{9D^2 - 4} \\
 &= \frac{4(-3 \sin 2x + \cos 2x)}{9(-4) - 4} \\
 &= \frac{4(-3 \sin 2x + \cos 2x)}{-40}
 \end{aligned}$$

\therefore

$$p_2 = \frac{3 \sin 2x - \cos 2x}{10}$$

Complete solution: $y = y_c + y_p$ where $y_p = p_1 + p_2$

$$\text{Thus } y = c_1 e^{-x} + c_2 e^{-2x} + 1 + \frac{3 \sin 2x - \cos 2x}{10}$$

Problem 34. $y'' - 4y' + 4y = e^{2x} + \cos 2x + 4$ (VTU July 2008, Feb 2005, Jan 2015)

Solution :

We have $(D^2 - 4D + 4)y = e^{2x} + \cos 2x + 4$

AE is given by $m^2 - 4m + 4 = 0$ or $(m - 2)^2 = 0 \Rightarrow m = 2, 2$

$$\therefore y_c = (c_1 + c_2 x) e^{2x}$$

$$\begin{aligned}
 y_p &= \frac{e^{2x}}{D^2 - 4D + 4} + \frac{\cos 2x}{D^2 - 4D + 4} + \frac{4}{D^2 - 4D + 4} \\
 &= p_1 + p_2 + p_3 \text{ (say)}
 \end{aligned}$$

Now,

$$p_1 = \frac{e^{2x}}{D^2 - 4D + 4} = \frac{e^{2x}}{2^2 - 4(2) + 4} \quad (Dr. = 0)$$

$$= x \cdot \frac{e^{2x}}{2D - 4} = x \cdot \frac{e^{2x}}{2(2) - 4} \quad (Dr. = 0)$$

$$p_1 = x^2 \frac{e^{2x}}{2}$$

$$\begin{aligned}
 p_2 &= \frac{\cos 2x}{D^2 - 4D + 4} \text{ Replace } D^2 \text{ by } -4 \\
 p_2 &= \frac{\cos 2x}{-4 - 4D + 4} \\
 &= \frac{\cos 2x}{-4D} \\
 &= -\frac{1}{4} \int \cos 2x dx \\
 &= \frac{-\sin 2x}{8}
 \end{aligned}$$

$$\begin{aligned}
 p_3 &= \frac{4e^{0x}}{D^2 - 4D + 4} \\
 &= \frac{4e^{0x}}{0 - 0 + 4} \\
 &= \frac{4}{4} = 1
 \end{aligned}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2 + p_3$ Thus

$$y = (c_1 + c_2x) e^{2x} + \frac{x^2 e^{2x}}{2} - \frac{\sin 2x}{8} + 1$$

Problem 35. $(D^3 - D)y = 2e^x + 4\cos x$ (VTU June 2014, July 2011)

Solution : The AE is $m^3 - m = 0$

$$\begin{aligned}
 m(m^2 - 1) &= 0 \\
 m(m+1)(m-1) &= 0 \\
 m &= 0, -1, 1
 \end{aligned}$$

Therefore, C.F. = $c_1 e^{0x} + c_2 e^{-x} + c_3 e^x = c_1 + c_2 e^{-x} + c_3 e^x$

$$y_p = x e^x - 2 \sin x \text{ (prove this !)}$$

Hence, the general solution is

$$y = c_1 + c_2 e^{-x} + c_3 e^x + x e^x - 2 \sin x$$

Problem 36. $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$

Solution : The given equation can be written as $(D^2 + 2D + 1)y = e^{2x} - \cos^2 x$

The AE is $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

Therefore. CF = $(c_1 + c_2x) e^{-x}$

$$\begin{aligned}
\text{PI.} &= \frac{1}{D^2 + 2D + 1} (e^{2x} - \cos^2 x) \\
&= \frac{1}{D^2 + 2D + 1} \left(e^{2x} - \frac{1}{2}(1 + \cos 2x) \right) \\
&= \frac{1}{D^2 + 2D + 1} - \frac{1}{2} \frac{1}{(D^2 + 2D + 1)} e^{2x} - \frac{1}{2} \frac{1}{(D^2 + 2D + 1)} \cos 2x \\
&= \frac{1}{9} e^{2x} - \frac{1}{2} \frac{1}{[0 + 2(0) + 1]} - \frac{1}{2} \frac{\cos 2x}{[-2^2 + 2D + 1]} \\
&= \frac{1}{9} e^{2x} - \frac{1}{2} - \frac{1}{2} \frac{(2D + 3)}{(2D - 3)(2D + 3)} \cos 2x \\
&= \frac{1}{9} e^{2x} - \frac{1}{2} - \frac{1}{2} \frac{(2D + 3)}{(4D^2 - 9)} \cos 2x \\
&= \frac{1}{9} e^{2x} - \frac{1}{2} - \frac{1}{2} \frac{(2D + 3)}{[4(-2^2) - 9]} \cos 2x \\
&= \frac{1}{9} e^{2x} - \frac{1}{2} + \frac{1}{50} (-4 \sin 2x + 3 \cos 2x)
\end{aligned}$$

Hence, the general solution is

$$y = (c_1 + c_2 x) e^{-x} + \frac{1}{9} e^{2x} - \frac{1}{2} + \frac{1}{50} (3 \cos 2x - 4 \sin 2x)$$

4.11 Practice Problems

Solve the following Differential Equations.

1. $\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$ (VTU Jan 2017, July 2016, Dec 2011, 2004)

2. $y'' - 4y' + 4y = e^{2x} + \sin x$ (VTU July 2019, Jan 2013)

3. $\frac{d^3 y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$ (VTU Jan 2010)

Ans : $y = c_1 e^{0x} + (c_2 \cos 2x + c_3 \sin 2x) - \frac{x \sin 2x}{8}$

4. $\frac{d^2 x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$

Ans : $x = e^{-t}(c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t) + \frac{1}{4}(\sin t - \cos t)$

5. $(D^3 - 1)y = 3 \cos 2x$ (VTU Dec June 2012, 2010)

Ans : $y = c_1 e^x + e^{-\frac{1}{2}x}(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x) - \frac{3}{65}(8 \sin 2x + \cos 2x)$

6. $y'' - 3y' + 2y = 2 \sin x \cos x$ (VTU Jan 2014)

Ans : $y = (c_1 e^x + c_2 e^{2x} + \frac{1}{40}(6 \cos 2x - 2 \sin 2x))$

7. $(D^3 + 1)y = \cos(2x - 1)$ (VTU Jan 2013)

Ans : $y = c_1 e^{-x} + e^{-\frac{x}{2}} \left(c_2 \cos \left(\frac{\sqrt{3}}{2} \right) x + c_3 \sin \left(\frac{\sqrt{3}}{2} \right) x \right) + \frac{1}{65} [\cos(2x - 1) - 8 \sin(2x - 1)]$

8. $(D^3 - 1)y = 3\cos 2x$ (VTU Dec 2010, June 2012)
9. $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = \sin 2x$ (VTU 2013, Jan 2007)
 Ans : $y = (c_1e^{-x} + c_2e^{2x} + \frac{1}{40}(2\cos 2x - 6\sin 2x))$
10. $(D^2 + 9)y = \cos 2x \cos x$ (VTU Dec 2010, July 2007)
 Ans : $y = c_1\cos 3x + c_2\sin 3x + \frac{1}{16}\cos x + x\frac{1}{12}\sin x$
11. $(D^2 + D + 1)y = \sin 2x$
 Ans : $y = e^{-\frac{1}{2}x} \left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x \right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$
12. $y'' + 4y' + 4y = 3\sin x + 4\cos x, y(0) = 1, y'(0) = 0$
 Ans : $y = (1 + x)e^{-2x} + \sin x$
13. $y'' - 4y' + 4y = e^{2x} + \sin x$ (VTU July 2019, Jan 2013)
 Ans : $y = (c_1 + c_2x)e^{2x} + \frac{1}{2}x^2e^{2x} - \frac{\sin}{19} - \frac{\cos x}{19}$
14. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$
 Ans : $y = c_1e^{-x} + c_2e^{-2x} + 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$
15. $(D^2 - 1)y = \cos 3x + \sin x$
 Ans : $y = c_1e^x + c_2e^{-x} - \frac{1}{10}\cos 3x - \frac{1}{2}\sin x + \frac{13}{232}(15\cos 3x - 3\sin 3x)$
16. $y'' - 5y' + 6y = 13\sin 3x$
 Ans : $y = c_1e^{2x} + c_2e^{3x} - 195/234\cos 3x - 39/234\sin 3x$
17. $(D^3 + 1)y = 65\cos(2x + 1)$
 Ans : $y = c_1e^{-x} + e^{x/2}(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x) + \cos(2x + 1) - 8\sin(2x + 1)$
18. $(D^2 + 9)y = \cos 2x \cos x$
 Ans : $y = c_1\cos 3x + c_2\sin 3x + \frac{1}{16}\cos x + x\frac{1}{12}\sin 3x$
19. $y'' + y = \sin x \sin 2x$
 Ans : $y = c_1\cos x + c_2\sin x + \frac{1}{16}(4x\sin x + \cos 3x)$
20. $D^2(D^2 + 4)(D^2 + 9)y = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$
 Ans : $y = c_1 + c_2x + c_3\cos 2x + c_4\sin 2x + c_5\cos 3x + c_6\sin 3x + \frac{1}{24}\sin x$

4.12 Type 3 : If $R(x)$ is a polynomial

If $R(x)$ is a polynomial of degree n ,

(say, $R(x) = k_0 + k_1x + k_2x^2 + k_3x^3 + \dots + k_nx^n$) then

$y_p = \frac{1}{f(D)}R(x)$ Take out the lowest degree term(or constant term) from the Denominator and express the Denominator in the form $(1 + \psi(D))$.

Then y_p reduces to any one of the following forms.

- $[1 + \psi(D)]^{-1} R(x)$
- $[1 - \psi(D)]^{-1} R(x)$

- $[1 + \psi(D)]^{-2} R(x)$
- $[1 - \psi(D)]^{-2} R(x)$

These are expanded as far as the term containing D^n by using following binomial expansions, and then operated on $R(x)$ term by term.

- $(1 + z)^{-1} = 1 - z + z^2 - z^3 + \dots$
- $(1 - z)^{-1} = 1 + z + z^2 + z^3 + \dots$
- $(1 + z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \dots$
- $(1 - z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \dots$

Example 4.12.1. Solve $y'' + 3y' + 2y = 12x^2$

(VTU Dec 2010)

Solution : In symbolic form the given Differential Equation can be written as $(D^2 + 3D + 2)y = 12x^2$

A.E. is $m^2 + 3m + 2 = 0$

Solving this we get $(m + 1)(m + 2) = 0 \implies m = -1, -2$

Hence the Complementary Function is

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

The Particular Integral is

$$\begin{aligned} y_p &= \frac{1}{D^2 + 3D + 2} 12x^2 \\ &= \frac{1}{2 \left(\frac{D^2}{2} + \frac{3D}{2} \right) + 1} 12x^2 \\ &= \frac{1}{2} \left(\frac{D^2}{2} + \frac{3D}{2} + 1 \right)^{-1} 12x^2 \\ &= \frac{1}{2} \left[1 + \left(\frac{D^2}{2} + \frac{3D}{2} \right) \right]^{-1} 12x^2 \end{aligned}$$

$$\begin{aligned} y_p &= \frac{1}{2} \left[1 - \left(\frac{D^2}{2} + \frac{3D}{2} \right) + \left(\frac{D^2}{2} + \frac{3D}{2} \right)^2 - \dots \right] 12x^2 \\ &= \frac{1}{2} \left[1 - \frac{D^2}{2} - \frac{3D}{2} + \left(\frac{D^2}{2} \right)^2 + \left(\frac{3D}{2} \right)^2 + 2 \left(\frac{D^2}{2} \right) \left(\frac{3D}{2} \right) - \dots \right] 12x^2 \\ &= \frac{1}{2} \left[1 - \frac{D^2}{2} - \frac{3D}{2} + \left(\frac{3D}{2} \right)^2 \right] 12x^2 \end{aligned}$$

($\because R(x) = 12x^2$ is a polynomial of degree 2, we can neglect terms with degree more than 2)

$$\begin{aligned} y_p &= \frac{1}{2} \left[1(12x^2) - \frac{D^2(12x^2)}{2} - \frac{3D(12x^2)}{2} + \frac{9D^2(12x^2)}{4} \right] \\ &= \frac{1}{2} \left[(12x^2) - \frac{24}{2} - \frac{3(24x)}{2} + \frac{9(24)}{4} \right] \\ &= 6x^2 - 18x + 21 \end{aligned}$$

Hence the General Solution is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 6x^2 - 18x + 21$$

Problem 37. $y'' + 3y' + 2y = 1 + 3x + x^2$ (VTU July 2017, July 2016, Dec 2011, Jan 2008)

Solution: The given equation is : $(D^2 + 3D + 2)y = 1 + 3x + x^2$

The AE is $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

Therefore, C.F. = $c_1 e^{-x} + c_2 e^{-2x}$

$$\begin{aligned} \text{P.I.} &= \frac{x^2 + 3x + 1}{D^2 + 3D + 2} \\ &= \frac{1}{2 \left(1 + \frac{D^2 + 3D}{2}\right)} (x^2 + 3x + 1) \\ &= \frac{1}{2} \left(1 + \frac{D^2 + 3D}{2}\right)^{-1} (x^2 + 3x + 1) \\ y_p &= \frac{1}{2} \left(1 - \frac{D^2 + 3D}{2} + \left(\frac{D^2 + 3D}{2}\right)^2 - + \dots\right) (x^2 + 3x + 1) \\ &= \frac{1}{2} \left(1 - \frac{D^2}{2} - \frac{3D}{2} + \frac{1}{4} (D^4 + 9D^2 + 6D^3) - + \dots\right) (x^2 + 3x + 1) \\ &= \frac{1}{2} \left(x^2 + 3x + 1 - \frac{1}{2}(2) - \frac{3}{2}(2x + 3) + \frac{1}{4}(0 + 9(2) + 0)\right) \\ &= \frac{1}{2} \left(x^2 + 3x + 1 - 1 - 3x - \frac{9}{2} + \frac{9}{2}\right) \\ \text{PI} &= \frac{1}{2} x^2 \end{aligned}$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2} x^2$$

Problem 38. Solve $\frac{d^2 y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ (VTU July 2013, Jan 2013)

Solution : Given equation in symbolic form is $(D^2 + D)y = x^2 + 2x + 4$

AE is $m^2 + m = 0$

$\Rightarrow m = 0, -1$

Hence $y_c = c_1 e^{0x} + c_2 e^{-x}$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D(D+1)} (x^2 + 2x + 4) \\
 &= \frac{1}{D} (1+D)^{-1} (x^2 + 2x + 4) \\
 &= \frac{1}{D} (1 - D + D^2 - \dots) (x^2 + 2x + 4) \\
 &= \frac{1}{D} [x^2 + 2x + 4 - (2x + 2) + 2] \\
 &= \int (x^2 + 4) dx \\
 &= \frac{x^3}{3} + 4x
 \end{aligned}$$

General Solution is $y = y_c + y_p = c_1 e^{0x} + c_2 e^{-x} + \frac{x^3}{3} + 4x$

Problem 39. Solve $(D^3 + 8)y = x^4 + 2x + 1$

Solution: The A.E. is $m^3 + 8 = 0$

$$\begin{aligned}
 m^3 + 2^3 &= 0 \\
 (m + 2)(m^2 - 2m + 2^2) &= 0 \\
 m = -2 \text{ or } m^2 - 2m + 4 &= 0 \\
 m &= \frac{2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} = \frac{2 \pm \sqrt{-12}}{2} = -\frac{2 \pm 2\sqrt{3}i}{2} \\
 m &= 1 \pm \sqrt{3}i
 \end{aligned}$$

Therefore, $m = -2, 1 \pm \sqrt{3}i$ are the roots of A.E

Hence, $C.F = c_1 e^{-2x} + e^x (c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^3 + 8} (x^4 + 2x + 1) \\
 &= \frac{1}{8 \left(1 + \frac{D^3}{8}\right)} (x^4 + 2x + 1) \\
 &= \frac{1}{8} \left(1 + \underbrace{\frac{D^3}{8}}\right)^{-1} (x^4 + 2x + 1) \\
 &= \frac{1}{8} \left[1 - \frac{D^3}{8} + \left(\frac{D^3}{8}\right)^2 \dots\right] (x^4 + 2x + 1)
 \end{aligned}$$

Thus, the general solution is

$$\begin{aligned}
 y_p &= \frac{1}{8} \left[x^4 + 2x + 1 - \frac{1}{8}(24x) \right] = \frac{1}{8} [x^4 + 2x + 1 - 3x] \\
 \text{P.I.} &= \frac{1}{8} (x^4 - x + 1)
 \end{aligned}$$

$$y = y_c + y_p = c_1 e^{-2x} + e^x \left(c_2 \cos \sqrt{3x} + c_3 \sin \sqrt{3x} \right) + \frac{1}{8} (x^4 - x + 1)$$

Problem 40. Find the P.I. of $(D^2 + 5D + 4)y = x^2 + 7x + 9$

Solution :

$$\begin{aligned} \text{Sol. P.I.} &= \frac{1}{D^2 + 5D + 4} (x^2 + 7x + 9) \\ &= \frac{1}{4 \left(1 + \frac{5D}{4} + \frac{D^2}{4} \right)} (x^2 + 7x + 9) \\ &= \frac{1}{4} \left[1 + \left(\frac{5D}{4} + \frac{D^2}{4} \right) \right]^{-1} (x^2 + 7x + 9) \\ y_p &= \frac{1}{4} \left[1 - \left(\frac{5D}{4} + \frac{D^2}{4} \right) + \left(\frac{5D}{4} + \frac{D^2}{4} \right)^2 - \dots \right] (x^2 + 7x + 9) \\ &= \frac{1}{4} \left(1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} \dots \right) (x^2 + 7x + 9) \\ &= \frac{1}{4} \left(1 - \frac{5D}{4} + \frac{21D^2}{16} \dots \right) (x^2 + 7x + 9) \\ &= \frac{1}{4} \left[(x^2 + 7x + 9) - \frac{5}{4} D(x^2 + 7x + 9) + \frac{21}{16} D^2(x^2 + 7x + 9) \right] \\ &= \frac{1}{4} \left[(x^2 + 7x + 9) - \frac{5}{4} (2x + 7) + \frac{21}{16} (2) \right] \\ &= \frac{1}{4} \left(x^2 + \frac{9}{2}x + \frac{23}{8} \right) \end{aligned}$$

Problem 41. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$

(VTU Jan 2006)

Solution : We have $(D^2 - 6D + 25)y = e^{2x} + \sin x + x$

AE is given by $m^2 - 6m + 25 = 0$

On solving, $m = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

$\therefore y_c = e^{3x} (c_1 \cos 4x + c_2 \sin 4x)$

$y_p = \frac{1}{D^2 - 6D + 25} e^{2x} + \frac{1}{D^2 - 6D + 25} \sin x + \frac{1}{D^2 - 6D + 25} x$
 $= p_1 + p_2 + p_3$ (say)

$$\begin{aligned}
 p_1 &= \frac{1}{D^2 - 6D + 25} e^{2x} \\
 &= \frac{1}{2^2 - 6(2) + 25} e^{2x} \\
 &= \frac{e^{2x}}{17} \\
 p_2 &= \frac{1}{D^2 - 6D + 25} \sin x \quad \text{Replace } D^2 \text{ by } -1^2 = -1 \\
 &= \frac{1}{24 - 6D} \sin x \\
 &= \frac{1}{6(4 - D)} \sin x \\
 &= \frac{(4 + D) \sin x}{6(16 - D^2)} \\
 &= \frac{4 \sin x + \cos x}{6(17)} = \frac{4 \sin x + \cos x}{102} \\
 p_3 &= \frac{1}{D^2 - 6D + 25} x \\
 &= \frac{1}{25 \left(\frac{D^2}{25} - \frac{6D}{25} + 1 \right)} x \\
 &= \frac{1}{25} \left[1 + \left(\frac{D^2}{25} - \frac{6D}{25} \right) \right]^{-1} x \\
 &= \frac{1}{25} \left[1 - \left(\frac{D^2}{25} - \frac{6D}{25} \right) \right] x \\
 &= \frac{x}{25} + \frac{6}{625} \\
 &= \frac{1}{625} (25x + 6)
 \end{aligned}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2 + p_3$

Thus

$$y = e^{3x} (c_1 \cos 4x + c_2 \sin 4x) + \frac{e^{2x}}{17} + \frac{4 \sin x + \cos x}{102} + \frac{1}{625} (25x + 6)$$

Problem 42. $(D - 2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

(VTU 2012, Dec 2011)

Solution: (i) To find CF.

Its A.E. is $(D - 2)^2 = 0$,

$\therefore D = 2, 2$.

Thus CF, = $(c_1 + c_2 x) e^{2x}$

To find P.I.

$$\begin{aligned} \text{P.I.} &= 8 \left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2 \right] \\ &= p_1 + p_2 + p_3 \text{ (say)} \end{aligned}$$

$$\begin{aligned} p_1 &= \frac{1}{(D-2)^2} e^{2x} \\ &= x \frac{1}{2(D-2)} e^{2x} \\ &= -x^2 \frac{1}{2(1)} e^{2x} \\ &= \frac{x^2}{2} e^{2x} \end{aligned}$$

$$\begin{aligned} p_2 &= \frac{1}{(D-2)^2} \sin 2x \\ &= \frac{1}{D^2 - 4D + 4} \sin 2x \\ &= \frac{1}{(-2^2) - 4D + 4} \sin 2x \\ &= -\frac{1}{4} \int \sin 2x dx \\ &= -\frac{1}{4} \left(\frac{-\cos 2x}{2} \right) \\ &= \frac{1}{8} \cos 2x \end{aligned}$$

$$\begin{aligned} p_3 &= \frac{1}{(D-2)^2} x^2 \\ &= \frac{1}{4} \left(1 - \frac{D}{2} \right)^{-2} x^2 \\ &= \frac{1}{4} \left[1 + (-2) \left(\frac{D}{2} \right) + \frac{(-2)(-3)}{2!} \left(-\frac{D}{2} \right)^2 + \dots \right] x^2 \\ &= \frac{1}{4} \left(1 + D + \frac{3D^2}{4} + \dots \right) x^2 \\ &= \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right) \end{aligned}$$

Thus P.I. = $4x^2 e^{2x} + \cos 2x + \frac{1}{8}(2x^2 + 4x + 3)$ Hence the G.S, is

$$y = y_c + y_p = (c_1 + c_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Pratice Problems

- $y'' + 4y' - 12y = e^{2x} - 3\sin 2x$ (VTU Jan 2016, Dec.2015)
 Ans : $y = c_1 e^{2x} + c_2 e^{-6x} + \frac{x e^{2x}}{8} + \frac{3}{40}(\cos 2x + 2\sin 2x)$
- $(D^3 - 6D^2 + 11D - 6)y = 2^x + \cos 2x$ (VTU July 2013)
 Ans :
 $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + \frac{2^x}{(\log 2)^3 - 6(\log 2)^2 + 11(\log 2) - 6} + \frac{1}{520}(14\sin 2x + 18\cos 2x)$
- $\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$ (VTU Dec 2011, Jan 2009)
 Ans : $y = c_1 + (c_2 + c_3 x)e^{-x} - \frac{1}{2}x^2 e^{-x} + \frac{1}{50}(3\cos 2x - 4\sin 2x)$
- $\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$ (VTU Jan 2006) Ans :
 $e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + y_p$
- $(D - 2)^2 y = 8(e^{2x} + \sin 2x)$ (VTU Jan 2017, Jun 2012, Dec 2011, Dec 2014, Jun 2014)
 Ans : $y = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x$
- $(D - 2)^2 y = 8(e^{2x} + x + x^2)$ (VTU Jan 2017)
- $y'' - 4y' + 4y = e^{2x} + \cos x + 4$ (VTU Jan 2013) Ans :
 $y = (c_1 + c_2 x)e^{2x} + \frac{1}{2}x^2 e^{2x} - \frac{\cos x}{19} + \frac{\sin x}{19} + 1$
- $y'' - 4y' + 4y = e^{2x} + \cos 2x + 4$ (VTU July 2008, Feb 2005, Jan 2015)
 Ans : $y = (c_1 + c_2 x)e^{2x} + \frac{1}{2}x^2 e^{2x} - \frac{\sin 2x}{8} + 1$
- $(D^3 - D)y = 2e^x + 4\cos x$ (VTU June 2014, July 2011) Ans :
 $y = c_1 + c_2 e^x + c_3 e^{-x} + x e^x - 2\sin x$
- $\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$ (VTU Jan 2017, July 2016, Dec 2011, 2004)
 Ans : $y = c_1 + (c_2 + c_3 x)e^{-x} - \frac{x^2}{2}e^{-x} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$
- $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$ Ans : $y = (c_1 + c_2 x)e^{-x} + \frac{1}{2} + \frac{1}{5}(2\sin 2x + \cos 2x)$
- $(D^2 + 5D + 6)y = \cos x + e^{-2x}$ Ans :
 $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{10}(\sin x + \cos x) + x e^{-2x}$
- $y'' + 2y' + y = e^{2x} - \cos 2x$ $y = (c_1 + c_2 x)e^{-x} + \frac{1}{9}e^{2x} - \frac{1}{2} - \frac{1}{50}(4\sin 2x - 3\cos 2x)$
- $\frac{d^2 y}{dx^2} + 4y = x^2 + e^{-x}$ (VTU June 2012, June 2015) Ans :
 $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} - \frac{1}{8} + \frac{1}{5}e^{-x}$
- $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$ Ans :
 $y = c_1 + c_2 e^x + c_3 e^{-x} + x e^x - (x^2 + x) - 2\sin x$
- $\frac{d^2 y}{dx^2} + 4y = x^2 + 2^{-x} + \log 2$ (VTU Jan 2018)

17. $(D - 2)^2 y = 8(e^{2x} + x + x^2)$ (VTU Jan 2017)
18. $y''' - 6y'' + 11y' - 6y = 1 + x + \sin x$ (VTU July 2017)
19. $\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x} + \sin x$ (VTU Jan 2006) **Ans :**
 $y = c_1 e^{2x} + c_2 e^{3x} - x e^{2x} + \frac{1}{10}(\sin x + \cos x)$
20. $y'' - y' - 2y = x + \sin x$ (VTU 2015 Model) **Ans :**
 $y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4} - \frac{1}{2}x + \frac{1}{10}(\cos x - 3\sin x)$
21. $\frac{d^2 y}{dx^2} + y = 2\cos x$ (VTU July 2016, Dec 2015, June 2012) **Ans :**
 $y = c_1 \cos x + c_2 \sin x + x \sin x$
22. $y'' + 4y = e^{-x} + x^2$ (VTU July 2017, JULY 2013, June 2012) **Ans :**
 $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{-x}}{5} + \frac{1}{4}x^2 - \frac{1}{8}$
23. $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$ (VTU Jan 2018, Jan 2017, Jan 2013, Jan 2008)
Ans : $y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$
24. $y'' - 4y' + 3y = 20\cos x,$ (VTU July 2011) **Ans :**
 $c_1 e^x + c_2 e^{3x} + \frac{1}{20}(40\cos x - 80\sin x)$
25. $y'' - 5y' + 6y = e^{3x} + x$ (VTU Jan 2013) **Ans :** $y = c_1 e^{2x} + c_2 e^{3x} + x e^{3x} + \frac{x}{6} + \frac{5}{36}$
26. $y'' - 5y' + 6y = 2e^x + 4\cos 2x$ (VTU July 2016)
27. $y'' - 5y' + 6y = e^{3x}$ (VTU July 2008) **Ans :** $y = c_1 e^{2x} + c_2 e^{3x} + x e^{3x}$
28. $y'' - 5y' + 6y = x$ (VTU July 2007) **Ans :** $y = c_1 e^{2x} + c_2 e^{3x} + \frac{x}{6} + \frac{5}{36}$
29. $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 4y = 3e^{-x}$ (VTU Jan 2013) **Ans :**
 $y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + e^{-x}$
30. $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2$ (VTU JULY 2008) **Ans :** $y = c_1 e^x + c_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$
31. $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + e^x$ (VTU July 2017, July 2016, Dec 2010) **Ans :**
 $y = c_1 e^x + c_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4} - x e^x$
32. $y'' + 2y' + 3y = x^2 - \cos x$ (VTU July 2009)
Ans : $y = e^{-x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + \frac{x^2}{3} - \frac{4}{9}x + \frac{2}{27} + \frac{1}{4}\sin x + \frac{1}{4}\cos x$
33. $y'' + 2y' + 3y = x^2 + \cos x$ (VTU Feb 2005)
Ans : $y = e^{-x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + \frac{x^2}{3} - \frac{4}{9}x + \frac{2}{27} - \frac{1}{4}\sin x - \frac{1}{4}\cos x$
34. $(D^2 - D - 2)y = 1 - 2x - 9e^{-x}$ (VTU June 2010)
35. $\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = 1 - 2x$ (VTU Jan 2007) **Ans :** $c_1 e^{-x} + c_2 e^{2x} + x - 1$
36. $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$ (VTU JULY 2017)
37. $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 2x + x^2$ (VTU July 2011) **Ans :** $y = (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$

38. $(D^2 - 5D + 1)y = 1 + x^2$ (VTU July 2009) **Ans :**
 $y = c_1 e^{\left(\frac{5}{2} + \frac{\sqrt{21}}{2}\right)x} + c_2 e^{\left(\frac{5}{2} - \frac{\sqrt{21}}{2}\right)x} + x^2 + 10x + 49$
39. $y'' + 3y' + 2y = 1 + 3x + x^2$ (VTU July 2017, July 2016, Dec 2011, Jan 2008) **Ans :**
 $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x^2}{2}$
40. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2 + 2x + 6$ (VTU Feb 2005) **Ans :**
 $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x + \frac{x^2}{4} + \frac{11}{8}$
41. $y'' + 4y' + 4y = 8x^2$ Given $y(0) = 1, y'(0) = 2$ (VTU July 2016)
42. $(D^2 + D - 2)y = 1 - x + x^2$ **Ans :** $y = c_1 e^x + c_2 e^{-2x} - \frac{x^2}{2} - 1$
43. $(D^2 + 3D + 2)y = 2x^2 + 4x + 1$ **Ans :** $y = c_1 e^{-x} + c - 2e^{-2x} + x^2 - x + 1$
44. $(D - 2)^2 y = 8x^2$ **Ans :** $y = (c_1 + c_2 x)e^{2x} + 2x^2 + 4x + 3$

4.13 Method of Variation of Parameters

To solve a second order Linear differential equation, under the method of variation of parameters, we first find Complementary Function, which takes the form $y_c = c_1 y_1 + c_2 y_2$.

Then we find a term called Wronskian(W), which is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Then the P.I. is

$$y_p = A(x)y_1 + B(x)y_2$$

where $A(x)$ and $B(x)$ are given by,

$$A(x) = - \int \frac{R(x) y_2}{W} dx$$

$$B(x) = \int \frac{R(x) y_1}{W} dx$$

General Solution is $y = y_c + y_p$.

Example 4.13.1. Solve $\frac{d^2y}{dx^2} + y = \tan x$ by the method of Variation of Parameters (VTU July 2013, July 2009, Jan 2009, Dec 2010, 2015 Model)

Solution : The symbolic form of the given equation is $(D^2 + 1)y = \tan x$

A. E. is $m^2 + 1 = 0 \implies m = \pm i$ (one pair of imaginary roots)

Hence the Complementary Function is $y_c = c_1 \cos x + c_2 \sin x$

This is in the form $c_1 y_1 + c_2 y_2$.

$$\therefore y_1 = \cos x, y_2 = \sin x$$

The Wronskian is given by

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x = 1 \end{aligned}$$

The Particular integral is $y_p = A(x)y_1 + B(x)y_2$, where

$$\begin{aligned} A(x) &= - \int \frac{R(x) y_2}{W} dx \\ &= - \int \tan x \sin x dx \\ &= - \int \frac{\sin x}{\cos x} \sin x dx \\ &= - \int \frac{\sin^2 x}{\cos x} dx \\ &= - \int \frac{1 - \cos^2 x}{\cos x} dx \\ &= - \int \left(\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \right) dx \\ A(x) &= - \int (\sec x - \cos x) dx \\ &= - [\log(\sec x + \tan x) - \sin x] \\ &= \sin x - \log(\sec x + \tan x) \\ B(x) &= \int \frac{R(x) y_1}{W} dx \\ &= \int \tan x \cos x dx \\ &= \int \frac{\sin x}{\cos x} \cos x dx \\ &= \int \sin x dx = -\cos x \end{aligned}$$

Hence

$$\begin{aligned} y_p &= A(x)y_1 + B(x)y_2 \\ &= (\sin x - \log(\sec x + \tan x)) \cos x + (-\cos x) \sin x \\ &= \sin x \cos x - \cos x \log(\sec x + \tan x) - \sin x \cos x \\ &= -\cos x \log(\sec x + \tan x) \end{aligned}$$

The general Solution is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$$

Problem 43. $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ (VTU Jan 2017, July 2017, Dec 2014, June 2011, Dec 2011)

Solution: A.E. is $D^2 - 6D + 9 = 0$

i.e., $(D - 3)^2 = 0$

i.e., $D = 3, 3$

C.F. = $(c_1 + c_2x)e^{3x} = c_1e^{3x} + c_2xe^{3x} = c_1y_1 + c_2y_2$, where $y_1 = e^{3x}$, $y_2 = xe^{3x}$ and

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (1+3x)e^{3x} \end{vmatrix} \\ &= e^{6x} \end{aligned}$$

$$R(x) = \frac{e^{3x}}{x^2}$$

$$\text{P.I.} = A(x)y_1 + B(x)y_2,$$

$$\text{where } A(x) = - \int \frac{y_2 R(x)}{W} dx$$

$$\text{and } B(x) = \int \frac{y_1 R(x)}{W} dx$$

$$\begin{aligned} A(x) &= - \int \frac{xe^{3x} \frac{e^{3x}}{x^2}}{e^{6x}} dx \\ &= - \int \frac{xe^{3x} \cdot e^{3x}}{e^{6x} x^2} dx \\ &= - \int \frac{xe^{3x} \cdot e^{3x}}{e^{6x} \cdot x^2} dx \\ &= \int \frac{-1}{x} dx = -\log x \end{aligned}$$

$$\begin{aligned} B(x) &= \int \frac{e^{3x} \cdot e^{3x}}{e^{6x} \cdot x^2} dx \\ &= \int \frac{1}{x^2} dx = -\frac{1}{x} \end{aligned}$$

$$\text{P.I.} = A(x)y_1 + B(x)y_2,$$

$$= -e^{3x} \log x + xe^{3x} \left(-\frac{1}{x} \right)$$

$$= -e^{3x}(1 + \log x)$$

C.S. is :

$$y = (c_1 + c_2x)e^{3x} - e^{3x}(1 + \log x)$$

Problem 44. Solve by the method of variation of parameters

$$y'' + a^2y = \sec ax$$

Solution : The AE is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$ Therefore,

$$\text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

This is in the form $c_1y_1 + c_2y_2$

where $y_1 = \cos ax$, $y_2 = \sin ax$

Here

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} \\ &= a(\cos^2 ax + \sin^2 ax) \\ &= a \end{aligned}$$

$P.I. = A(x) \cos ax + B(x) \sin ax$ where

$$\begin{aligned} A(x) &= - \int \frac{y_2 R(x)}{w} dx \\ &= - \int \frac{\sin ax \sec ax}{a} dx \\ &= - \frac{1}{a} \int \tan ax dx \\ &= - \frac{1}{a} \log(\sec ax) \\ &= - \frac{1}{a^2} \log(\sec ax) \end{aligned}$$

$$\begin{aligned} B(x) &= \int \frac{y_1 R(x)}{w} dx \\ &= \int \frac{\cos ax \sec ax}{a} dx \\ &= \frac{1}{a} \int dx \\ &= \frac{x}{a} \end{aligned}$$

Therefore,

$$P.I. = -\frac{1}{a^2} \log(\sec ax) \cos ax + \frac{x}{a} \sin ax$$

$$G.S.y = y_c + y_p = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log(\sec ax) + \frac{x}{a} \sin ax$$

Problem 45. Solve by the method of variation of parameters

$$y'' + y = \operatorname{cosec} x$$

Solution : The A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Therefore,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

Here $y_1 = \cos x$ $y_2 = \sin x$

$$\begin{aligned} w &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x = 1 \end{aligned}$$

$$\text{P.I.} = A \cos x + B \sin x$$

$$\begin{aligned} A &= - \int \frac{y_2 R(x)}{w} dx \\ &= - \int \frac{\sin x \csc x}{1} dx \\ &= - \int 1 dx = -x \end{aligned}$$

$$\begin{aligned} B &= \int \frac{y_1 R(x)}{w} dx \\ &= \int \frac{\cos x \csc x}{1} dx \\ &= \int \cos x \cdot \frac{1}{\sin x} dx \\ &= \int \cot x dx = \log(\sin x) \end{aligned}$$

$$y_p = -x \cos x + \sin x \log(\sin x)$$

$$\text{G.S.}, y = y_c + y_p = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log(\sin x)$$

Problem 46. Solve : $\frac{d^2 y}{dx^2} + y = \sec x \tan x$

Solution : The A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Therefore,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

Here $y_1 = \cos x$ $y_2 = \sin x$

$$\begin{aligned} w &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x = 1 \end{aligned}$$

$$\begin{aligned}
 A &= - \int \frac{y_2 R(x)}{w} dx \\
 &= - \int \frac{\sin x \sec x \tan x}{1} dx \\
 &= - \int \tan^2 x dx \\
 &= - \int (\sec^2 x - 1) dx \\
 &= -[\tan x - x] \\
 &= x - \tan x \\
 B &= \int \frac{y_1 R(x)}{w} dx \\
 &= \int \frac{\cos x \sec x \tan x}{1} dx \\
 &= \int \tan x dx \\
 &= \log(\sec x)
 \end{aligned}$$

Therefore,

$$\text{P.I.} = (x - \tan x) \cos x + \log(\sec x) \cdot \sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + (x - \tan x) \cos x + \sin x \log(\sec x)$$

Problem 47. Solve: $\frac{d^2 y}{dx^2} + 4y = \tan 2x$ (VTU Jan 2020, Jan 2018, July 2017, Jan 2016, June 2012, Dec 2011, Dec 2014, July 2008)

Solution : A.E. is $D^2 + 4 = 0$, $\therefore D = \pm 2i$

Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $R(x) = \tan 2x$

\therefore

$$\begin{aligned}
 W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
 &= \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} \\
 &= 2
 \end{aligned}$$

$$\text{P.I.} = A(x)y_1 + B(x)y_2$$

$$\begin{aligned} A(x) &= - \int \frac{y_2 R(x)}{W} dx \\ &= - \int \frac{\sin 2x \tan 2x}{2} dx \\ &= - \frac{1}{2} \int \frac{\sin 2x \sin 2x}{\cos 2x} dx \\ &= - \frac{1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx \\ &= - \frac{1}{2} \int (\sec 2x - \cos 2x) dx \\ &= - \frac{1}{2} \frac{\log(\sec 2x + \tan 2x)}{2} + \frac{\sin 2x}{2} \end{aligned}$$

$$B(x) = \int \frac{y_1 R(x)}{W} dx$$

$$\begin{aligned} B(x) &= \int \frac{\cos 2x \tan 2x}{2} dx \\ &= \frac{1}{2} \int \sin 2x dx \\ &= - \frac{1}{4} \cos 2x \end{aligned}$$

$$y_p = A(x)y_1 + B(x)y_2$$

$$= - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$$

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$$

Problem 48. Solve by the method of variation of parameters : $y'' - 2y' + y = \frac{e^x}{x}$ (VTU July 2016, Jan 2013, July 2007)

Solution :

$$\text{A.E. is } m^2 - 2m + 1 = 0 \Rightarrow (m - 1)^2 = 0$$

$m = 1, 1$ are the roots of A . E

$$\text{C.F.} = (C_1 + C_2x) e^x = c_1 e^x + c_2 x e^x$$

Here, $y_1 = e^x$, $y_2 = x e^x$

$$\therefore y_1' = e^x, y_2' = (x + 1)e^x$$

$$W = y_1 y_2' - y_2 y_1' = e^{2x}$$

$$\text{Also } R(x) = \frac{e^x}{x}$$

$$\text{Further, we have } A(x) = \int \frac{-y_2 R(x)}{W} dx \text{ and } B(x) = \int \frac{y_1 R(x)}{W} dx$$

$$A = \frac{-xe^x \frac{e^x}{x}}{e^{2x}} dx$$

$$= \int -dx = -x$$

$$B(x) = \int \frac{e^x \frac{e^x}{x}}{e^{2x}} dx$$

$$= \int \frac{1}{x} dx$$

$$= \log x$$

$$y_p = (-x) e^x + (\log x) x e^x$$

$$GS y = y_c + y_p = (C_1 + C_2 x) e^x + (-x) e^x + (\log x) x e^x$$

Problem 49. Solve: $y'' - 2y' + y = e^x \log x$

(VTU July 2006)

Solution : To find C.F. Its A.E. is $(D - 1)^2 = 0$, $\therefore D = 1, 1$

Thus C.F. is $y = (c_1 + c_2 x) e^x$

To find P.I. Here $y_1 = e^x$, $y_2 = x e^x$ and $X = e^x \log x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x e^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x}$$

$$A(x) = - \int \frac{R(x) y_2}{W} dx$$

$$= - \int \frac{x e^x \cdot e^x \log x}{e^{2x}} dx$$

$$= - \int x \log x dx$$

Integrate by parts by taking $\log x$ as first function,

$$A(x) = - \left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right)$$

$$= - \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right)$$

$$B(x) = \int \frac{R(x) y_1}{W} dx$$

$$= \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx$$

$$= \int \log x dx$$

$$= (x \log x - x)$$

$$y_p = A(x) y_1 + B(x) y_2$$

$$= \frac{1}{4} x^2 e^x (2 \log x - 3)$$

Hence G.S. is

$$y = y_c + y_p = (c_1 + c_2x) e^x + \frac{1}{4}x^2 e^x (2 \log x - 3)$$

Problem 50. Solve: $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

(VTU July 2016, July 2005)

Solution : The A.E is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

Therefore,

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

Here $y_1 = e^x$ and $y_2 = e^{-x}$

$$\begin{aligned} w &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x \cdot e^{-x} - e^x e^{-x} = -2 \end{aligned}$$

$$\text{P.I.} = Ae^x + Be^{-x}$$

$$\begin{aligned} A &= - \int \frac{y_2 R(x)}{w} dx \\ &= - \int \frac{e^{-x} \left(\frac{2}{1+e^x} \right)}{-2} dx \\ &= \int \frac{e^{-x}}{1+e^x} dx \\ &= \int \frac{e^{-x}}{e^x(e^{-x}+1)} dx \\ &= \int \frac{e^{-x} \cdot e^{-x}}{e^{-x}+1} dx \quad \text{Put } e^{-x} + 1 = t \Rightarrow e^{-x} dx = -dt \end{aligned}$$

$$\begin{aligned} A &= \int \frac{(t-1)(-dt)}{t} \\ &= - \int \left(1 - \frac{1}{t} \right) dt \\ &= \int \left(\frac{1}{t} - 1 \right) dt \\ &= \log t - t \\ &= \log(e^{-x} + 1) - (e^{-x} + 1) \end{aligned}$$

$$\begin{aligned} B &= \int \frac{y_1 R(x)}{w} dx \\ &= \int \frac{e^x \left(\frac{2}{e^x+1} \right)}{-2} dx \\ &= - \int \frac{e^x}{e^x+1} dx \\ &= - \log(e^x + 1) \end{aligned}$$

Therefore,

$$\begin{aligned} \text{P.I.} &= [\log(e^{-x} + 1) - (e^{-x} + 1)] e^x - \log(e^x + 1) e^{-x} \\ y &= y_c + y_p \\ &= c_1 e^x + c_2 e^{-x} + e^x \log(e^{-x} + 1) - 1 - e^x - e^{-x} \log(e^x + 1) \end{aligned}$$

Problem 51. Solve: $y'' - 2y' + 2y = e^x \tan x$ (VTU Jan 2010, June 2015)

Solution : The A.E is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Therefore,

$$\text{C.F.} = e^x (c_1 \cos x + c_2 \sin x)$$

Hence $y_1 = e^x \cos x$, $y_2 = e^x \sin x$

$$\begin{aligned} w &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(-\sin x + \cos x) & e^x(\sin x + \cos x) \end{vmatrix} \\ &= e^{2x} (\cos x \sin x + \cos^2 x) - e^{2x} (-\sin^2 x + \sin x \cos x) \\ &= e^{2x} (\cos x \sin x + \cos^2 x + \sin^2 x - \sin x \cos x) \\ &= e^{2x} \end{aligned}$$

P.I. = $Ae^x \cos x + Be^x \sin x$ where

$$\begin{aligned} A &= - \int \frac{R(x)y_2}{w} dx \\ &= - \int \frac{e^x \tan x \times e^x \sin x}{e^{2x}} dx \\ &= - \int \sin x \cdot \frac{\sin x}{\cos x} dx \\ &= - \int \frac{\sin^2 x}{\cos x} dx \\ &= - \int \frac{1 - \cos^2 x}{\cos x} dx \\ &= - \int (\sec x - \cos x) dx \\ &= -[\log(\sec x + \tan x) - \sin x] = \sin x - \log(\sec x + \tan x) \end{aligned}$$

$$\begin{aligned}
 B &= \int \frac{y_1 R(x)}{w} dx \\
 &= \int \frac{e^x \cos x \times e^x \tan x}{e^{2x}} dx \\
 &= \int \sin x dx = -\cos x
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \text{P.I.} &= [\sin x - \log(\sec x + \tan x)]e^x \cos x + (-\cos x)e^x \sin x \\
 &= e^x [\sin x \cos x - \cos x \log(\sec x + \tan x) - \sin x \cos x]
 \end{aligned}$$

$$\text{P.I.} = -e^x \cos x \log(\sec x + \tan x)$$

Hence G.S. is

$$y = y_c + y_p = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$$

Problem 52. Solve: $\frac{d^2 y}{dx^2} + y = \frac{1}{1+\sin x}$ (VTU Model 2018, June 2015)

Solution: The AE is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Therefore,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

Here, $y_1 = \cos x$, $y_2 = \sin x$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

P.I. = $A \cos x + B \sin x$ where

$$\begin{aligned}
 A &= - \int \frac{y_2 R(x)}{w} dx \\
 &= - \int \frac{\sin x \left(\frac{1}{1+\sin x} \right)}{1} dx \\
 &= - \int \frac{\sin x}{1 + \sin x} dx \\
 &= - \int \sin x \frac{(1 - \sin x)}{1 - \sin^2 x} dx \\
 &= - \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx \\
 &= - \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx \\
 &= - \int (\tan x \sec x - \tan^2 x) dx
 \end{aligned}$$

$$\begin{aligned}
 A &= - \int [\sec x \tan x - (\sec^2 x - 1)] dx \\
 &= -[\sec x - (\tan x - x)] = \tan x - \sec x - x \\
 B &= \int \frac{y_1 R(x)}{w} dx \\
 &= \int \frac{\cos x \left(\frac{1}{1+\sin x} \right)}{1} dx \\
 &= \int \frac{\cos x}{1 + \sin x} dx
 \end{aligned}$$

Put $1 + \sin x = t \Rightarrow \cos x dx = dt$

$$B = \int \frac{dt}{t} = \log t = \log(1 + \sin x)$$

Therefore,

$$\text{P.I.} = (\tan x - \sec x - x) \cos x + \log(1 + \sin x) \sin x$$

Hence, the general solution

$$y = c_1 \cos x + c_2 \sin x + \cos x(\tan x - \sec x - x) + \sin x \log(1 + \cos x)$$

Problem 53. Solve: $y'' + y = x \sin x$

(VTU Jan 2016)

Solution : The A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Therefore,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = A \cos x + B \sin x$$

Here $y_1 = \cos x$, $y_2 = \sin x$

$$\begin{aligned}
 w &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
 &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\
 &= \cos^2 x + \sin^2 x = 1
 \end{aligned}$$

$$\begin{aligned}
 A &= - \int \frac{y_2 R(x)}{w} dx \\
 &= - \int \frac{\sin x \cdot x \sin x}{1} dx \\
 &= - \int x \sin^2 x dx \\
 &= - \int x \frac{(1 - \cos 2x)}{2} dx \\
 &= - \frac{1}{2} \int (x - x \cos 2x) dx \\
 &= - \frac{1}{2} \left[\frac{x^2}{2} - \left\{ x \left(\frac{\sin 2x}{2} \right) - (1) \left(\frac{-\cos 2x}{2^2} \right) \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 A &= -\frac{x^2}{4} + \frac{1}{2} \left\{ \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \right\} \\
 &= -\frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x \\
 B &= \int \frac{R(x)y_1}{W} dx \\
 &= \int \frac{x \sin x \cos x}{1} dx \\
 &= \int \frac{x \sin 2x}{2} dx \\
 &= \frac{1}{2} \left[x \frac{-\cos 2x}{2} - \frac{-\sin 2x}{4} \right] \\
 &= \left(-\frac{x}{4} \cos 2x + \frac{1}{8} \sin 2x \right)
 \end{aligned}$$

Hence, the general solution is

$$\begin{aligned}
 \text{P.I.} &= \left(-\frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x \right) \cos x \\
 &\quad + \left(-\frac{x}{4} \cos 2x + \frac{1}{8} \sin 2x \right) \sin x \\
 y &= c_1 \cos x + c_2 \sin x + \left(-\frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x \right) \cos x \\
 &\quad + \left(-\frac{x}{4} \cos 2x + \frac{1}{8} \sin 2x \right) \sin x
 \end{aligned}$$

4.14 Practice problems :

Solve the following differential equations by the method of Variation of Parameters.

1. Solve $\frac{d^2y}{dx^2} + y = \tan x$ (VTU July 2013, July 2009, Jan 2009, Dec 2010, 2015 Model)
2. Solve: $\frac{d^2y}{dx^2} + y = \sec x$ (VTU June 2019, July 2008)
3. Solve: $\frac{d^2y}{dx^2} + a^2y = \sec ax$ (VTU Jan 2017, July 2016, Jan 2015, Jan 2014, July 2011, Dec 2010, Jan 2015)
4. Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ (VTU July 2016)
5. Solve: $\frac{d^2y}{dx^2} + 4y = \tan 2x$ (VTU Jan 2020, Jan 2018, July 2017, Jan 2016, June 2012, Dec 2011, Dec 2014, July 2008)
6. Solve: $\frac{d^2y}{dx^2} + a^2y = \tan ax$ (VTU July 2005)
7. Solve: $(D^2 + 1)y = \sec x \tan x$ (VTU Model 2018, July 2013)
8. Solve: $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ (VTU July 2017, Jan 2017, Dec 2014, June 2011, Dec 2011)

9. Solve: $y'' - 2y' + y = \frac{e^x}{x}$ (VTU July 2016, Jan 2013, July 2007)
10. Solve: $y'' + 2y' + y = e^{-x} \log x$ (VTU July 2017, July 2014, Jan 2007)
11. Solve: $y'' - 2y' + y = e^x \log x$ (VTU July 2006)
12. Solve: $y'' - 2y' + 2y = e^x \tan x$ (VTU Jan 2010, June 2015)
13. Solve: $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$ (VTU Model 2018, June 2015)
14. Solve: $(D^2 - 3D + 2)y = \frac{1}{1+e^{-x}}$ (VTU July 2017, June 2010)
15. Solve: $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ (VTU July 2016, July 2005)
16. Solve: $(D^2 + 4)y = 4 \sec^2 2x$ (VTU Dec 2015)

4.15 Linear Differential Equation with variable coefficients

In this section, we shall study two special forms of linear equations with variable coefficients, namely Cauchy's homogeneous linear differential equations and Legendre's homogeneous linear differential equations, which can be reduced to linear differential equations with constant coefficients by a suitable substitution.

4.16 Cauchy's Homogeneous Linear Differential Equations

A linear differential equation of the form,

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = R(x) \quad (1)$$

where a_1, a_2, \dots, a_n are constants and $R(x)$ is either a constant or a function of x only, is called Cauchy's homogeneous linear differential equation. Note that the index of x and order of derivative is same in each term of such equations.

The above equation can be reduced to linear differential equation with constant coefficients, by substituting

$$x = e^z \quad (\text{or}) \quad \log x = z$$

so that

$$\frac{dz}{dx} = \frac{1}{x}$$

Using chain rule for differentiation we obtain

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$$

$$\Rightarrow \boxed{x \frac{dy}{dx} = \frac{dy}{dz} = Dy}$$

where

$$D = \frac{d}{dz}$$

Similarly, for the second order derivative

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right) \\ &= \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dz} \right) \\ &= \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dz} \left(\frac{dy}{dz} \right) \frac{dz}{dx} \\ &= \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2y}{dz^2} \frac{1}{x} \\ &= \frac{1}{x^2} \left(\frac{d^2y}{dz^2} - \frac{dy}{dz} \right) \\ &= \frac{1}{x^2} (D^2 - D)y \end{aligned}$$

$$\Rightarrow \boxed{x^2 \frac{d^2y}{dx^2} = D(D-1)y} \text{ where } D = \frac{d}{dz}$$

Similarly, $\boxed{x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y}$

and so on.

In general, we have the relationship

$$x^n \frac{d^n y}{dx^n} = D(D-1)(D-2) \cdots (D-n+1)y$$

Substituting the above values of $x, x \frac{dy}{dx}, x^2 \frac{d^2y}{dx^2}, \dots, x^n \frac{d^n y}{dx^n}$ in the Equation (1), we get a linear differential equation with constant coefficients which can be solved with the methods discussed in previous lessons. Finally, by replacing z by $\log x$ we obtain the desired solution.

Example 4.16.1. Solve : $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2$

Solution: The given D.E. is a cauchy's linear D.E.

To solve this we use the substitution,

$$\boxed{x = e^t} \quad (\text{or}) \quad \boxed{\log x = t}$$

Then, $\boxed{x \frac{dy}{dx} = Dy}$, $\boxed{x^2 \frac{d^2y}{dx^2} = D(D-1)y}$ \cdots where $D = \frac{d}{dt}$

Substituting the above values of x , $x \frac{dy}{dx}$, $x^2 \frac{d^2y}{dx^2}$ in the given Differential Equation, we get

$$D(D-1)y - 4Dy + 6y = (e^t)^2$$

$$(D^2 - D - 4D + 6)y = e^{2t}$$

$$\text{A.E. is } m^2 - 5m + 6 = 0$$

$$\text{Roots are } m = 2, 3$$

$$\therefore y_c = c_1 e^{2t} + c_2 e^{3t}$$

$$y_p = \left[\frac{1}{D^2 - 5D + 6} \right]_{D=2} e^{2t} \quad (\text{Type I})$$

$$= t \left[\frac{1}{2D - 5} \right]_{D=2} e^{2t}$$

$$= t \frac{1}{4 - 5} e^{2t}$$

$$= -te^{2t}$$

$$\therefore \text{G.S. is } y = y_c + y_p$$

$$= c_1 e^{2t} + c_2 e^{3t} - te^{2t}$$

Resubstituting $t = \log x$ and $e^t = x$,

$$y = c_1 x^2 + c_2 x^3 - (\log x)x^2$$

Problem 54. Solve : $x \frac{d^2y}{dx^2} - 2y/x = x + 1/x^2$

Solution : The given equation can be written as

$$x^2 y'' - 2y = x^2 + \frac{1}{x}$$

Put $\log x = z$ or $x = e^z$, then

$$xy' = Dy \text{ and}$$

$$x^2 y'' = D(D-1)y$$

where $D = \frac{d}{dz}$ Now the given DE can be written as

$$|D(D-1) - 2| y = (e^z)^2 + (e^z)^{-1}$$

$$(D^2 - D - 2) y = e^{2z} + e^{-z}$$

This is a linear differential equation with constant coefficients.

$$\text{The AE is } m^2 - m - 2 = 0$$

$$\Rightarrow (m-2)(m+1) = 0 \Rightarrow m = 2, -1$$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{-z} = c_1 x^2 + c_2 x^{-1}$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - D - 2} e^{2z} + \frac{1}{D^2 - D - 2} e^{-z} \\
 &= z \cdot \frac{1}{2D - 1} e^{2z} + z \cdot \frac{1}{2D - 1} e^{-z} \\
 &= \frac{ze^{2z}}{3} + \frac{ze^{-z}}{-3} \\
 &= \frac{1}{3} z (e^{2z} - e^{-z}) \\
 &= \frac{1}{3} \log x (x^2 - x^{-1})
 \end{aligned}$$

general solution is

$$y = c_1 x^2 + c_2 x^{-1} + \frac{1}{3} \log x (x^2 - x^{-1})$$

Problem 55. Solve : $x^2 y'' - xy' + y = 2 \log x$

Solution: This is a cauchy's linear differential equation.

Put $\log x = z$ or $x = e^z$, then $xy' = Dy$, $x^2 y'' = D(D-1)y$ where $D = \frac{d}{dz}$

The equation becomes,

$$\begin{aligned}
 [D(D-1) - D + 1] y &= 2z \\
 (D^2 - 2D + 1) y &= 2z
 \end{aligned}$$

The AE is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

Therefore, C.F. = $(c_1 + c_2 z) e^z = (c_1 + c_2 \log x) x$

$$\begin{aligned}
 \text{P.I.} &= \frac{2z}{D^2 - 2D + 1} = \frac{2z}{1 + (D^2 - 2D)} \\
 &= [1 + (D^2 - 2D)]^{-1} 2z \\
 &= [1 - (D^2 - 2D) + \dots] 2z \\
 &= [2z - (0 - 2(2))] = 2z + 4 \\
 \text{P.I.} &= 2 \log x + 4
 \end{aligned}$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x) x + 2 \log x + 4$$

Problem 56. Solve: $x^2 y'' - xy' + y = 2 \cos^2(\log x)$

(VTU June 2014)

Solution: Put $\log x = t$ or $e^t = x$

$$xy' = Dy, x^2y'' = D(D-1)y \quad \text{where } D = \frac{d}{dt}$$

$$D(D-1)y - Dy + y = 2 \cos^2 t$$

$$(D(D-1) - D + 1)y = 2 \cos^2 t$$

$$[D^2 - 2D + 1]y = 2 \cos^2 t$$

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$$

$$y_c = (c_1 + c_2 t) e^t$$

$$= (c_1 + c_2 \log x) x$$

$$\begin{aligned} y_p &= \frac{1}{D^2 - 2D + 1} 2 \cos^2 t \\ &= \frac{1}{D^2 - 2D + 1} (1 + \cos 2t) \\ &= \frac{1}{D^2 - 2D + 1} e^{0t} + \frac{\cos 2t}{D^2 - 2D + 1} \\ &= P_1 + P_2 \text{ (say)} \end{aligned}$$

$$\begin{aligned} P_1 &= \frac{1}{D^2 - 2D + 1} \\ &= \frac{1}{D^2 - 2D + 1} e^{0t} \\ &= \frac{1}{0 - 2(0) + 1} e^{0t} = \frac{1}{1} = 1 \end{aligned}$$

$$\begin{aligned} P_2 &= \frac{1}{D^2 - 2D + 1} \cos 2t \\ &= \frac{1}{-4 - 2D + 1} \cos 2t \\ &= \frac{1}{-2D - 3} \cos 2t \\ &= \frac{(2D - 3)}{-(2D + 3)(2D - 3)} \cos 2t \\ &= \frac{2D(\cos 2t) - 3 \cos 2t}{-(4D^2 - 9)} \\ &= \frac{-4 \sin 2t - 3 \cos 2t}{25} \\ &= \frac{-4 \sin(2 \log x) - 3 \cos(2 \log x)}{25} \end{aligned}$$

$$\begin{aligned} y_p &= p_1 + p_2 \\ &= 1 + \frac{-4 \sin(2 \log x) - 3 \cos(2 \log x)}{25} \end{aligned}$$

$$\begin{aligned} y &= y_c + y_p \\ &= (c_1 + c_2 \log x) x + 1 + \frac{-4 \sin(2 \log x) - 3 \cos(2 \log x)}{25} \end{aligned}$$

Problem 57. $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x$

(VTU July 2017)

Sol: Setting $z = \log x$, the given equation becomes

$$\begin{aligned} | D(D-1) - D + 1] y &= z, \text{ where } D = \frac{d}{dz} \\ (D-1)^2 y &= z \end{aligned}$$

For this equation,

$$\text{C.F.} = (c_1 + c_2 z) e^z$$

and

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D-1)^2} z = (1-D)^{-2} z \\ &= (1 + 2D + 3D^2 + \dots) z = z + 2 \end{aligned}$$

Therefore, the general solution of (i) is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 z) e^z + z + 2$$

Since $z = \log x$, this becomes

$$y = (c_1 + c_2 \log x) x + \log x + 2$$

Problem 58. Solve $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10 \left(x + \frac{1}{x} \right)$

Solution : Putting $z = \log x$ (or $x = e^z$), the given equation becomes

$$\begin{aligned} [D(D-1)(D-2) + 2D(D-1) + 2] y &= 10 (e^z + e^{-z}) \\ \text{i.e. } (D^3 - D^2 + 2) y &= 10 (e^z + e^{-z}) \end{aligned}$$

where $D = \frac{d}{dz}$. For this equation, the A.E. is

$$m^3 - m^2 + 2 \equiv (m+1)(m^2 - 2m + 2) = 0,$$

whose roots are $-1, 1 \pm i$. Therefore,

$$\text{C.F.} = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D+1)(D^2 - 2D + 2)} (10e^z + 10e^{-z}) \\ &= \frac{10}{2} e^z + 10 \cdot \frac{1}{5(D+1)} e^{-z} \\ &= 5e^z + 2ze^{-z} \cdot \frac{1}{1} = 5e^z + 2ze^{-z} \end{aligned}$$

GS is

$$y = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z) + 5e^z + 2ze^{-z}$$

Since $z = \log x$, this becomes

$$y = \frac{c_1}{x} + x (c_2 \cos \log x + c_3 \sin \log x) + 5x + \frac{2}{x} \log x$$

4.17 Legendre's Homogeneous Linear Differential Equation

A linear differential equation of the form is

$$\boxed{a_0(ax + b)^n \frac{d^n y}{dx^n} + a_1(ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2(ax + b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = R(x)} \quad (2)$$

where a_1, a_2, \dots, a_n are constants and $R(x)$ is either a constant or a function of x only, is called a Legendre's homogeneous linear differential equation. Note that the index of $(ax + b)$ and the order of derivative is same in each term of such equation.

To solve the Equation (2), we introduce a new independent variable t such that $ax + b = e^t$, or $\log(ax + b) = t$ so that

$$\begin{aligned} (ax + b) \frac{dy}{dx} &= a Dy \\ (ax + b)^2 \frac{d^2 y}{dx^2} &= a^2 D(D - 1)y \\ (ax + b)^3 \frac{d^3 y}{dx^3} &= a^3 D(D - 1)(D - 2)y \end{aligned}$$

Substituting the above values of $(ax + b) \frac{dy}{dx}$, $(ax + b)^2 \frac{d^2 y}{dx^2}$, \dots , $(ax + b)^n \frac{d^n y}{dx^n}$ in the Equation (2), we get a linear differential equation with constant coefficients which can be solved easily. Finally, by replacing t by $\log x$ we obtain the desired solution of the given differential equation.

Problem 59. Solve: $(1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 2 \sin \log(1 + x)$. (VTU June 2012, Dec 2014)

Solution: This is Legendre's Linear DE.

put $t = \log(1 + x)$ or $(x + 1) = e^t$ then

$$(1 + x) \frac{dy}{dx} = 1 \cdot Dy, \quad (1 + x)^2 \frac{d^2 y}{dx^2} = 1D(D - 1)y; \quad \text{where } D = \frac{d}{dt}$$

Hence the given equation becomes

$$(D(D - 1) + D + 1)y = 2 \sin t \Rightarrow (D^2 + 1)y = 2 \sin t$$

$$\text{A.E is } m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\begin{aligned} y_c &= c_1 \cos t + c_2 \sin t \\ &= c_1 \cos \log(1 + x) + c_2 \sin \log(1 + x) \end{aligned}$$

$$\begin{aligned} y_p &= \frac{1}{(D^2 + 1)} 2 \sin t, \quad D^2 \rightarrow -(1^2) \\ &= \frac{1}{2D} 2 \sin t \\ &= -t \cos t \\ &= -\log(1 + x) \cos \log(1 + x) \end{aligned}$$

complete solution: $y_c + y_p$

$$\text{Thus } y = c_1 \cos \log(1+x) + c_2 \sin \log(1+x) - \log(1+x) \cos \log(1+x)$$

Problem 60. $(3x+2)^2 y'' + 3(3x+2)y' - 36y = 3x^2 + 4x + 1$ (VTU July 2016)

Solution : This is a Legendre's Linear Differential equation.

Put $\log(3x+2) = z$ or $3x+2 = e^z$

$$\Rightarrow x = \frac{1}{3}(e^z - 2)$$

then

$$(3x+2)y' = 3Dy, \quad (3x+2)^2 y'' = 3^2 D(D-1)y \text{ where } D = \frac{d}{dz}$$

Given DE can be written as

$$[3^2 D(D-1) + 3 \cdot (3D) - 36] y = 3 \left[\frac{1}{3} (e^z - 2) \right]^2 + 4 \left[\frac{1}{3} (e^z - 2) \right] + 1$$

$$(9D^2 - 9D + 9D - 36) y = 3 \left[\frac{1}{9} (e^{2z} - 4e^z + 4) \right] + \frac{4}{3} e^z - \frac{8}{3} + 1$$

$$(9D^2 - 36) y = \frac{1}{3} e^{2z} - \frac{4}{3} e^z + \frac{4}{3} + \frac{4}{3} e^z - \frac{8}{3} + 1$$

$$9(D^2 - 4) y = \frac{1}{3} (e^{2z} - 1)$$

$$(D^2 - 4) y = \frac{1}{27} (e^{2z} - 1)$$

The AE is $m^2 - 4 = 0 \Rightarrow m = 2, -2$ Therefore,

$$\text{C.F} = c_1 e^{2z} + c_2 e^{-2z} = c_1 (3x+2)^2 + c_2 (3x+2)^{-2}$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{27} \frac{1}{(D^2 - 4)} e^{2z} - \frac{1}{27} \frac{1}{(D^2 - 4)} \\ &= \frac{1}{27} z \cdot \frac{1}{(2D)} e^{2z} - \frac{1}{27} \frac{1}{(0 - 4)} \\ &= \frac{1}{108} z e^{2z} + \frac{1}{108} \\ &= \frac{1}{108} (z e^{2z} + 1) \\ &= \frac{1}{108} [\log(3x+2)[3x+2]^2 + 1] \\ &= \frac{1}{108} [(3x+2)^2 \log(3x+2) + 1] \end{aligned}$$

G.S. is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 (3x+2)^2 + c_2 (3x+2)^{-2} + \frac{1}{108} [(3x+2)^2 \log(3x+2) + 1] \end{aligned}$$

Problem 61. Solve: $(2x-1)^2 \frac{d^2 y}{dx^2} + (2x-1) \frac{dy}{dx} - 2y = 8x^2 - 2x + 3$

$$\text{Put } (2x - 1) = e^t \Rightarrow t = \log(2x - 1)$$

$$(2x - 1) \frac{dy}{dx} = 2Dy; (2x - 1)^2 \frac{d^2y}{dx^2} = 4D(D - 1)y$$

$$\text{Here } 2x - 1 = e^t \Rightarrow x = \frac{e^t + 1}{2}$$

Given Equation is:

$$\begin{aligned} [4D(D - 1) + 2D - 2]y &= 8 \left(\frac{e^t + 1}{2} \right)^2 - 2 \left(\frac{e^t + 1}{2} \right) + 3 \\ \Rightarrow (2D^2 - D - 1)y &= e^{2t} + \frac{3}{2}e^t + 2 \end{aligned}$$

$$\text{Its AE is } 2m^2 - m - 1 = 0 \Rightarrow m = 1, -\frac{1}{2}.$$

$$\begin{aligned} C.F. &= c_1 e^t + c_2 e^{-\frac{1}{2}t} \\ &= c_1(2x - 1) + c_2(2x - 1)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} P.I. &= \frac{1}{2D^2 - D - 1} \left(e^{2t} + \frac{3}{2}e^t + 2 \right) \\ &= \frac{1}{2D^2 - D - 1} e^{2t} + \frac{3}{2} \frac{1}{2D^2 - D - 1} e^t + (2) \frac{1}{2D^2 - D - 1} e^{0t} \\ &= \frac{1}{8 - 3} e^{2t} + \frac{3}{2} \cdot \frac{1}{4D - 1} e^t + 2 \frac{1}{-1} \\ &= \frac{e^{2t}}{5} + \frac{3}{2} t \left(\frac{e^{3t}}{3} \right) + (-2) \\ &= \frac{e^{2t}}{5} + \frac{te^t}{2} - 2 \\ &= \frac{(2x - 1)^2}{5} + \frac{\log(2x - 1)(2x - 1)}{2} - 2 \end{aligned}$$

G.S. is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1(2x - 1) + c_2(2x - 1)^{-\frac{1}{2}} \\ &\quad + \frac{(2x - 1)^2}{5} + \frac{\log(2x - 1)(2x - 1)}{2} - 2 \end{aligned}$$

Problem 62. Solve $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x) \frac{dy}{dx} + y = \sin \{ \log(1 + x)^2 \}$

Solution : Putting $z = \log(1 + x)$, the equation becomes

$$\text{or } D(D - 1)y + Dy + y = \sin 2z$$

$$(D^2 + 1)y = \sin 2z,$$

$$\text{where } D = \frac{d}{dz} \text{ AE. is } m^2 + 1 = 0 \Rightarrow m = \pm i$$

This yields

$$C.F. = c_1 \cos z + c_2 \sin z = c_1 \cos \log(1 + x) + c_2 \sin \log(1 + x)$$

and

$$\begin{aligned}
 P.I. &= \frac{1}{D^2 + 1}(\sin 2z) \\
 &= \frac{1}{-2^2 + 1}(\sin 2z) \\
 &= -\frac{1}{3} \sin 2z \\
 &= -\frac{1}{3} \sin \{\log(1+x)^2\}
 \end{aligned}$$

Hence G.S. is

$$\begin{aligned}
 y &= y_c + y_p \\
 &= c_1 \cos \log(1+x) + c_2 \sin \log(1+x) - \frac{1}{3} \sin \{\log(1+x)^2\}
 \end{aligned}$$

Problem 63. Solve: $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3) \frac{dy}{dx} - 12y = 6x$

Solution : Putting $z = \log(2x+3)$, or $2x+3 = e^z$, the given equation becomes

$$2^2 D(D-1)y - 2Dy - 12y = 3(e^z - 3)$$

AE is $4m^2 - 6m - 12 = 0 \Rightarrow m = \frac{3 \pm \sqrt{57}}{4}$

$$C.F. = c_1 e^{\frac{3+\sqrt{57}}{4}z} + c_2 e^{\frac{3-\sqrt{57}}{4}z}$$

$$= c_1 (2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right) \log(2x+3)} + c_2 (2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right) \log(2x+3)}$$

$$\begin{aligned}
 P.I. &= \frac{1}{4D^2 - 6D - 12} 3(e^z - 3) \\
 &= \frac{1}{4D^2 - 6D - 12} 3e^z - \frac{1}{4D^2 - 6D - 12} 9e^{0z} \\
 &= \frac{3}{-14} e^z - \frac{9}{-12} e^{0z} \\
 &= -\frac{3}{14} (2x+3) + \frac{3}{4}
 \end{aligned}$$

GS is

$$\begin{aligned}
 y &= c_1 (2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right) \log(2x+3)} + c_2 (2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right) \log(2x+3)} \\
 &\quad - \frac{3}{14} (2x+3) + \frac{3}{4}
 \end{aligned}$$

Problem 64. Solve $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1) \frac{dy}{dx} + 16y = 8(2x+1)^2$

Sol:

$$\text{Put } z = \log(2x+1) \Rightarrow 2x+1 = e^z$$

$$(2x+1) \frac{dy}{dx} = 2Dy, \quad D = \frac{d}{dz}$$

$$(2x+1)^2 \frac{d^2y}{dx^2} = 4D(D-1)y$$

$$\begin{aligned} \Rightarrow (4D(D-1) - 6(2D) + 16)y &= 8(2x+1)^2 \\ [4(D^2 - D) - 12D + 16]y &= 8(e^z)^2 \\ (4D^2 - 4D - 12D + 16)y &= 8e^{2z} \\ (4D^2 - 16D + 16)y &= 8e^{2z} \\ 4(D^2 - 4D + 4)y &= 8e^{2z} \\ (D^2 - 4D + 4)y &= 2e^{2z} \end{aligned}$$

$$\text{AE is } m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$$

$$\text{C.F.} = (c_1 + c_2 z)e^{2z} = (c_1 + c_2 \log(2x+1))(2x+1)^2$$

$$\begin{aligned} P.I &= \frac{1}{D^2 - 4D + 4} 2e^{2z} \\ &= 2 \left\{ \frac{1}{D^2 - 4D + 4} \right\} e^{2z} \\ &= 2z \left[\frac{1}{2D - 4} \right] e^{2z} \\ &= 2z^2 \left[\frac{1}{2} \right] e^{2z} \\ &= z^2 e^{2z} \\ &= [\log(2x+1)]^2 (2x+1)^2 \end{aligned}$$

\therefore The general solution is:

$$y = (c_1 + c_2 z) e^{2 \log(2x+1)} + (2x+1)^2 \cdot \log(2x+1)^2$$

$$y = (c_1 + c_2 \log(2x+1))(2x+1)^2 + (2x+1)^2 [\log(2x+1)]^2$$

4.18 Practice Problems

1. Solve: $x^2 y'' + xy' + y = 2\cos^2(\log x)$. (VTU June 2014)

$$\text{Ans : } y = c_1 \cos(\log x) + c_2 \sin(\log x) + 1 - \frac{\cos 2(\log x)}{3}$$

2. Solve $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1+x)^2$ (VTU July 2016)

Ans :

3. $\frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 2 \log x$

$$\text{Ans : } y = x(c_1 + c_2 \log x) + 2 \log x + 4$$

4. $x \frac{d^2 y}{dx^2} - 2y/x = x + 1/x^2$

$$\text{Ans : } Y = c_1 x^2 + \frac{c_2}{x} + \frac{\log x}{3} (x^2 - \frac{1}{x})$$

5. $x^4 \frac{d^3 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = \sin(\log x)$

$$\text{Ans : } y = \frac{c_1}{x} + (c_2 + c_3 \log x)x + \frac{(-3\cos(\log x) + 4\sin(\log x))}{25x}$$

6. $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$

$$\text{Ans : } Y = \frac{c_1}{x^2} + x \{ c_2 \cos(\sqrt{3} \log x) + c_3 \sin(\sqrt{3} \log x) \} + 8 \cos(\log x) - \sin(\log x)$$

$$7. x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 9y = 3x^2 + \sin(3 \log x)$$

$$\text{Ans : } y = c_1 \cos(3 \log x) + c_2 \sin(3 \log x) + \frac{(3x^2)}{13} - \frac{(\log x \cos(3 \log x))}{6}$$

$$8. x^3 \frac{d^3 y}{dx^3} + 9x^2 \frac{d^2 y}{dx^2} + 18x \frac{dy}{dx} + 6y = \frac{1}{x}$$

$$\text{Ans : } y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \frac{\log x}{2x}$$

$$9. x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10(x + \frac{1}{x})$$

$$\text{Ans : } y = \frac{c_1}{x} + x(c_2 \cos(\log x) + c_3 \sin(\log x)) + 5x + 2 \frac{\log x}{x}$$

$$10. x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x$$

$$\text{Ans : } y = x^{\frac{3}{2}} [c_1 \cos^{\frac{7}{2}}(\log x) + c_2 \sin^{\frac{7}{2}}(\log x)] + \frac{x^2}{2} + \frac{\log x}{2} + \frac{3}{8}$$

$$11. x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1 + x^2)$$

$$\text{Ans : } y = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + 2x + x^2 \frac{(\log x)^2}{2}$$

$$12. x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4$$

$$\text{Ans : } Y = ax^4 + \frac{b}{x} + \frac{(\log x)x^4}{5}$$

$$13. x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = \cos[2 \log x]$$

$$\text{Ans : } Y = c_1 x^2 + c_2 x^3 - \frac{(\cos 2(\log x))}{2}$$

$$14. x^2 y'' - 5xy' + 8y = 2 \log x$$

$$\text{Ans : } Y = c_1 x^4 + c_2 x^2 + \frac{\log x}{4} + \frac{3}{16}$$

$$15. x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$$

(VTU Jan 2018)

$$16. x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10(x + \frac{1}{x})$$

(VTU July 2017)

$$17. x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x$$

(VTU July 2017)

$$18. x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 13y = \log x + x^2$$

(VTU Jan 2017, July 2016)

$$19. (3x + 2)^2 y'' + 3(3x + 2)y' - 36y = 8x^2 + 4x + 1$$

(VTU July 2016)

$$\text{Ans : } y = (3x + 2)^2 + \frac{c_2}{(3x+2)^2} + \frac{1}{81} [2 \log(3x + 2) (3x + 2)^2 + \frac{20}{3}(3x + 2) - \frac{17}{4}]$$

$$20. (1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = \sin 2(\log(1 + x))$$

(VTU June 2012, Dec 2014)

$$\text{or } (1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = \sin \{ \log(1 + x)^2 \}$$

$$\text{Ans : } c_1 \cos(\log(1 + x)) + c_2 \sin(\log(1 + x)) - \frac{\sin 2[\log(1+x)]}{3}$$

$$21. (x + a)^2 y'' - 4(x + a)y' + 6y = x$$

(VTU Jan 2016)

Ans :

$$22. (1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 4 \cos[\log(1 + x)]$$

$$\text{Ans : } Y = c_1 \cos[\log(1 + x)] + c_2 \sin[\log(1 + x)] + 2 \log(1 + x) \sin \log(1 + x)$$

$$23. (3x + 2)^2 \frac{d^2 y}{dx^2} + 3(3x + 2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1$$

$$\text{Ans : } Y = c_1 (3x + 2)^2 + \frac{c_2}{(3x+2)^2} + \frac{(3x+2)^2 \log(3x+2) + \frac{20}{3}(3x+2)}{81}$$

$$24. (2x + 1)^2 \frac{d^2 y}{dx^2} - 2(2x + 1) \frac{dy}{dx} - 12y = 6x + 5$$

$$\text{Ans : } Y = c_1 (2x + 1)^3 + \frac{c_2}{(2x+1)} - \frac{3(2x+1)}{16} - \frac{1}{6}$$

$$25. (2x + 1)^2 \frac{d^2y}{dx^2} - 6(2x + 1) \frac{dy}{dx} + 16y = 8(2x + 1)^2$$

$$\text{Ans : } Y = [c_1 + c_2 \log(1 + 2x)] (2x + 1)^2 + [\log(1 + 2x)]^2 (2x + 1)^2$$

$$26. (2x + 3)^2 \frac{d^2y}{dx^2} - (2x + 3) \frac{dy}{dx} - 12y = 6x$$

$$\text{Ans : } Y = c_1 (2x + 3)^{\frac{3+\sqrt{57}}{4}} + c_2 (2x + 3)^{\frac{3-\sqrt{57}}{4}} - \frac{3}{14} (2x + 3) + \frac{3}{4}$$

$$27. (2x + 3)^2 \frac{d^2y}{dx^2} + 5(2x + 3) \frac{dy}{dx} + y = 4x$$

$$\text{Ans : } Y = (2x + 3)^{\frac{(-1)}{8}} [c_1 \cos\left(\frac{\sqrt{15}}{8} \log(2x + 3)\right) + c_2 \sin\left(\frac{\sqrt{15}}{8} \log(2x + 3)\right)] + \frac{(2x+3)}{3} - 6$$

$$28. (2x - 1)^2 \frac{d^2y}{dx^2} + (2x - 1) \frac{dy}{dx} - 2y = 8x^2 - 2x + 3$$

$$\text{Ans : } Y = \frac{c_1}{\sqrt{(2x-1)}} + c_2 (2x - 1) + \frac{(2x-1)^2}{5} + \frac{(2x-1)\log(2x-1)}{2} - 2$$

$$29. (2x + 3)^2 \frac{d^2y}{dx^2} - 2(2x + 3) \frac{dy}{dx} - 12y = 6x$$

(July 2017)

AJIET



Lecture Notes

**Mathematics I for Civil Stream (Subject Code -
BMATC101)- Module 5- Linear Algebra**

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Module - 5 Linear Algebra

5.1 Elementary row transformation

An elementary row transformation is an operation of any one of the following types.

- (i) Interchange of two rows i.e. $R_i \longleftrightarrow R_j$
- (ii) The multiplication of the elements of rows by a non-zero number. $R_i \longrightarrow cR_i$ for the multiplication of i th row by a non zero constant c .
- (iii) The addition to the elements of a row, the corresponding elements of a row multiplied by any number. $R_i \longrightarrow R_i + cR_j$ for the addition to the i th row to the products of the j th row by c .

Note : A row of only zero entries is called a zero row

5.1.1 The Row - Echelon Form

A rectangular matrix is in row echelon form if it has the following three properties:

- (i) All zero rows are at the bottom of the matrix (i.e. If there is a row where every entry is zero, then this row lies below any other row that contains a non zero entry.)
- (ii) In any non zero row the leading entry (i.e. first non-zero entry) is 1
- (iii) The leading entry of each nonzero row after the first row occurs to the right of the leading entry of the previous row

Example 5.1.1. The following matrix is in row echelon form: $A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note :

In echelon form the leftmost non zero entry of a non zero row is a leading 1. A column containing a leading 1 will be called a **pivot column**.

Minor : Let A be a matrix of order $m \times n$. Let B be a submatrix of A of order r obtained by deleting few rows and columns from A . Then the determinant of this matrix B is called a minor of A of order

r.

5.1.2 Rank of a Matrix:

Definition: A matrix is said to be of rank r if

- (i) It has at least one non-zero minor of order r and
- (ii) Every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix. The rank of matrix A is denoted by $\rho(A)$.

Note : Using row-operations if we can reduce a matrix to echelon-form, then rank of the matrix is given by

Rank = Number of non-zero rows in echelon form.

Example 5.1.2. Find the rank of the matrix $A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$

Solution :

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$$

using row operation $R_1 \leftrightarrow R_2$

$$A \simeq \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$

$$A \simeq \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$A \simeq \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & 4 \end{bmatrix}$$

This matrix is in Echelon form and number of non-zero rows = 3

Hence $\rho(A) = 3$

Example 5.1.3. Find the rank of

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(VTU Jan 2017, Jun 2012)

Sol :

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Using $R_1 \leftrightarrow R_2$,

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Using $R_2 \rightarrow R_2 - 2R_1$

$$R_3 \rightarrow R_3 - 3R_1$$

$$R_4 \rightarrow R_4 - 6R_1$$

$$\sim A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Using $R_3 \rightarrow 5R_3 - 4R_2$

$$R_4 \rightarrow 5R_4 - 9R_2$$

$$\sim A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

Using $R_4 \rightarrow R_4 - R_3$

$$\sim A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 3$$

Example 5.1.4. Find the rank of $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ (VTU Jan 2019, June 2019, Jan 2015, Jan 2013)

Sol: Given $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{matrix} R_2 : R_2 - 2R_1 \\ R_3 : R_3 + R_1 \\ R_4 : R_4 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow \frac{1}{3}R_3$

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

\sim (Echelom form) $\rho(A) =$ Number of non – zero rows $= 4$

Example 5.1.5. Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (VTU Jan 2020, Jun 2014)

Sol : Given $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Applying, $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Applying, $R_3 \rightarrow R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly number of non-zero rows=2, then $\rho(A) = 2$

Exercise 5.1

Find the rank of the following matrices by reducing them to echelon form.

1. $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

(VTU Jun 2013, Jun 2011)

Ans : 2

2. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

(VTU Jan 2016)

Ans : 3

3. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Ans : 2

4. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(VTU Jun 2014, Dec 2011)

Ans : 2

5. $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Ans : 1

6. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

Ans : 2

$$7. \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

(VTU Jan 2016)

Ans : 3

$$8. \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(VTU Jan 2017, Jun 2012)

Ans : 3

$$9. \begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \\ 95 & 96 & 97 & 98 & 99 \end{bmatrix}$$

(VTU July 2014)

Ans : 2

$$10. \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$$

(VTU Jan 2015, Jan 2013)

Ans : 4

$$11. \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

Ans : 3

$$12. \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

(VTU Jan 2015)

Ans :

$$13. \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

(VTU Jun 2014)

Ans :

$$14. \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

(VTU Jan 2014)

Ans :

Augmented Matrix : Suppose we have a system of m equations in n variables, with coefficient matrix A and constants B . Then the augmented matrix of the system of equations is the $m \times (n + 1)$ matrix whose first n columns are the columns of A and whose last column (i.e. $(n+1)$ th column) is the column matrix B . This matrix will be written as $[A|B]$

Row-Equivalent Matrices Two matrices, A and B, are row-equivalent if one can be obtained from the other by a sequence of row operations.

5.2 Solution of system of linear equations - Consistency and Gauss-Elimination method

Gauss Elimination method applied to three linear equations : First we explain this method applied to a particular systems of order three given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of linear equations can be written as $AX = B$ where A is the coefficient matrix

given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ X is a column vector(column-matrix) containing the variables, i.e.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and B is the matrix of right hand side constants. i.e. $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ The co-efficient matrix

A written along with a constant matrix B is called an Augmented matrix and is denoted by [A:B].

The augmented matrix of the above system is $[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Following Steps are used to solve the above system of equations by using Gauss-Elimination Method :

Step 1: Use the element $a_{11} \neq 0$ to make the remaining elements $a_{21} = 0$ and $a_{31} = 0$ by elementary row transformation operation.

$$\text{i.e. } R_2 \rightarrow R_2 - \frac{a_{21}}{a_{11}}R_1, R_3 \rightarrow R_3 - \frac{a_{31}}{a_{11}}R_1$$

$$\text{Then } [A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a'_{22} & a'_{23} & : & b'_2 \\ 0 & a'_{32} & a'_{33} & : & b'_3 \end{bmatrix}$$

Step 2: Use the element $a'_{22} \neq 0$ to make the remaining elements $a'_{32} = 0$ by elementary row transformation operation.

$$\text{i.e. } R_3 \rightarrow R_3 - \frac{a'_{32}}{a'_{22}}R_2$$

$$\text{Then } [A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a'_{22} & a'_{23} & : & b'_2 \\ 0 & 0 & a''_{33} & : & b''_3 \end{bmatrix} \quad (*)$$

There are 3 possibilities:

- (i) If **Rank of A = Rank of the Augmented Matrix [A:B] = no.of unknowns**, then the system is consistent and has a unique solution.
- (ii) If **Rank of A = Rank of the Augmented Matrix [A:B] ; no.of unknowns**, then the system is consistent and has infinite number of solutions.
- (iii) If **Rank of A \neq Rank of the Augmented Matrix [A:B]**, then there is no solution. i.e. the system is inconsistent.

If the system is consistent, then (*) can be written as,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$

$$a'_{33}x_3 = b''_3$$

By back substitution we get the values of x_1, x_2, x_3

Example 5.2.1. Test for consistency and solve the following system of equation by using Gauss-elimination method.

$$2x + y + 4z = 12; 4x + 11y - z = 33; 8x - 3y + 2z = 20$$

Solution : The Augmented matrix is given by,

$$[A : B] = \begin{bmatrix} 2 & 1 & 4 : 12 \\ 4 & 11 & -1 : 33 \\ 8 & -3 & 2 : 20 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 4R_1$$

$$\sim \begin{bmatrix} 2 & 1 & 4 : 12 \\ 0 & 9 & -9 : 9 \\ 0 & -7 & -14 : -28 \end{bmatrix}$$

$$R_2 \rightarrow R_2 \div 9, R_3 \rightarrow R_3 \div -7$$

$$\sim \begin{bmatrix} 2 & 1 & 4 : 12 \\ 0 & 1 & -1 : 1 \\ 0 & 1 & 2 : 4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 4 : 12 \\ 0 & 1 & -1 : 1 \\ 0 & 0 & 3 : 3 \end{bmatrix}$$

We can observe that Rank of A = Rank of the Augmented Matrix $[A : B] = \text{no.of unknowns}$, and hence the system is consistent and has a unique solution.

$$\therefore \text{the system of equation reduces to } 2x + y + 4z = 12; y - z = 1; 3z = 3$$

By using back substitution we get $z = 1, y = 2, x = 3$

Example 5.2.2. Solve by using Gauss Elimination method, $x + y + z = 9$, $2x + y - z = 0$, $2x + 5y + 7z = 52$ (VTU Jan 2019)

Sol :

$$(A, B) = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$$R_2 : R_2 - 2R_1$$

$$R_3 : R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$$R_3 : R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 3 & : & 18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$\therefore \rho(A) = \rho(A, B) = 3 \quad \text{Number of unknown}$$

Hence system is consistent and unique solution exists.

From 3rd row, $z = 5$ From 2nd row, $y + 3z = 18 \Rightarrow y + 3(5) = 18 \Rightarrow y = 3$

from first row, $x + 3 + 5 = 9 \Rightarrow x = 9 - 8 \Rightarrow x = 1$

$\therefore x = 1, y = 3, z = 5$

Example 5.2.3. Find the values of λ and μ so that the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. (VTU July 2021)

Sol : The augmented matrix $[A, B]$ is

$$[A, B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$R_2 \rightarrow 2R_2 - 7R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$[A, B] \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix}$$

The above matrix is in echelon form (i) When $\lambda = 5, \mu \neq 9$,

$$\rho(A) = 2, \rho(A, B) = 3 \text{ (i.e.) } \rho(A, B) \neq \rho(A)$$

The system is inconsistent and it has no solution. (ii) When $\lambda \neq 5, \mu \in R,$

$$\rho(A, B) = \rho(A) = 3 = \text{number of unknowns}$$

The system is consistent with a unique solution.

(iii) When $\lambda = 5, \mu = 9,$ then $\rho(A, B) = \rho(A) = 2 < \text{number of unknowns}$

The system is consistent with infinite number of solutions.

Example 5.2.4. Find the values of λ and μ so that the equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. (VTU Jan 2021)

Solution : The given system of linear equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The Augmented matrix is :

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

Applying elementary row transformations $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ we get

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix}$$

Using $R_3 \rightarrow R_3 - R_2$

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix},$$

This is in echelon form.

(i) For no solution, we must have $\rho(A) \neq \rho(A, B)$

$$\text{i.e. } \lambda - 3 = 0 \text{ or } \lambda = 3 \text{ and } \mu - 10 \neq 0 \Rightarrow \mu \neq 10$$

(ii) for unique solution, we must have $\rho(A) = \rho(A, B) = 3$

$$\text{i.e. } \lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$$

$$\text{and } \mu - 10 \neq 0 \Rightarrow \mu \neq 10$$

(iii) for infinite solutions, we must have $\rho(A) = \rho(A, B) < 3$

$$\text{i.e. } \lambda - 3 = 0 \Rightarrow \lambda = 3 \text{ and } \mu - 10 = 0 \Rightarrow \mu = 10$$

Exercise 5.2

1. Test for consistency and hence solve $4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21$
 Ans : $x = 1, y = 3k - 2, z = k$

2. Solve $x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0$ Ans :
 $x = \frac{-10}{7}k, y = \frac{8}{7}k, z = k$
3. Solve: $x + 3y + 2z = 0, 2x - y + 3z = 0, 3x - 5y + 4z = 0, x + 17y + 4z = 0$ Ans :
 $x = 11k, y = k, z = -7k$
4. Find the values of k for which the equations $x + y + z = 1, 2x + y + 4z = k$ and $4x + y + 10z = k^2$ has a solution and solve them in each case. Ans :
 $k = 1, x = -3n, y = 2n + 1, z = n$ and when $k = 2, x = 1 - 3m, y = 2m, z = m$
5. Find the values of λ for which the system $x + y + z = 1, x + 2y + 4z = \lambda, x + 4y + 10z = \lambda^2$ has a solution. Solve it in each case. Ans : when $\lambda = 1, x = 2k_1 + 1, y = -3k_1, z = k_1$ and when $\lambda = 2, x = 2k_2, y = 1 - 3k_2, z = k_2$
6. Find the values of λ and μ so that the equations $2x + 3y + 5z = 9, 7x + 3y - 2z = 8, 2x + 3y + \lambda z = \mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. Ans: (i) $\lambda = 5, \mu \neq 9$, (ii) $\lambda \neq 5$ (iii) $\lambda = 5$ and $\mu = 9$
7. Solve by using Gauss Elimination method, $x + 2y + z = 3, 2x + 3y + 2z = 5, 3x - 5y + 5z = 2$. (VTU July 2017)
8. Solve by using Gauss Elimination method, $2x - 3y + 4z = 7, 5x - 2y + 2z = 7, 6x - 3y + 10z = 23$ VTU July 2017
9. $3x - y + 2z = 12, x + 2y + 3z = 11, 2x - 2y - z = 2$ (VTU Jan 2015)
10. $5x_1 + x_2 + x_3 + x_4 = 4, x_1 + 7x_2 + x_3 + x_4 = 12, x_1 + x_2 + 6x_3 + x_4 = -5, x_1 + x_2 + x_3 + 4x_4 = -6$ (VTU Jun 2015) Ans :
 $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$
11. Test for consistency and solve by Gauss Elimination method : $4x + y + z = 4, x + 4y - 2z = 4, 3x + 2y - 4z = 6$ (VTU Mar 2001) Ans : $x = 1, y = \frac{1}{2}, z = -\frac{1}{2}$
12. $x + 2y + z = 3, 2x + 3y + 3z = 10, 3x - y + 3z = 13$ (VTU Model 2015)
13. $4x_1 + x_2 + x_3 = 4, x_1 + 4x_2 - 2x_3 = 4, 3x_1 + 2x_2 - 4x_3 = 6$ Ans :
 $x_1 = 1, x_2 = \frac{1}{2}, x_3 = -\frac{1}{2}$
14. $2x + 4y + z = 3, 3x + 2y - 2z = -2, x - y + z = 6$ Ans : $x = 2, y = -1, z = 3$
15. $x + 2y - z = 3, 3x - y + 2z = 1, 2x - 2y + 3z = 2$ Ans : $x = -1, y = 4, z = 4$
16. $x_1 + x_2 + x_3 = 4, 2x_1 + x_2 - x_3 = 1, x_1 - x_2 + 2x_3 = 2$ (VTU Jan 2014) Ans :
 $x_1 = \frac{3}{7}, x_2 = \frac{13}{7}, x_3 = \frac{12}{7}$
17. $2x_1 - x_2 + 3x_3 = 1, -3x_1 + 4x_2 - 5x_3 = 0, x_1 + 3x_2 - 6x_3 = 0$ (VTU jun 2014, Jun 2012) Ans : $x_1 = \frac{9}{34}, x_2 = \frac{23}{34}, x_3 = \frac{13}{34}$

18. Test the consistency and solve $x + 2y + 2z = 1, 2x + y + z = 2, 3x + 2y + 2z = 3, x + z = 0$
Ans : $x = 1, y = -k, z = k$

19. Test the consistency of the following equations, $4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21$

20. $x + 4y - z = -5, x + y - 6z = -12, 3x - y - z = 4$ (VTU Model 2014)

21. $2x + 3y - z = 5, 4x + 4y - 3z = 3, 2x - 3y + 2z = 2$ (VTU Jan 2015, July 2014)
Ans : $x = 1, y = 2, z = 3$

22. $2x - y + 3z = 1, -3x + 4y - 5z = 0, x + 3y - 6z = 0$ (VTU Jan 2012)

23. Solve by using Gauss Elimination method, $2x + y + 4z = 12, 4x + 11y - z = 33, 8x - 3y + 2z = 20$ (VTU June 2018)

24. Solve by using Gauss Elimination method, $x + y + z = 9, 2x + y - z = 0, 2x + 5y + 7z = 52$ (VTU Jan 2019)

25. Solve by using Gauss Elimination method, $x - 2y + 3z = 2, 3x - y + 4z = 4, 2x + y - 2z = 5$ (VTU Jan 2020)

26. Solve by using Gauss Elimination method, $x + y + z = 9, x - 2y + 3z = 8, 2x + y - z = 3$ (VTU July 2021)

5.3 Gauss-Jordan method

This method aims in reducing the coefficient matrix A to a diagonal matrix by using only row-operations.

Example 5.3.1. Solve the following system of equation by using Gauss- Jordan method

$$2x + y + 4z = 12; 4x + 11y - z = 33; 8x - 3y + 2z = 20$$

Solution : The Augmented matrix is given by,

$$[A : B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 4R_1$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix}$$

$$R_2 \rightarrow R_2 \div 9, R_3 \rightarrow R_3 \div -7$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & -1 & : & 1 \\ 0 & 1 & 2 & : & 4 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2; R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 0 & 5 & : & 11 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & 3 & : & 3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 \div 3$$

$$\sim \begin{bmatrix} 2 & 0 & 5 & : & 11 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 5R_3; R_2 \rightarrow R_2 + R_3$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & : & 6 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

\therefore the system of equation reduces to $2x = 6; y = 2; z = 1$

By solving, we get $x = 3, y = 2, z = 1$.

Example 5.3.2. Solve by using Gauss Jordan, $x + y + z = 9, 2x + y - z = 0, 2x + 5y + 7z = 52$
(VTU June 2019)

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$$R_2 : R_2 - 2R_1$$

$$R_3 - 2R_1$$

Solution : $[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$

$$R_1 \rightarrow R_1 + R_2 : R_3 + 3R_2$$

$$[A, B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$$R_3 : \left(\frac{1}{4}\right) R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_1 : R_1 + 2R_3$$

$$R_2 : R_2 + 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Rewriting the equations, we get

$$x = 1, \quad -y = -3 \Rightarrow y = 3 \text{ and } z = 5$$

Example 5.3.3. Apply Gauss - Jordan method to solve the system of equations, $2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16$ (VTU Jan 2015)

Solution : The Augmented matrix of the system is

$$[A | B] = \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - \frac{3}{2}R_1$ and $R_3 \rightarrow R_3 - \frac{1}{2}R_1$, we get

$$[A | B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 7R_2$, we get

$$[A | B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix}$$

Applying $R_1 \rightarrow R_1 + \frac{1}{2}R_3$, $R_2 \rightarrow R_2 + \frac{3}{4}R_3$ and $R_1 \rightarrow R_1 - 2R_2$ we get

$$[A | B] \sim \begin{bmatrix} 2 & 0 & 0 & 14 \\ 0 & 1/2 & 0 & -9/2 \\ 0 & 0 & -2 & -10 \end{bmatrix}$$

We see that A is reduced to diagonal form.

From this, we get $2x = 14 \Rightarrow x = 7$;

$\frac{1}{2}y = \frac{-9}{2} \Rightarrow y = -9$;

$-2z = -10 \Rightarrow z = 5$

Thus $x = 7, y = -9, z = 5$ is the solution.

Example 5.3.4. Solve by Gauss Jordan method : $x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40$

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$ and

$R_3 \rightarrow R_3 - 3R_1$

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & 9 & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}$$

$R_1 \rightarrow 5R_1 + R_2$ and

$R_3 \rightarrow 5R_3 + R_2$

Sol :

$$[A : B] \sim \begin{bmatrix} 5 & 0 & 7 & : & 40 \\ 0 & -5 & 2 & : & -5 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$$R_3 \Rightarrow \frac{R_3}{12}$$

$$[A : B] \sim \begin{bmatrix} 5 & 0 & 7 & : & 40 \\ 0 & -5 & 2 & : & -5 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_2 \Rightarrow R_2 - 2R_3 \text{ and } R_1 \Rightarrow R_1 - 7R_3 [A : B] \sim \begin{bmatrix} 5 & 0 & 0 & : & 5 \\ 0 & -5 & 0 & : & -15 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$5x = 5 \Rightarrow x = 1$

$-5y = -15 \Rightarrow y = 3$

$z = 5$

Exercise 5.3

Apply Gauss - Jordan method to solve the system of equations.

1. Apply Gauss - Jordan method to solve the system of equations, $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$ (VTU Jan 2017, Jan 2015) Ans : $x = 7$, $y = -9$, $z = 5$
2. $x + y + z = 9$, $x - 2y + 3z = 8$, $2x + y - z = 3$ (VTU Jan 2021, Jan 2018, July 2017)
3. Solve by using Gauss Jordan method, $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$ (VTU Jan 2017, Jan 2015)
4. Solve by using Gauss Jordan method, $2x + 5y + 7z = 52$, $2x + y - z = 0$, $x + y + z = 9$ (VTU Jan 2016, June 2014) Ans : $x = 1$, $y = 3$, $z = 5$
5. Solve by using Gauss Jordan method, $x + 2y + z = 3$, $2x + 3y + 3z = 10$, $3x - y + 3z = 13$ (VTU Jan 2018)
6. Solve by using Gauss Jordan, $x + y + z = 9$, $2x + y - z = 0$, $2x + 5y + 7z = 52$ (VTU June 2019)
7. Solve by using Gauss Jordan method,
 $2x_1 + x_2 + 3x_3 = 1$, $4x_1 + 4x_2 + 7x_3 = 1$, $2x_1 + 5x_2 + 9x_3 = 3$ (VTU Jan 2020)
8. $x + y + z = 9$, $2x - 3y + 4z = 13$, $3x + 4y + 5z = 40$ Ans : $x = 1$, $y = 3$, $z = 5$
9. $x_1 + 2x_2 + x_3 = 8$, $2x_1 + 3x_2 + 4x_3 = 20$, $4x_1 + 3x_2 + 2x_3 = 16$ Ans :
 $x = 1$, $y = 2$, $z = 3$
10. $10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$ Ans : $x = 1$, $y = 1$, $z = 1$
11. $x + y + z = 4$, $2x + y - z = 1$, $x - y + 2z = 2$ Ans : $x = \frac{3}{7}$, $y = \frac{13}{7}$, $z = \frac{12}{7}$
12. $2x + 3y - z = 5$, $4x + 4y - 3z = 3$, $2x - 3y + 2z = 2$ (VTU Dec 2011) Ans :
 $x = 1$, $y = 2$, $z = 3$

5.4 Gauss-Seidel Iteration Method:

Iterative methods provide an alternative approach. An iterative method starts with an approximate solution, and uses it by means of a recurrence formula to provide another approximate solution; by repeatedly applying the formula, a sequence of solutions is obtained which converges to the exact solution.

First we explain this method applied to a particular systems of order three given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

To begin, solve the 1st equation for x_1 , the 2nd equation for x_2 and third equation for x_3 and obtain the rewritten equations :

$$\begin{aligned}x_1 &= \frac{1}{a_{11}} [b_1 - a_{12}x_2 - a_{13}x_3] \\x_2 &= \frac{1}{a_{22}} [b_2 - a_{21}x_1 - a_{23}x_3] \\x_3 &= \frac{1}{a_{33}} [b_3 - a_{31}x_1 + a_{32}x_2]\end{aligned}$$

Then make an initial guess of the solution $x_1^{(0)} = y_1^{(0)} = z_1^{(0)} = 0$ Substitute these values into the right hand side the of the rewritten equations to obtain the first approximations as

$$\begin{aligned}x_1^{(1)} &= \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}] \\x_2^{(1)} &= \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}] \\x_3^{(1)} &= \frac{1}{a_{33}} [b_3 - a_{31}x_1^{(1)} + a_{32}x_2^{(1)}]\end{aligned}$$

In each iteration we use the new values(i.e.latest values) as soon as they are known. i.e. we calculate $x_1^{(1)}$ from the first equation, its value is then used in the second equation to obtain the new $x_2^{(1)}$, and these new values of $x_1^{(1)}$ and $x_2^{(1)}$ are then used in the third equation to obtain the new $x_3^{(1)}$. To find the second approximations we use the above set of equations by replacing $x_1^{(0)}, y_1^{(0)}, z_1^{(0)}$ by $x_1^{(1)}, y_1^{(1)}, z_1^{(1)}$. We continue like this by obtaining each successive approximation, using a previous approximation until the values of x_1, x_2 and x_3 obtained in two successive approximations are equal.

Note : The following condition is essential to use Gauss-Seidel Iteration Method.

- The Coefficient matrix of given system of equations must be **diagonally dominant**.

(A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row.

Example 5.4.1. Solve the following system of equation by using Gauss- Seidel method. $x + y + 54z = 110; 27x + 6y - z = 85; 6x + 15y + 2z = 72$

Solution : The system of equation is not in diagonal form by interchanging the rows.

$$27x + 6y - z = 85; 6x + 15y + 2z = 72; x + y + 54z = 110$$

We can observe , $|27| > +|-1|$; $|15| > |6| + |2|$; $|54| > |1| + |1|$

∴ the system of equation is diagonally dominant.

$$\begin{aligned}\text{From(1)}x &= \frac{1}{27}[85 - 6y + z] \\ \text{From(2)}y &= \frac{1}{15}[72 - 6x - 2z] \\ \text{From(3)}z &= \frac{1}{54}[110 - x - y]\end{aligned}$$

First Iteration :

$$x^{(1)} = \frac{1}{27}[85 - 0 + 0] = \frac{85}{27} = 3.1481$$

$$y^{(1)} = \frac{1}{15}[72 - 6x^{(1)} - 2z] = \frac{1}{15}[72 - 6(3.1481) - 0] = 3.5481$$

$$z^{(1)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = 1.913$$

Second Iteration :

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = 2.432$$

$$y^{(2)} = \frac{1}{15}[72 - 6x^{(2)} - 2z^{(1)}] = 3.572$$

$$z^{(2)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = 1.925$$

Third Iteration :

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = 2.4256$$

$$y^{(3)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(2)}] = 3.5730$$

$$z^{(3)} = \frac{1}{54}[110 - x^{(3)} - y^{(3)}] = 1.925$$

Third Iteration :

$$x^{(4)} = \frac{1}{27}[85 - 6y^{(3)} + z^{(3)}] = 2.425$$

$$y^{(4)} = \frac{1}{15}[72 - 6x^{(4)} - 2z^{(3)}] = 3.573$$

$$z^{(4)} = \frac{1}{54}[110 - x^{(4)} - y^{(4)}] = 1.925$$

Since $x^{(3)} \approx x^{(4)} = 2.425$, $y^{(3)} \approx y^{(4)} = 3.573$, $z^{(3)} \approx z^{(4)} = 1.925$, the solution is $x = 2.425$, $y = 3.573$, $z = 1.925$

Example 5.4.2. Solve the following system of equations by Gauss-Seidal iteration method, $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$ (VTU Jan 2020, June 2019, Jan 2018, July 2017, Jan 2017, Model 2015)

$$\text{Given, } 2x + y - 2z = 17 \longrightarrow (1)$$

$$3x + 20y - z = -18 \longrightarrow (2)$$

$$2x - 3y + 20z = 25 \longrightarrow (3)$$

\therefore The equation are diagonally dominant.

$$(1) \Rightarrow x = \frac{1}{20}[17 - y + 2z]$$

$$(2) \Rightarrow y = \frac{1}{20}[-18 - 3x + z]$$

$$(3) \Rightarrow z = \frac{1}{20}[25 - 2x + 3y]$$

Let $x = 0$, $y = 0$, $z = 0$

First Iteration :

$$x^{(1)} = \frac{1}{20}[17 - 0 + 0] = 0.85$$

$$y^{(1)} = \frac{1}{20}[-18 - 2.55 + 0] = -1.0275$$

$$z^{(1)} = \frac{1}{20}[20 - 1.7 - 3(0.825)] = 1.0109$$

Second iteration :

$$x^{(2)} = \frac{1}{20}[17 + 1.025 - 12(1.0109)] = 1.0025$$

$$y^{(2)} = \frac{1}{20}[-18 - 3(1.0025) + 1.0109] = -0.999$$

$$z^{(2)} = \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998$$

Third iteration :

$$x^{(3)} = \frac{1}{20}[17 + 0.9998 + 2(0.9998)] = 1$$

$$y^{(3)} = \frac{1}{20}[-18 - 3(1) + 0.9998] = -1$$

$$z^{(3)} = \frac{1}{20}[25 - 2(1) + 3(-1)] = 1$$

$$\therefore x = 1, y = -1, z = 1$$

Example 5.4.3. Solve the system of equations $12x + y + z = 31, 2x + 8y - z = 24, 3x + 4y + 10z = 58$ using Gauss Seidal method. (VTU Jan 2019)

Sol : \Rightarrow Given equation are diagonally dominant.

$$x = \frac{1}{12}[31 - y - z]$$

$$y = \frac{1}{8}[24 - 2x + z]$$

$$z = \frac{1}{10}[58 - 3x - 4y]$$

$$x^{(1)} = \frac{1}{12}[31 - 0 - 0] = \frac{31}{12} = 2.583$$

$$y^{(1)} = \frac{1}{8}[24 - 2(2.583) + 0] = 2.354$$

$$z^{(1)} = \frac{1}{10}[58 - 3(2.583) - 4(2.354)] = 4.0835$$

$$x^{(2)} = \frac{1}{12}[31 - (2.354) - (4.0835)] = 2.046$$

$$y^{(2)} = \frac{1}{8}[24 - 2(2.046) + 4.0835] = 2.99 = 3$$

$$z^{(2)} = \frac{1}{10}[58 - 3(2.046) - 4(3)] = 3.98 = 4$$

$$x^{(3)} = \frac{1}{12}[31 - (3) - (4)] = 2$$

$$y^{(3)} = \frac{1}{8}[24 - 2(2) + 4] = 3 \quad \therefore x = 2, y = 3, z = 4$$

$$z^{(3)} = \frac{1}{10}[58 - 3(2) - 4(3)] = 4$$

Example 5.4.4. Solve the system of equations $83x + 11y - 4z = 95$, $7x + 52y + 13z = 104$, $3x + 8y + 29z = 71$ using Gauss Seidal method. (VTU Model 2018)

Sol:

The given equation can be written in the iteration form as

$$x = \frac{1}{83}(95 - 11y + 4z)$$

$$y = \frac{1}{52}(104 - 7x - 13z)$$

$$z = \frac{1}{29}(71 - 3x - 8y)$$

Taking first $x^{(1)} = 0$, $y^{(1)} = 0$, $z^{(1)} = 0$ and put these values in (i), we get

$$\begin{aligned} x^{(2)} &= \frac{1}{83}(95 - 11y^{(1)} - 4z^{(1)}) \\ &= \frac{1}{83}(95 - 11 \times 0 + 4 \times 0) = \frac{95}{83} = 1.14 \end{aligned}$$

$x^{(2)} = 1.14$, $y^{(1)} = 0$, $z^{(1)} = 0$, in (ii), we get

$$\begin{aligned} y^{(2)} &= \frac{1}{52}(104 - 7x^{(2)} - 13z^{(1)}) = \frac{1}{52}(104 - 7 \times 1.14 - 13 \times 0) \\ &= \frac{96.02}{52} = 1.85 \end{aligned}$$

Put $x^{(2)} = 1.14$, $y^{(2)} = 1.85$, $z^{(1)} = 0$ in (iii), we get

$$\begin{aligned} z^{(2)} &= \frac{1}{29}(71 - 3x^{(2)} - 8y^{(2)}) \\ &= \frac{1}{29}(71 - 3 \times 1.14 - 8 \times 1.85) = \frac{52.78}{29} = 1.82 \end{aligned}$$

Now put $x^{(2)} = 1.14, y^{(2)} = 1.85, z^{(2)} = 1.82$ in (i), we get

$$\begin{aligned}x^{(3)} &= \frac{1}{83} (95 - 11y^{(2)} + 4z^{(2)}) \\x^{(3)} &= \frac{1}{83} [95 - 11 \times 1.85 + 4 \times 1.82] \\&= \frac{81.93}{83} = 0.99\end{aligned}$$

$x^{(3)} = 0.99, y^{(2)} = 1.85, z^{(2)} = 1.82$ in (ii), we get

$$\begin{aligned}y^{(3)} &= \frac{1}{52} [104 - 7x^{(3)} - 13z^{(3)}] = \frac{1}{52} [104 - 7 \times 0.99 - 13 \times 1.82] \\&= \frac{73.41}{52} = 1.41.\end{aligned}$$

Put $x^{(3)} = 0.99, y^{(3)} = 1.41, z^{(2)} = 1.82$ in (iii), we get

$$\begin{aligned}z^{(3)} &= \frac{1}{29} (71 - 3x^{(3)} - 8y^{(3)}) \\&= \frac{1}{29} (71 - 3 \times 0.99 - 8 \times 1.41) \\&= \frac{56.75}{29} = 1.95\end{aligned}$$

Now put $x^{(3)} = 0.99, y^{(3)} = 1.41, z^{(3)} = 1.95$ in (i), we get

$$\begin{aligned}x^{(4)} &= \frac{1}{83} (95 - 11y^{(3)} + 4z^{(3)}) \\&= \frac{1}{83} (95 - 11 \times 1.41 + 4 \times 1.95) \\&= \frac{1}{83} (87.29) = 1.05\end{aligned}$$

$= \frac{1}{83} (95 - 11 \times 1.41 + 4 \times 1.94.59) = 1.05 = \frac{1}{83} (87.29)$ $x^{(4)} = 1.05, y^{(3)} = 1.41, z^{(3)} = 1.95$ in (ii), we get $y^{(4)} = \frac{1}{52} (104 - 7x^{(3)} - 13z^{(3)})$ Put

$$\begin{aligned}&= \frac{1}{52} (104 - 7 \times 1.05 - 13 \times 1.95) \\&= \frac{71.3}{52} = 1.37\end{aligned}$$

Put $x^{(4)} = 1.05, y^{(4)} = 1.37, z^{(3)} = 1.95$ in (iii), we get

$$\begin{aligned}z^{(4)} &= \frac{1}{29} (71 - 3x^{(4)} - 8y^{(4)}) \\&= \frac{1}{29} (71 - 3 \times 1.05 - 8 \times 1.37) \\&= \frac{56.89}{29} = 1.96\end{aligned}$$

$$\begin{aligned}
 z^{(4)} &= \frac{1}{29} (71 - 3x^{(4)} - 8y^{(4)}) \\
 &= \frac{1}{29} (71 - 3 \times 1.05 - 8 \times 1.37) \quad \text{Here The values are sufficiently close to } x^{(3)}, y^{(3)}, z^{(3)} \\
 &= \frac{56.89}{29} = 1.96
 \end{aligned}$$

$x^{(4)} = 1.05, y^{(4)} = 1.37, z^{(4)} = 1.96$
 respectively. Hence the solution is

$$x = 1.05, y = 1.37, z = 1.96.$$

5.5 Eigen values and Eigen Vectors :

Let A be an $n \times n$ matrix. Then a real number λ is called an eigenvalue of the matrix A, if and only if, there is a n-dimensional nonzero vector, X for which

$$AX = \lambda X \quad \dots (1)$$

Any such vector, X is called an **eigenvector** of the matrix A, associated with the eigenvalue λ .

5.6 Rayleigh's power method

To find largest eigen value and corresponding eigen vector of a given square matrix A, we use the following steps.

Step1. To find the largest eigen value and corresponding eigen vector of a given square matrix 'A' of order 3, we initially assume the eigen vector in the form $X^{(0)} = [1 \ 0 \ 0]^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$$\text{or } X^{(0)} = [0 \ 0 \ 1]^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\text{or } X^{(0)} = [0 \ 1 \ 0]^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Step2. Evaluate the matrix product $AX^{(0)}$, which is a column matrix of order 3.

Step3. Take out the numerically largest value and write it in the form $\lambda^{(1)} X^{(1)}$

Step4. Compute $AX^{(1)}$ and re-write it in the form $\lambda^{(2)} X^{(2)}$

Step5. Similarly Compute $AX^{(2)} = \lambda^{(3)} X^{(3)}$ and so on.

Continue this iterative process until we get two consecutive values of λ and X with desired accuracy.

The values so obtained are largest eigen value and corresponding eigen vectors respectively for the given square matrix A .

Example 5.6.1. Use power method to find the largest eigen value and the corresponding eigen vectors

of the matrix A by using power method. Take $[1 \ 0 \ 0]'$ as the initial eigen vector. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(Apply 4 iterations)

(VTU Jan 2020, June 2019, June 2018, Jan 2017, Jan 2016)

SOL : Given,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Ax^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)}x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)}x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.9286 \end{bmatrix} = \lambda^{(3)}x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9228 \end{bmatrix} = \begin{bmatrix} 2.9286 \\ 0 \\ 2.8572 \end{bmatrix} = 2.9286 \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \lambda^{(4)}(x)^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \begin{bmatrix} 2.9756 \\ 0 \\ 2.9512 \end{bmatrix} = 2.9756 \begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix}$$

$$Ax^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix} = \begin{bmatrix} 2.9918 \\ 0 \\ 2.9836 \end{bmatrix} = 2.9918 \begin{bmatrix} 1 \\ 0 \\ 0.9973 \end{bmatrix}$$

∴ The large Given value $\lambda = 2.9918 \sim 3$ and $\lambda^{(6)}x^{(6)}$

it's eigen value $\begin{bmatrix} 1 \\ 0 \\ 0.9973 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Example 5.6.2. Find the Dominant eigen value and the corresponding eigen vectors of the matrix

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by power method taking the initial eigen vector as $(1 \ 1 \ 1)'$ (VTU Jan 2018, July 2017, Jun 2015)

Sol : Given,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}, x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ax^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \lambda^{(1)}x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 7.34 \\ -2.67 \\ 4.01 \end{bmatrix} = 7.34 \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \lambda_2 x_2$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 7.82 \\ -3.63 \\ 4.01 \end{bmatrix} = 7.82 \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \lambda_3 X_3$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.94 \\ -3.89 \\ 3.99 \end{bmatrix} = 7.94 \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \lambda_4 X_4$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7.98 \\ -3.97 \\ 3.99 \end{bmatrix} = 7.98 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda_5 x_5$$

$$\begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda_6 x_6$$

∴ The large Given value of $\lambda = 7.9970 \sim 8$ and it's eigen value is $\begin{bmatrix} 1 \\ -0.4994 \\ 0.5002 \end{bmatrix}$

Example 5.6.3. Determine the largest eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ using Rayleigh's power method. (VTU Model 2014)

Sol :

$$AX^0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

Repeating the above process, we get

$$AX^{(2)} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} X^{(3)};$$

$$AX^{(3)} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} X^{(5)};$$

$$AX^{(5)} = 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)} X^{(6)};$$

$$AX^{(6)} = 3.41 \begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen vector is $[0.74, -1, 0.67]'$.

Example 5.6.4. Find the largest eigen value and the corresponding eigen vector of the matrix $A =$

$$\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \text{ by using Power method.}$$

Solution : Let $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$, and $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

1st iteration :

$$\begin{aligned} AX^{(0)} &= \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} \\ &= 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} \\ &= \lambda^{(1)} X^{(1)} \end{aligned}$$

2nd iteration :

$$\begin{aligned} AX^{(1)} &= \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} \\ &= \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} \\ &= 25.2 \begin{bmatrix} 1 \\ 0.04 \\ 0.0666 \end{bmatrix} \\ &= \lambda^{(2)} X^{(2)} \end{aligned}$$

3rd iteration:

$$\begin{aligned} AX^{(2)} &= \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.0666 \end{bmatrix} \\ &= \begin{bmatrix} 25.173 \\ 1.12 \\ 1.7336 \end{bmatrix} \\ &= 25.173 \begin{bmatrix} 1 \\ 0.0444 \\ 0.0689 \end{bmatrix} \\ &= \lambda^{(3)} X^{(3)} \end{aligned}$$

4th iteration :

$$\begin{aligned}
 AX^{(3)} &= \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.0689 \end{bmatrix} \\
 &= \begin{bmatrix} 25.1822 \\ 1.1332 \\ 1.7244 \end{bmatrix} \\
 &= 25.1822 \begin{bmatrix} 1 \\ 0.045 \\ 0.069 \end{bmatrix} \\
 &= \lambda^{(4)} X^{(4)}
 \end{aligned}$$

5th iteration:

$$\begin{aligned}
 AX^{(4)} &= \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.045 \\ 0.069 \end{bmatrix} \\
 &= \begin{bmatrix} 25.183 \\ 1.135 \\ 1.724 \end{bmatrix} \\
 &= 25.183 \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix} \\
 &= \lambda^{(5)} X^{(5)}
 \end{aligned}$$

6th iteration :

$$\begin{aligned}
 AX^{(5)} &= \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.045 \\ 0.0685 \end{bmatrix} \\
 &= \begin{bmatrix} 25.182 \\ 1.135 \\ 1.726 \end{bmatrix} \\
 &= 25.182 \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix} \\
 &= \lambda^{(6)} X^{(6)}
 \end{aligned}$$

$$\text{Here } \lambda^{(5)} = \lambda^{(6)} = 25.183 \text{ and } X^{(5)} = X^{(6)} = \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix}$$

Hence largest eigen value is $\lambda = 25.183$ and largest eigen vector is $X = \begin{bmatrix} 1 \\ 0.0450 \\ 0.0689 \end{bmatrix}$

Exercise 5.4

1. Use power method to find the largest eigen value and the corresponding eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ taking $[0 \ 1]'$ as the initial eigen vector. (VTU Model 2015)

2. Find the Dominant eigen value and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by power method taking the initial eigen vector as $(1 \ 1 \ 1)'$ (VTU July 2017, Jun 2015) **Ans :** 8, $[1, -0.5, 0.5]'$

3. Determine the largest eigen value and the corresponding eigen vectors of the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ taking $[0 \ 0 \ 1]'$ as the initial eigen vector. Perform 5 iterations. (VTU Jan 2015)

4. Use power method to find the largest eigen value and the corresponding eigen vectors of the matrix A by using power method. Take $[1 \ 0 \ 0]'$ as the initial eigen vector. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (VTU Jan 2017, Jan 2016) **Ans :** 2.99, $[1 \ 0 \ 1]'$

5. Use power method to find the largest eigen value and the corresponding eigen vectors of the matrix A by using power method. Given $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ Take $[1 \ 0.8 \ -0.8]'$ as the initial eigen vector. **Ans :** 5.994, $[1, 0.999, -0.999]'$

6. Determine the largest eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ using Rayleigh's power method. (VTU Model 2014) **Ans :** 3.41 and $[0.74, -1, 0.67]^T$

7. reduce the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 9 \end{bmatrix}$ to diagonal form. (VTU June 2011)