Lecture Notes

BMATE201

Mathematics-II for EEE stream

Module 1 : Vector Calculus

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Module 1

Vector Calculus:

Syllabus:

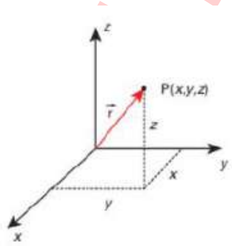
Introduction to Vector Calculus in Mechanical Engineering applications
Vector Differentiation: Scalar and vector fields. Gradient, directional derivative, curl and divergence – physical interpretation, solenoidal and irrotational vector fields. Problems. Vector Integration: Line integrals, Surface integrals. Applications to work done by a force and flux. Statement of Green's theorem and Stoke's theorem. Problems.
Self-Study: Volume integral and Gauss divergence theorem.
Applications: Heat and mass transfer, oil refinery problems, environmental engineering, velocity and acceleration of moving particles, analysis of streamlines. (RBT Levels: L1, L2 and L3)

1.1 Basic Concepts

- **Vector :** A vector is a physical quantity having both magnitude and direction. Example : Force, Acceleration, Velocity etc.

- A vector is represented by a directed line segment with an arrow over it. i.e. $\vec{PQ}$ is a vector with initial point P and terminal point Q whose direction is from P to Q and magnitude is the length PQ.

- **Position vector of a point P :**Position vector is a vector which represents the position of a point in a space with respect to the origin. The position vector of a point P(x,y,z) in space is given by $\boxed{\vec{r} = xi + yj + zk}$



- **Note :** If x, y, z are functions of a single variable 't' then the vector $\vec{r}$ is said to be vector point function of 't'(i.e. vector equation of a curve at any point 't'). i.e. $\vec{r} = r(\vec{t}) = x(t)i + y(t)j + z(t)k$.

- **Magnitude** of a vector $\vec{r}$ denoted by $|\vec{r}|$ or r is defined as $\boxed{r = \sqrt{x^2 + y^2 + z^2}}$

- **Dot product :** If $\vec{A} = a_1i + a_2j + a_3k$ and $\vec{B} = b_1i + b_2j + b_3k$ are two vectors, then the dot product is given by the formula,
$$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$$
Sometimes the dot product is called the scalar product.

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

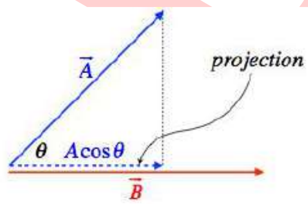
- $i \cdot i = j \cdot j = k \cdot k = 1$

- $i \cdot j = j \cdot k = k \cdot i = 0$

- If $\vec{A} = a_1i + a_2j + a_3k$ and $\vec{B} = b_1i + b_2j + b_3k$ then $\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$
- $i \times i = j \times j = k \times k = 0$
- $i \times j = k, j \times k = i, k \times i = j$
- $j \times i = -k, k \times j = -i, i \times k = -j$

- If θ is the angle between two vectors $\vec{A}$ and $\vec{B}$, then $cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$
- If $\vec{A} \cdot \vec{B} = 0$, then $\vec{A}$ and $\vec{B}$ are perpendicular.
- If $\vec{A} \times \vec{B} = 0$, then $\vec{A}$ and $\vec{B}$ are parallel.
- **Component of a vector along a given direction:** A **component** of a vector $\vec{A}$ along a given direction $\vec{B}$ is the resolved part of $\vec{A}$ and is given by $\boxed{\vec{A} \cdot \hat{n}}$ where $\boxed{\hat{n} = \frac{\vec{B}}{|\vec{B}|}}$



1.2 Vector differential operator

The vector differential operator is denoted by ∇ (Read it as **del** or **nabla**) and is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

1.3 The gradient of the scalar point function:

The gradient of the scalar point function Φ denoted by $grad\Phi$ or $\nabla\Phi$ is a vector function defined by

$$\begin{aligned} grad\,\Phi = \nabla\Phi &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\Phi \\ &= \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k \\ &= \sum \frac{\partial\Phi}{\partial x}i \end{aligned}$$

Note : Gradient of a scalar function is a vector quantity

1.4 Divergence of a vector point function :

The divergence of a vector point function $\vec{F} = f_1 i + f_2 j + f_3 k$ denoted by $divF$ is defined by,

$$\begin{aligned} div\,\vec{F} &= \nabla\cdot\vec{F} \\ &= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot(f_1 i + f_2 j + f_3 k) \\ &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ &= \sum \frac{\partial f_1}{\partial x} \end{aligned}$$

Note : Divergence of a vector point function is a scalar quantity

1.5 Curl of a vector point function

The curl of a continuously differentiable vector point function $\vec{F} = f_1 i + f_2 j + f_3 k$ denoted by $curl\vec{F}$ is defined by

$$\begin{aligned} curl\ \vec{F} &= \nabla \times \vec{F} \\ &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (f_1 i + f_2 j + f_3 k) \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \\ &= \sum \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i \end{aligned}$$

Note : Curl of a vector point function is a vector quantity.

Problem 1.5.1. *Find grad* Φ *when* $\Phi = 3x^2y - y^3z^2$ *at the point (1,-2,-1) (VTU July 2017)*

Solution:

$$\begin{aligned} \nabla\phi &= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left(3x^2y - y^3z^2\right) \\ &= \frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right)i + \frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right)j \\ &\quad + \frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)k \\ &= (6xy)i + \left(3x^2 - 3y^2z^2\right)j + \left(-2y^3z\right)k \qquad \cdots (1) \end{aligned}$$

Thus $\nabla\phi$ at the point $(1, -2, -1)$ is obtained by putting these values of x, y, z in $R.H.S.$ of (1) Thus

$\nabla\phi\ |_{(1,-2,-1)}$

$$\begin{aligned} &= 6(1)(-2)i + \left[3 - 3(-2)^2(-1)^2\right]j + \left[-2(-2)^3(-1)\right]k \\ &= -12i + j(3 - 12) + k(-16) \\ &= -12i - 9j - 16k \end{aligned}$$

Problem 1.5.2. *If* $\Phi = log(x^2 + y^2 + z^2)$, *then find* $\nabla\Phi$ *&* $|\nabla\Phi|$ *at (1 , 2 , 1)*

Solution: Given

$$\phi(x,y,z) = \log\left(x^2+y^2+z^2\right)$$

$$\text{grad}\,\phi = \nabla\phi$$

$$= \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get,

$$\frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2+z^2}\cdot 2x$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{x^2+y^2+z^2}\cdot 2y$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{x^2+y^2+z^2}\cdot 2z$$

At the point $(1,2,1)$

$$\frac{\partial\phi}{\partial x} = \frac{2\cdot 1}{1^2+2^2+1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{\partial\phi}{\partial y} = \frac{2\cdot 2}{1^2+2^2+1^2} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{\partial\phi}{\partial z} = \frac{2\cdot 1}{1^2+2^2+1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore\ ,\text{grad}\,\phi\,|_{(1,2,1)} = \frac{1}{3}i + \frac{2}{3}j + \frac{1}{3}k$$

$$= \frac{1}{3}[i+2j+k]$$

$$|\ grad\phi\ | = \sqrt{\frac{1^2+2^2+1^2}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

Problem 1.5.3. *Find* $divF$ *and* $curlF$ *where* $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$ **(VTU Jul 2015. Jul 2013, Jan 2008, Aug 2002)**

Solution: Let $\phi = x^3 + y^3 + z^3 - 3xyz$

The given expression can be written as

$$\begin{aligned}
\vec{F} &= grad\phi = \nabla\phi \\
&= \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k \\
&= \frac{\partial(x^3+y^3+z^3-3xyz)}{\partial x}i + \frac{\partial(x^3+y^3+z^3-3xyz)}{\partial y}j \\
&\quad + \frac{\partial(x^3+y^3+z^3-3xyz)}{\partial z}k \\
\vec{F} &= (3x^2-3yz)i + (3y^2-3xz) + (3z^2-3xy)k
\end{aligned}$$

$$\begin{aligned}
div(\vec{F}) &= \nabla\cdot\vec{F} \\
&= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left((3x^2-3yz)i + (3y^2-3xz) + (3z^2-3xy)k\right) \\
&= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy) \\
&= 6x+6y+6z \\
&= 6(x+y+z)
\end{aligned}$$

$$\begin{aligned}
curl(\vec{F}) &= \nabla\times\vec{F} \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix} \\
&= i\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right] \\
&\quad - j\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right] \\
&\quad + k\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right] \\
&= i(-3x+3x) - j(-3y+3y) + k(-3z+3z) \\
&= 0i - 0j + 0k = \vec{0}
\end{aligned}$$

Problem 1.5.4. *If* $\Phi = 2x^3y^2z^4$, *find* $div(grad\Phi)$ *at (1,1,1)* *(VTU Jan 2018)*

Solution:

$$\begin{aligned}\text{grad}\,\varphi &= \nabla\varphi \\ &= \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z} \\ &= \hat{i}\frac{\partial}{\partial x}\left(2x^3y^2z^4\right) + \hat{j}\frac{\partial}{\partial y}\left(2x^3y^2z^4\right) + \hat{k}\frac{\partial}{\partial z}\left(2x^3y^2z^4\right) \\ \text{grad}\,\varphi &= \hat{i}\left(6x^2y^2z^4\right) + \hat{j}\left(4x^3yz^4\right) + \hat{k}\left(8x^3y^2z^3\right)\end{aligned}$$

and

$$\begin{aligned}\text{div}(\text{grad}\,\varphi) &= \nabla\cdot\nabla\varphi \\ &= \frac{\partial}{\partial x}\left(6x^2y^2z^4\right) \\ &\quad + \frac{\partial}{\partial y}\left(4x^3yz^4\right) \\ &\quad + \frac{\partial}{\partial z}\left(8x^3y^2z^3\right) \\ &= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 \\ \text{div}(\text{grad}\,\varphi)\,|_{(1,1,1)} &= 12 + 4 + 24 = 40\end{aligned}$$

Problem 1.5.5. *Find* $div\vec{F}$ *and* $curl F$ *at the point (1,2,3) if* $\vec{F} = grad(x^3y + y^3z + z^3x - x^2y^2z^2)$ *(VTU 2007)*

Solution: Let $\Phi = x^3y + y^3z + z^3x - x^2y^2z^2$

$$\begin{aligned}\text{Given }\ \vec{F} &= \text{grad}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right) \\ &= \nabla(\Phi) \\ &= \frac{\partial}{\partial x}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right)\hat{i} \\ &\quad + \frac{\partial}{\partial y}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right)\hat{j} \\ &\quad + \frac{\partial}{\partial z}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right)\hat{k}\end{aligned}$$

$$
\begin{aligned}
\vec{F} &= \left(3x^2y + z^3 - 2xy^2z^2\right)\hat{i} + \left(x^3 + 3y^2z - 2x^2yz^2\right)j \\
&\qquad + \left(y^3 + 3z^2x - 2x^2y^2z\right)\hat{k} \\
\therefore \operatorname{div}\vec{F} &= \nabla\cdot\vec{F} \\
&= \frac{\partial}{\partial x}\left(3x^2y + z^3 - 2xy^2z^2\right) \\
&\qquad + \frac{\partial}{\partial y}\left(x^3 + 3y^2z - 2x^2yz^2\right) \\
&\qquad + \frac{\partial}{\partial z}\left(y^3 + 3z^2x - 2x^2y^2z\right) \\
&= 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6zx - 2x^2y^2 \\
\operatorname{div}\vec{F}\,|_{(1,2,3)} &= 12 - 36 + 54 - 72 + 18 - 8 = -32
\end{aligned}
$$

$$
Curl(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \begin{matrix}(3x^2y + z^2 \\ -2xy^2z^2)\end{matrix} & \begin{matrix}(x^3 + 3y^2z \\ -2xy^2z^2)\end{matrix} & \begin{matrix}(y^3 + 3z^2x \\ -2x^2y^2z)\end{matrix} \end{vmatrix}
$$

$$
\begin{aligned}
&= i\left[3y^2 - 4x^2yz - 3y^2 + 4x^2yz\right] \\
&\quad - j\left[3z^2 - 4xy^2z - 3z^2 + 4xy^2z\right) \\
&\quad + k\left[3x^2 - 4xyz^2 - 3x^2 + 4xyz^2\right] \\
&= 0\hat{i} + 0\hat{j} + 0\hat{k} \\
&= \vec{0}
\end{aligned}
$$

Problem 1.5.6. *Prove that* $\nabla\cdot\vec{r} = 3$ *and* $\nabla\times\vec{r} = 0$

Solution: (i)

$$
\begin{aligned}
\nabla\cdot\vec{r} &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(xi + yj + zk) \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\
&= 1 + 1 + 1 = 3
\end{aligned}
$$

$$\nabla \times \vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times (xi + yj + zk)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - j\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + k\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= 0i + 0j + 0k = \vec{0}$$

Problem 1.5.7. *If* $\vec{V} = \vec{w} \times \vec{r}$, *P.T.* $w = \frac{1}{2}(\nabla \times \vec{V})$ *or* $curl\vec{V} = 2\vec{w}$ *where* $\vec{w}$ *is a constant vector.* **(VTU Jan 2015)**

Solution: Given $\vec{v} = \vec{w} \times \vec{r}$

$\vec{r} = xi + yj + zk$ $\quad [\because \vec{r}$ is the position vector of $(x, y, z)]$

and let $\vec{w} = w_1 i + w_2 j + w_3 k$, $\quad w_1, w_2, w_3$ are constants.

$$\vec{w} \times \vec{r} = \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= i[w_2 z - w_3 y] - j[w_1 z - w_3 x] + k[w_1 y - w_2 x]$$

$$
\begin{aligned}
curl\vec{v} &= \nabla \times (\vec{w} \times \vec{r}) \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix} \\
&= i\left[\frac{\partial}{\partial y}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_3 x - w_1 z)\right] \\
&\quad - j\left[\frac{\partial}{\partial x}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_2 z - w_3 y)\right] \\
&\quad + k\left[\frac{\partial}{\partial x}(w_3 x - w_1 z) - \frac{\partial}{\partial y}(w_2 z - w_3 y)\right] \\
&= i\left[w_1 + w_1\right] - j\left[-w_2 - w_2\right] + k\left[w_3 + w_3\right] \\
&= 2\left[w_1 i + w_2 j + w_3 k\right] \\
&= 2\vec{w} \Rightarrow \vec{w} = \frac{1}{2}\,\text{curl}\,\vec{v}
\end{aligned}
$$

Problem 1.5.8. *Find divergence and curl of the vector* $\vec{V} = xyzi + 3x^2yj + (xz^2 - y^2z)k$ *at (2,-1,1).* *(VTU Jan 2017)*

Solution:

$$
\begin{aligned}
\vec{v} &= (xyz)i + \left(3x^2y\right)j + \left(xz^2 - y^2z\right)k \\
\text{Div. } \vec{v} &= \nabla \cdot \vec{v} \\
&= \left[\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right] \cdot \vec{v} \\
&= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}\left(3x^2y\right) + \frac{\partial}{\partial z}\left(xz^2 - y^2z\right) \\
&= yz + 3x^2 + 2xz - y^2 \\
\text{Div. } \vec{v}\,|_{(2,-1,1)} &= -1 + 12 + 4 - 1 = 14
\end{aligned}
$$

$$\begin{aligned}
\text{Curl } \vec{v} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} \\
&= -2yzi - \left(z^2 - xy\right) j + (6xy - xz)k \\
&= -2yzi + \left(xy - z^2\right) j + (6xy - xz)k \\
\text{Curl at } (2,-1,1) &= -2(-1)(1)i + \{(2)(-1) - 1\}j \\
&\quad + 6(2)(-1) - 2(1)k \\
&= 2\hat{i} - 3\hat{j} - 14\hat{k}
\end{aligned}$$

Problem 1.5.9. *Find curl (curl $\vec{A}$) where $\vec{A} = xy\hat{i} + y^2z\hat{j} + z^2y\hat{k}$*

$$\begin{aligned}
\vec{A} &= xy\hat{i} + y^2z\hat{j} + z^2y\hat{k} \\
\text{curl } \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2y \end{vmatrix} \\
&= \hat{i}\left(z^2 - y^2\right) - \hat{j}(0-0) + \hat{k}(0-x) \\
\text{curl } \vec{A} &= \left(z^2 - y^2\right)\hat{i} - 0j - x\hat{k} \\
\therefore \text{curl}(\text{curl } \vec{A}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & 0 & -x \end{vmatrix} \\
&= \hat{i}(0-0) - \hat{j}(-1-2z) + \hat{k}(0+2y) \\
\text{curl } (\text{curl } \vec{A}) &= 0i + (2z+1)j + 2yk
\end{aligned}$$

Problem 1.5.10. *If $F = (x+y+1)i + j - (x+y)k$, then show that $\vec{F} \cdot curl\vec{F} = 0$* **(VTU 2018, Jan 2014, July 2013)**

Solution:

$\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ and

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= \hat{i}[-1-0] - \hat{j}[-1-0)] + \hat{k}[0-1]$$

$$\text{curl } \vec{F} = -\hat{i} + \hat{j} - \hat{k}$$

$$\therefore \vec{F} \cdot \text{curl } \vec{F} = ((x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}) \cdot (-\hat{i} + \hat{j} - \hat{k})$$

$$= -(x+y+1) + 1 + (x+y)$$

$$= 0$$

Problem 1.5.11. *If* $\vec{r} = xi + yj + zk$ *and* $r = |\vec{r}|$ *then prove that* $\nabla r^n = nr^{n-2}\vec{r}$ **(VTU Jul 2015, Jul 2013)**

Solution: We have $\vec{r} = xi + yj + zk$ and $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\nabla r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(r^n)$$

$$= i\frac{\partial}{\partial x}(r^n) + j\frac{\partial}{\partial y}(r^n) + k\frac{\partial}{\partial z}(r^n)$$

$$= i\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + j\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + k\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \nabla r^n = nr^{n-1}\left[\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right]$$

$$= \frac{nr^{n-1}}{r}[xi + yj + zk]$$

$$= nr^{n-2}\vec{r}$$

Problem 1.5.12. *Prove that* $\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^3}\vec{r} = \frac{-1}{r^2}\hat{r}$

Solution: We have $\vec{r} = xi + yj + zk$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\nabla\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= i\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + j\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + k\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\left[\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right]$$

$$= -\frac{1}{r^3}(xi + yj + zk)$$

$$= -\frac{\vec{r}}{r^3} = -\frac{1}{r^2}\frac{\vec{r}}{r} = \frac{-1}{r^2}\hat{r}$$

Problem 1.5.13. *Compute curl of* $V(x, y, z) = xyz^2\hat{\imath} + xy^2z\hat{\jmath} + x^2yz\hat{k}$

$$\text{Curl}\, V = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & xy^2z & x^2yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(x^2yz\right) - \frac{\partial}{\partial z}\left(xy^2z\right)\right)\hat{\imath} + \left(\frac{\partial}{\partial z}\left(xyz^2\right) - \frac{\partial}{\partial x}\left(x^2yz\right)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}\left(xy^2z\right) - \frac{\partial}{\partial y}\left(xyz^2\right)\right)\hat{k}$$

$$= \left(x^2z - xy^2\right)\hat{\imath} + (2xyz - 2xyz)\hat{\jmath} + \left(y^2z - xz^2\right)\hat{k}$$

$$= \left(x^2z - xy^2\right)\hat{\imath} + \left(y^2z - xz^2\right)\hat{k}$$

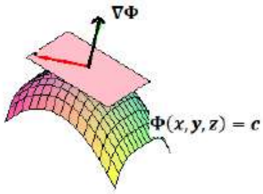
Normal vector :

If $\Phi(x, y, z)$ represents equation of a surface, then $\vec{\nabla}\Phi$ is a vector normal to the surface.

Thus

$$\boxed{\text{normal vector} = \vec{\nabla}\Phi}$$

$$\text{unit normal vector, } \hat{n} = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|}$$



Problem 1.5.14. *Find a unit normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.*

Solution: Given surface is

$$xy^3z^2 = 4$$

$$\Rightarrow \phi(x, y, z) = xy^3z^2$$

Hence the normal vector is given by

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = y^3z^2,$$

At the point $(-1, -1, 2)$,

$$\frac{\partial\phi}{\partial x} = (-1)^3 \cdot 2^2 = -4$$

$$\frac{\partial\phi}{\partial y} = 3(-1)(-1)^2 \cdot 2^2 = -12$$

$$\frac{\partial\phi}{\partial z} = 2(-1)(-1)^3 \cdot 2 = 4$$

$\therefore$ at the point $(-1, -1, 2)$,

$\nabla\phi = -4i - 12j + 4k = -4(i + 3j - k)$

$\therefore$ unit normal to the given surface at the point $(-1, -1, 2)$ is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{-4(i + 3j - k)}{4\sqrt{1 + 9 + 1}} = -\frac{(i + 3j - k)}{\sqrt{11}}$$

1.6 Angle between two surfaces :

If $\Phi_1(x, y, z) = c_1$ and $\Phi_2(x, y, z) = c_2$ are two surfaces, then the angle between these two surfaces is equal to the angle between their normals $\nabla\Phi_1$ and $\nabla\Phi_2$.

Hence the angle is given by

$$\boxed{cos\theta = \frac{\nabla\Phi_1 \cdot \nabla\Phi_2}{|\nabla\Phi_1|\,|\nabla\Phi_2|}}$$

Problem 1.6.1. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2, -1, 2)$.*

Solution: The given surfaces are

$$x^2 + y^2 + z^2 = 9 \cdots (1) \text{ and } x^2 + y^2 - z = 3 \cdots (2)$$

Let

$$\Phi_1 = x^2 + y^2 + z^2 \text{ and } \Phi_2 = x^2 + y^2 - z$$

Now, vector normal to the first surface is given by

$$\begin{aligned} \nabla\Phi_1 &= i\frac{\partial\Phi_1}{\partial x} + j\frac{\partial\Phi_1}{\partial y} + k\frac{\partial\Phi_1}{\partial z} \\ &= 2xi + 2yj + 2zk \\ \nabla\Phi_1 \,|_{(2,-1,2)} &= 4i - 2j + 4k \end{aligned}$$

Now, vector normal to the second surface is given by

$$\begin{aligned} \nabla\Phi_2 &= i\frac{\partial\Phi_2}{\partial x} + j\frac{\partial\Phi_2}{\partial y} + k\frac{\partial\Phi_2}{\partial z} \\ &= 2xi + 2yj - k \\ \nabla\Phi_2 \,|_{(2,-1,2)} &= 4i - 2j - k \end{aligned}$$

If θ is the angle between the surfaces (1) and (2) at $(2, -1, 2)$, then

$$\begin{aligned}\cos\theta &= \frac{\nabla\Phi_1 \cdot \nabla\Phi_2}{|\nabla\Phi_1||\nabla\Phi_2|}\\ &= \frac{(4i - 2j + 4k)\cdot(4i - 2j - k)}{\sqrt{16+4+16}\ \sqrt{16+4+1}}\\ &= \frac{16+4-4}{\sqrt{36}\sqrt{21}}\\ &= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}\\ \Rightarrow \theta &= \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)\end{aligned}$$

Problem 1.6.2. *Find the angle between the surfaces* $x\log z = y^2 - 1$ *and* $x^2y = 2 - z$ *at the point* $(1, 1, 1)$

Recall that angle between the surfaces at a point of their intersection is defined to be the angle between the normals to the surfaces at the point of intersection. Thus we need to find the normals to both the surfaces at $(1, 1, 1)$.

Let $\varphi_1(x, y, z) = x\log z - y^2 + 1$ and $\phi_2(x, y, z) = x^2y - 2 + z$

Normal to the first surface is given by

$$\begin{aligned}grad\varphi_1 &= \nabla\varphi_1\\ &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\\ &= \frac{\partial}{\partial x}\left(x\log z - y^2 + 1\right)\hat{i}\\ &\quad + \frac{\partial}{\partial y}\left(x\log z - y^2 + 1\right)\hat{j}\\ &\quad + \frac{\partial}{\partial z}\left(x\log z - y^2 + 1\right)\hat{k}\\ &= (\log z)\hat{i} + (-2y)\hat{j} + \left(\frac{x}{z}\right)\hat{k}\\ \therefore grad\varphi_1 |_{(1,1,1)} &= -2\hat{j} + \hat{k}\end{aligned}$$

Normal to the second surface is given by

$$\begin{aligned}
grad\phi_2 &= \nabla\phi_2 \\
&= \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} + \frac{\partial\phi_2}{\partial z}\hat{k} \\
&= \frac{\partial}{\partial x}\left(x^2y - 2 + z\right)\hat{i} + \frac{\partial}{\partial y}\left(x^2y - 2 + z\right)\hat{j} \\
&\quad + \frac{\partial}{\partial z}\left(x^2y - 2 + z\right)\hat{k} \\
&= 2xy\hat{i} + \left(x^2\right)\hat{j} + (1)\hat{k} \\
\nabla\phi_2 \,|_{(1,1,1)} &= 2\hat{i} + \hat{j} + \hat{k}
\end{aligned}$$

Angle between the surfaces is given by

$$\begin{aligned}
\cos\theta &= \frac{\nabla\Phi_1 \cdot \nabla\Phi_2}{|\nabla\Phi_1||\nabla\Phi_2|} \\
&= \frac{\left[0i - 2\hat{j} + \hat{k}\right] \cdot \left[2\hat{i} + \hat{j} + \hat{k}\right]}{\sqrt{0^2 + (-2)^2 + 1^2}\,\sqrt{2^2 + 1^2 + 1^2}} \\
&= \frac{0 - 2 + 1}{\sqrt{5}\,\sqrt{6}} = -\frac{1}{\sqrt{30}} \\
\Rightarrow \theta &= \cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)
\end{aligned}$$

Problem 1.6.3. *Find the angle between the normal to the surface $xy = z^2$ at (1,4,2) and (-3,-3,3)*

Solution: The given surface is $xy - z^2 = 0 \therefore$

$$\phi = xy - z^2$$

We know $\nabla\phi$ is normal to the surface at the point (x, y, z)

Let $\vec{n}_1, \vec{n}_2$, be the normals to the surface at the points $(1, 4, 2)$ and $(-3, -3, 3)$ respectively.

Now

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \qquad \cdots (1)$$

$$\frac{\partial\phi}{\partial x} = y, \quad \frac{\partial\phi}{\partial y} = x, \quad \text{and} \quad \frac{\partial\phi}{\partial z} = -2z$$

Substituting in (1) we get

$$\nabla\phi = yi + xj - 2zk$$

At the point $(1, 4, 2)$, $\nabla\phi = 4i + j - 4k$

and at the point $(-3, -3, 3)$, $\nabla\phi = -3i - 3j - 6k$

$\therefore\ \vec{n}_1 = 4i + j - 4k,\quad \vec{n}_2 = -3i - 3j - 6k$

If θ is the angle between the normals, then

$$\begin{aligned}\cos\theta &= \frac{\vec{n}_1}{|\vec{n}_1|} \cdot \frac{\vec{n}_2}{|\vec{n}_2|}\\ &= \frac{(i + j - 4k)}{\sqrt{16+1+16}} \cdot \frac{(-3i - 3j - 6k)}{\sqrt{9+9+36}}\\ &= \frac{-12 - 3 + 24}{\sqrt{33}\sqrt{54}}\\ &= \frac{9}{\sqrt{3}\ \sqrt{11}\ 3\sqrt{3}\ \sqrt{2}}\\ &= \frac{1}{\sqrt{22}}\end{aligned}$$

Hence $\theta = cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$

Problem 1.6.4. *Find the constants* a *&* b *so that the surface* $ax^2 - byz = (a+2)x$ *is orthogonal to the surface* $4x^2y + z^3 = 4$ *at the point* $(1, -1, 2)$. ***Ans :***

Solution: The given surfaces are

$$ax^2 - byz - (a+2)x = 0$$

$$4x^2y + z^3 - 4 = 0$$

$$\text{Let}\quad \phi_1 = ax^2 - byz - (a+2)x \qquad \cdots(1)$$

and

$$\phi_2 = 4x^2y + z^3 - 4 \qquad \cdots(2)$$

Given that surfaces (1) and (2) cut orthogonally at the point $(1, -1, 2)$.

$$\therefore\ \nabla\phi_1 \cdot \nabla\phi_2 = 0$$

we have

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$\text{where } \frac{\partial \phi_1}{\partial x} = 2ax - a - 2,$$
$$\frac{\partial \phi_1}{\partial y} = -bz \text{ and}$$
$$\frac{\partial \phi_1}{\partial z} = -by$$
$$\therefore \nabla \phi_1 = (2ax - a - 2)i - bzj - byk$$

Similarly,

$$\nabla \phi_2 = i\frac{\partial \phi_2}{\partial x} + j\frac{\partial \phi_2}{\partial y} + k\frac{\partial \phi_2}{\partial z}$$
$$\text{where } \frac{\partial \phi_2}{\partial x} = 8xy,$$
$$\frac{\partial \phi_2}{\partial y} = 4x^2$$
$$\text{and } \frac{\partial \phi_2}{\partial z} = 3z^2$$
$$\therefore \nabla \phi_2 = 8xyi + 4x^2j + 3z^2k$$

At the point (1,-1,2),

$$\nabla \phi_1 = (2a - a - 2)i - b(2)j - b(-1)k$$
$$= (a-2)i - 2bj + bk \text{ and}$$
$$\nabla \phi_2 = -8i + 4j + 12k$$
$$\nabla \phi_1 \cdot \nabla \phi_2 = 0$$
$$\Rightarrow ((a-2)i - 2bj + bk) \cdot (-8i + 4j + 12k) = 0$$
$$\Rightarrow -8(a-2) - 8b + 12b = 0$$
$$\Rightarrow -8a + 4b + 16 = 0$$
$$\Rightarrow \quad 2a - b = 4 \qquad \cdots (3)$$

Since the point of intersection $(1, -1, 2)$ is a common point on the surfaces (1) and (2), it should satisfy (1). Hence we get

$$\begin{aligned} a + 2b - (a + 2) &= 0 \\ \Rightarrow \quad 2b &= 2 \\ \Rightarrow \quad b &= 1 \\ \therefore \Rightarrow \quad 2a &= 4 + b = 4 + 1 = 5 \text{ (from (3))} \\ \Rightarrow \quad a &= \frac{5}{2} \\ \therefore \quad a &= \frac{5}{2}, \quad b = 1 \end{aligned}$$

1.7 Directional Derivative :

The directional derivative of a scalar point function ϕ in a given direction $\vec{a}$ is the rate of change of ϕ in that direction. It is given by the component of $\nabla\phi$ in the direction of $\vec{a}$

$$\therefore \text{ the directional derivative } = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

We can write

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{|\nabla\phi|\,|\vec{a}|}{|\vec{a}|} \cos\theta = |\nabla\phi| \cos\theta$$

, where θ is the angle between $\nabla\phi$ and $\vec{a}$.

So, the directional derivative at a given point is maximum if $\cos\theta$ is maximum. i.e., $\cos\theta = 1 \Rightarrow \theta = 0$

$\therefore$ **The maximum directional derivative at a point is in the direction of $\nabla\phi$ and the maximum directional derivative is $|\nabla\phi|$.**

Problem 1.7.1. *Find the directional derivative of* $\Phi = x^2yz + 4xz^2$ *at* $(1, -2, -1)$ *in the direction of the vector* $2i - j - 2k$ **(VTU Aug 2005, Aug 2004, Mar 2000)**

Solution: Given

$$\phi(x, y, z) = x^2yz + 4xz^2$$

We know

$$\text{grad}\,\phi = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\phi}{\partial y} = x^2z, \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz$$

At the point $(1, -2, -1)$,

$$\frac{\partial\phi}{\partial x} = 2\cdot 1(-2)(-1) + 4(-1)^2 = 8$$

$$\frac{\partial\phi}{\partial y} = 1^2\cdot(-1) = -1$$

$$\frac{\partial\phi}{\partial z} = 1^2(-2) + 8\cdot 1(-1) = -2 - 8 = -10$$

$\therefore$ at the point $(1, -2, -1)$, $\nabla\phi = 8i - j - 10k$

Given direction is $\vec{a} = 2i - j - 2k$

$\therefore$ the directional derivative of ϕ at the point $(1, -2, -1)$ in the direction of $\vec{a}$ is given by

$$\nabla\phi\cdot\frac{\vec{a}}{|\vec{a}|} = (8i - j - 10k)\cdot\frac{(2i - j - 2k)}{\sqrt{4+1+4}} = \frac{16+1+20}{\sqrt{9}} = \frac{37}{3}$$

Problem 1.7.2. *Find the maximum value of the directional derivative of* $\phi = x^3yz$ *at the point* $(1, 4, 1)$

Solution: Given

$$\phi = x^3yz$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

We know, The directional derivative is maximum in the direction of $\nabla\phi$ and the maximum value $= |\nabla\phi|$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 3x^2yz, \quad \frac{\partial\phi}{\partial y} = x^3z, \quad \frac{\partial\phi}{\partial z} = x^3y$$

At the point $(1, 4, 1)$,
$\frac{\partial\phi}{\partial x} = 3(1)(4)(1) = 12,$
$\frac{\partial\phi}{\partial y} = 1^3(1) = 1,$
$\frac{\partial\phi}{\partial z} = 1^3(4) = 4$
$\therefore$ at the point $(1, 4, 1)$, $\quad \nabla\phi = 12i + j + 4k$
Maximum value of the directional derivative is

$$\begin{aligned} \mid \nabla\phi \mid &= |12i + j + 4k| \\ &= \sqrt{144 + 1 + 16} \\ &= \sqrt{161} \end{aligned}$$

Problem 1.7.3. *Find the directional derivative of* $f = xy^2 + yz^3$ *at the point* $(2, -1, 1)$ *in the direction of the normal to the surface* $xlogz - y^2 = -4$ *at the point* $(-1, 2, 1)$. **(VTU Feb 2004)**

Solution:: Given

$$\begin{aligned} f &= xy^2 + yz^3 \\ \therefore \nabla f &= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \\ &= y^2 i + \left(2xy + z^3\right) j + 3yz^2 k \end{aligned}$$

At the point $(2, -1, 1)$,

$$\nabla f = (-1)^2 i + (-4 + 1)j + 3(-1)1^2 k$$

$$\Rightarrow \nabla f = i - 3j - 3k$$

The directional derivative of f in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point $(-1, 2, 1)$ is required.
Let $g = x \log z - y^2 + 4$.
Normal to this surface is,

$$\begin{aligned} \nabla g &= i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z} \\ &= \log z i - 2yj + \frac{x}{z}k \end{aligned}$$

At the point $(-1, 2, 1)$,

$$\begin{aligned}\vec{a} &= \nabla g \\ &= \log 1i - 4j + \left(\frac{-1}{1}\right) k \\ &= 0i - 4j - k = -4j - k \\ \therefore \vec{a} &= -4j - k\end{aligned}$$

Required directional derivative is

$$\begin{aligned}\nabla f \cdot \frac{\vec{a}}{|\vec{a}|} &= (i - 3j - 3k) \cdot \frac{(-4j - k)}{\sqrt{16+1}} \\ &= \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}\end{aligned}$$

Problem 1.7.4. *Find the directional derivative of $\varphi(x, y, z) = 2xy + 3y^2 - z^3$ at the point $P(1, -1, 2)$ in the direction of the vector $\vec{v} = \hat{i} - \hat{j} + \hat{k}$.*

Solution: First we find that

$$\begin{aligned}\operatorname{grad} \varphi &= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \\ &= 2y\hat{i} + (2x + 6y)\hat{j} + \left(-3z^2\right)\hat{k} \\ \nabla \varphi|_{(1,-1,2)} &= -2\hat{i} - 4\hat{j} - 12\hat{k}\end{aligned}$$

The directional derivative of ϕ in the direction of v is given by

$$\begin{aligned}\nabla \varphi \cdot \frac{\vec{v}}{|\vec{v}|} &= (-2\hat{i} - 4\hat{j} - 12\hat{k}) \cdot \left(\frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}\right) \\ &= \frac{-2 + 4 - 12}{\sqrt{3}} \\ &= -\frac{10}{\sqrt{3}}\end{aligned}$$

12. Find the directional derivative of the function $\phi = xy + yz + zx$ in the direction of the vector $2i + 3j + 6k$ at the point $(3, 1, 2)$. Ans : $45/7$

13. Find the directional derivative of the function $2yz + z^2$ in the direction of the vector $i + 2j + 2k$ at the point $(1, -1, 3)$ Ans : $\frac{20}{3}$

14. Find the directional derivative of $x^3 + y^3 + z^3$ at the point $(1, -1, 2)$ in the direction of $i + 2j + k$ Ans : $\frac{7\sqrt{6}}{2}$

1.8 Solenoidal and irrotational vector fields

A vector $\vec{F}$ said to be **solenoidal** if $div\vec{F} = 0$. i. e.

$$\boxed{\nabla \cdot \vec{F} = 0}$$

A vector $\vec{F}$ said to be **irrotational** if $curl\vec{F} = \vec{0}$. i. e.

$$\boxed{\nabla \times \vec{F} = \vec{0}}$$

Note : (1) An irrotational vector field is also called as conservative field or potential field.

(2) If $\vec{F}$ is irrotational, then there always exists a scalar function Φ such that $\nabla\phi = \vec{F}$

Problem 1.8.1. *Show that* $V(x, y, z) = (4xy - z^3)\,\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$ *is irrotational*

Solution:

$$\begin{aligned} Curl\ \vec{V} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}\left(-3xz^2\right) - \frac{\partial}{\partial z}\left(2x^2\right)\right)\hat{i} \\ &\quad + \left(\frac{\partial}{\partial z}\left(4xy - z^3\right) - \frac{\partial}{\partial x}\left(-3xz^2\right)\right)\hat{j} \\ &\quad + \left(\frac{\partial}{\partial x}\left(2x^2\right) - \frac{\partial}{\partial y}\left(4xy - z^3\right)\right)\hat{k} \\ &= (0-0)\hat{i} + \left(-3z^2 - \left(-3z^2\right)\right)\hat{j} + (4x - 4x)\hat{k} = 0 \end{aligned}$$

Hence $\vec{V}$ is irrotational.

Problem 1.8.2. *Show that* $V(x, y, z) = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}$ *is solenoidal.*

Solution:

$$\begin{aligned} \text{div}\ \vec{V} &= \nabla \cdot \vec{V} \\ &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}\right) \\ &= \frac{\partial}{\partial x}\left(3y^4z^2\right) + \frac{\partial}{\partial y}\left(4x^3z^2\right) + \frac{\partial}{\partial z}3\left(x^2y^2\right) \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

Hence, $\boldsymbol{V}$ is solenoidal.

Problem 1.8.3. *Find the value of* $\boldsymbol{a}$ *so that* $\boldsymbol{V}(x, y, z) = (x+3y)\hat{\imath}+(y-2z)\hat{\jmath}+(x-az)\hat{k}$ *is solenoidal.*

Solution: Given that $\vec{V}$ is solenoidal.

Hence div $\boldsymbol{V} = 0$

$$\Rightarrow \nabla \cdot \boldsymbol{V} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot ((x+3y)\hat{i} + (y-2z)\hat{j} + (x-az)\hat{k}) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x-az) = 0$$

$$\Rightarrow 1+1-a = 0$$

$$\Rightarrow a = 2$$

Problem 1.8.4. *Prove that* $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ *is both solenoidal and irrotational.* *(VTU Jan 2016)*

Solution: For solenoidal, we have to prove $\nabla \cdot \vec{F} = 0$.

Now,

$\nabla \cdot \vec{F}$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left[\begin{array}{c} (y^2 - z^2 + 3yz - 2x)\,\hat{i} \\ +(3xz + 2xy)\hat{j} \\ +(3xy - 2xz + 2z)\hat{k} \end{array}\right]$$

$$= \frac{\partial}{\partial x}\left(y^2 - z^2 + 3yz - 2x\right) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= (-2) + (2x) + (-2x + 2)$$

$$= 0$$

Thus, $\vec{F}$ is solenoidal.

For irrotational, we have to prove Curl $\vec{F} = 0$.

$$\begin{aligned}
&\text{Curl } \vec{F} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix} \\
&= (3x - 3x)\hat{i} - (-2z + 3y - 3y + 2z)\hat{j} \\
&\qquad + (3z + 2y - 2y - 3z)k \\
&= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0
\end{aligned}$$

Hence $\vec{F}$ is irrotational.

Problem 1.8.5. *Find a, b, c if* $(x+y+az)i+(bx+2y-z)j+(-x+cy+2z)k$ *is irrotational.*

Solution: Let $\vec{F} = (x + y + az)i + (bx + 2y - z)j + (-x + cy + 2z)k$
Given $\vec{F}$ is irrotational.

$$\begin{aligned}
\therefore \quad & \nabla \times \vec{F} = \vec{0} \\
& \Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & -x + cy + 2z \end{vmatrix} = \vec{0} \\
& \Rightarrow i\left[\frac{\partial}{\partial y}(-x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z)\right] \\
& \quad - j\left[\frac{\partial}{\partial x}(-x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az)\right] \\
& \quad + k\left[\frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az)\right] = 0 \\
& \Rightarrow i(c + 1) - j(-1 - a) + k(b - 1) = \vec{0} \\
& \Rightarrow (c + 1)i + (1 + a)j + (b - 1)k = \vec{0} \\
& \Rightarrow c + 1 = 0, 1 + a = 0, b - 1 = 0 \\
& \Rightarrow a = -1, b = 1 \text{ and } c = -1
\end{aligned}$$

Problem 1.8.6. *Prove that* $\frac{xi+yj}{x^2+y^2}$ *is both solenoidal & irrotational.* *(VTU Model 2014)*

Solution::

$$\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$$
$$= \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$$
$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$
$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$
$$= \frac{(y^2-x^2)}{(x^2+y^2)^2} + \frac{(x^2-y^2)}{(x^2+y^2)^2}$$
$$= 0$$

$\Rightarrow \vec{F}$ is solenoidal.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$
$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}\left(\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right)$$
$$= (0)\hat{i} + (0)\hat{j} + (0)\hat{k} = \vec{0}$$

curl $\vec{F} = \vec{0} \Rightarrow \vec{F}$ is irrotational.

Problem 1.8.7. *A vector field is given by* $\vec{A} = (x^2 + xy^2)i + (y^2 + x^2y)j$. *Show that the field is irrotational and find the scalar potential.* *(VTU Jan 2017)*

Solution:

$$Curl\vec{A} = \nabla \times \vec{A}$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$
$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy - 2xy) = 0$$

Hence, $\vec{A}$ is irrotational. If ϕ is the scalar potential, then

$$\vec{A} = \text{grad}\,\phi$$

$$i.e.\ \vec{A} = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$(x^2 + xy^2)i + (y^2 + x^2y)j = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

Equating the components on both sides

$$\frac{\partial\phi}{\partial x} = (x^2 + xy^2) \qquad \cdots(1)$$

$$\frac{\partial\phi}{\partial y} = (y^2 + x^2y) \qquad \cdots(2)$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad \cdots(3)$$

Integrating (1), only w.r.to x, treating y and z as constants,

$$\phi = \frac{x^3}{3} + \frac{x^2}{2}y^2 + f(y,z) \qquad \cdots(4)$$

Integrating (1), only w.r.to y, treating x and z as constants,

$$\phi = \frac{y^3}{3} + \frac{y^2}{2}x^2 + g(x,z) \qquad \cdots(5)$$

Integrating (1), only w.r.to z, treating x and y as constants,

$$\phi = 0 + h(x,y) \qquad \cdots(6)$$

Equating all the expressions for ϕ from equations (4), (5) and (6), we get

$$\frac{x^3}{3} + \frac{x^2}{2}y^2 + f(y,z) = \frac{y^3}{3} + \frac{y^2}{2}x^2 + g(x,z) = 0 + h(x,y)$$

Comparing we get

$$f(y,z) = \frac{y^3}{3} + 0$$

$$g(x,z) = \frac{x^3}{3} + 0$$

$$h(x,y) = \frac{y^3}{3} + \frac{y^2}{2}x^2 + \frac{x^3}{3}$$

Substituting these values in (4), (5) and (6), we get a unique expression for ϕ as

$$\phi = \frac{y^3}{3} + \frac{y^2}{2}x^2 + \frac{x^3}{3}$$

Problem 1.8.8. *Show that* $\vec{F} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ *is irrotational and hence find its scalar potential.* **(VTU July 2015, Jan 2013)**

Solution: let

$$\vec{f} = \left(x^2 - yz\right) i + \left(y^2 - zx\right) j + \left(z^2 - xy\right) k$$

$$\text{Then curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \sum i(-x + x) = \vec{0}$$

$\therefore \vec{f}$ is Irrotational. Then there exists ϕ such that $\vec{f} = \nabla\phi$

$$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = \left(x^2 - yz\right) i + \left(y^2 - zx\right) j + \left(z^2 - xy\right) k$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int \left(x^2 - yz\right) dx = \frac{x^3}{3} - xyz + f_1(y, z) \qquad \cdots (1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z, x) \qquad \cdots (2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x, y) \qquad \cdots (3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = \frac{x^3 + y^3 + z^3}{3} - xyz + \text{Constant}$$

Which is the required scalar potential.

Problem 1.8.9. *Find the constants* a *and* b *such that* $F = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ *is irrotational. Find* Φ *such that* $F = \nabla\Phi$ **(VTU July 2014, Jan 2014, Jun 2012, Feb 2005)**

Solution:: Given that $\boldsymbol{F}$ is irrotational. Hence

$$\nabla \times \vec{F} = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = 0$$

$$\Rightarrow i(-1+1) - \hat{j}\left(bz^2 - 3z^2\right) + \hat{k}(6x - ax) = 0$$

$$\Rightarrow 0i - z^2(b-3)\hat{j} + x(6-a)\hat{k} \qquad = 0$$

$$\Rightarrow b = 3 \text{ and } a = 6$$

$$\therefore \vec{F} = \left(6xy + z^3\right)\hat{i} + \left(3x^2 - z\right)\hat{j} + \left(3xz^2 - y\right)\hat{k}$$

Given, $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \left(6xy + z^3\right)\hat{i} + \left(3x^2 - z\right)\hat{j} + \left(3xz^2 - y\right)\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy + z^3 \Rightarrow \phi = 3x^2y + xz^3 + f_1(y,z) \qquad \cdots(1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \Rightarrow \phi = 3x^2y - yz + f_2(x,z) \qquad \cdots(2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \Rightarrow \phi = xz^3 - yz + f_3(x,y) \qquad \cdots(3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = 3x^2y + xz^3 - yz + c$$

Problem 1.8.10. *Find the constants* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *such that the vector field* $\vec{f} = (x + y + az)i + (x + cy + 2z)k + (bx + 2y - z)j$ *is irrorational. Also find* Φ *such that* $\vec{F} = \nabla\Phi$ **(VTU Jul 2015)**

Solution: Given that the vector field is irrotational. Therefore $\nabla \times \vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & x + cy + 2z \end{vmatrix} = 0$$

$$\hat{i}(c+1) - \hat{j}(1-a) + \hat{k}(b-1) = 0$$

$c + 1 = 0, \quad 1 - a = 0, \quad b - 1 = 0$ Hence $\vec{f} = (x+y+z)i + (x+2y-$

$c = -1, \quad a = 1, \quad b = 1$

$z)j + (x - y + 2z)k$

Then there exists ϕ such that $\vec{f} = \nabla\phi$

$$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$
$$= (x+y+z)i + (x+2y-z)j + (x-y+2z)k$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x+y+z \Rightarrow \phi = \frac{x^2}{2} + xy + xz + f_1(y,z) \qquad \cdots(1)$$
$$\frac{\partial\phi}{\partial y} = x+2y-z \Rightarrow \phi = xy + 2\frac{y^2}{2} - zy + f_2(x,z) \qquad \cdots(2)$$
$$\frac{\partial\phi}{\partial z} = x-y+2z \Rightarrow \phi = xz - yz + 2\frac{z^2}{2} + f_3(x,y) \qquad \cdots(3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2 + \text{constant}$$

Problem 1.8.11. *Find constants a, b and c if the vector*
$\vec{f} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$ *is Irrotational.*

Solution: Given

$$\vec{f} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$$

$$\text{Curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$
$$= (c-3)i - (2-a)j + (b-3)k$$

If the vector is irrotational then curl $\vec{f} = \vec{0}$

$$\therefore 2-a=0 \Rightarrow a=2, b-3=0 \Rightarrow b=3, c-3=0 \Rightarrow c=3$$

Problem 1.8.12. *Find the constants a, b, c such that the vector field $\vec{f} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$ is irrorational. Hence find its scalar potential.* *(VTU July 2014)*

Solution:: Given vector is $\vec{A} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is Irrotational.

$$\Rightarrow \text{curl } \vec{A} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow (c + 1)i + (a - 4)j + (b - 2)k = \vec{0}$$

$$\Rightarrow (c + 1)i + (a - 4)j + (b - 2)k = 0i + 0j + 0k$$

Comparing both sides,

$$c + 1 = 0, a - 4 = 0, b - 2 = 0$$

$$c = -1, a = 4, b = 2$$

Now $\vec{A} = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k$,

on substituting the values of a, b, c

we have $\vec{A} = \nabla\phi$.

$$\Rightarrow \vec{A} = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k$$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + bk\frac{\partial\phi}{\partial z}$$

Comparing both sides, we have

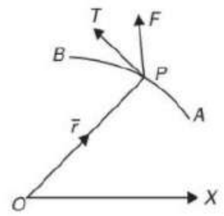
$$\frac{\partial\phi}{\partial x} = x + 2y + 4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4zx + f_1(y, z)$$

$$\frac{\partial\phi}{\partial y} = 2x - 3y - z \Rightarrow \phi = 2xy - 3y^2/2 - yz + f_2(x, z)$$

$$\frac{\partial\phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2 + f_3(y, x)$$

Hence $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

1.9 Line Integrals-Vector line Integral:



Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.
Line integral of a vector function $\vec{F}$ along the curve $C = AB$ is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

$$\therefore \boxed{\text{Line integral } = \int_c \vec{F} \cdot d\vec{R} = \int_c \vec{F} \cdot d\vec{r}}$$

Note : When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$

If $F(R)$or$\vec{F} = f_1(x, y, z)i + f_2(x, y, z)j + f_3(x, y, z)k$

and $\vec{r} = xi + yj + zk$

Then $d\vec{R} = dxi + dyj + dzk$

and $\displaystyle\int_c \vec{F} \cdot d\vec{r} = \int_c (f_1dx + f_2dy + f_3dz)$

This is a scalar.

Note:

- **Circulation :** If $\vec{F}$ represents the velocity of a fluid particle then the line integral $\int_C \vec{F} \cdot d\vec{R}$ is called the circulation of $\vec{F}$ around the curve. When the circulation of $\vec{F}$ around every closed curve in a region E vanishes, $\vec{F}$ is said to be irrotational in E.

- **Work :** If $\vec{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement $\delta R = \vec{F} \cdot \delta R$. Therefore, the total work done by $\vec{F}$ during the displacement from A to B is given by the integral $\int_A^B \vec{F} \cdot d\vec{R}$

Problem 1.9.1. *If* F $= \left(5xy - 6x^2\right) i + (2y - 4x)j$, *evaluate* $\int_C F \cdot dr$ *along the curve* C *the* xy *-plane,* $y = x^3$ *from the point* $(1, 1)$ *to* $(2, 8)$.

Solution:: Given $\vec{F} = \left(5xy - 6x^2\right)\hat{i} + (2y - 4x)\hat{j}$ Let

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ then}$$
$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$
$$\vec{F} \cdot d\vec{r} = \left(5xy - 6x^2\right) dx + (2y - 4x)dy$$
$$\int_C \vec{F} \cdot dr = \int_C \left(5xy - 6x^2\right) dx + (2y - 4x)dy$$

Along the curve $y = x^3$ $\left(\therefore dy = 3x^2dx\right)$, x varies from $x = 1$ to $x = 2$ and we get

$$\begin{aligned}
\int_C F \cdot dr &= \int_1^2 \left(5x\left(x^3\right) - 6x^2\right) dx + \left(2x^3 - 4x\right) 3x^2dx \\
&= \int_1^2 \left(5x^4 - 6x^2 + 6x^5 - 12x^3\right) dx \\
&= \left[5\frac{x^5}{5} - 6\frac{x^3}{3} + 6\frac{x^6}{6} - 12\frac{x^4}{4}\right]_1^2 \\
&= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2 \\
&= (32) - (3) = 35
\end{aligned}$$

Problem 1.9.2. *Find the total work done in moving a particle in the force field* $F = 3xyi - 5zj + 10xk$ *along the curve* $x = t^2 + 1, y = 2t^2, z = t^3$ *from* $t = 1$ *to* $t = 2$ *(VTU June 2019)*

Solution:: We have $\vec{r} = xi + yj + zk$ will give $d\vec{r} = dxi + dyj + dzk$

$$\begin{aligned}
&\vec{F} \cdot \vec{dr} \\
&= (3xyi - 5zj + 10xk) \cdot (dxi + dyi + dzk) \\
&= 3xydx - 5zdy + 10xdz \\
&= 3(t^2 + 1)(2t^2)(2tdt) - 5(t^3)(4tdt) + 10(t^2 + 1)(3t^2dt) \\
&= (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)dt
\end{aligned}$$

$$\begin{aligned}
\vec{F}\cdot\vec{dr} &= (12t^5+10t^4+12t^3+30t^2)dt \\
\int_C F\cdot dr &= \int_1^2 (12t^5+10t^4+12t^3+30t^2)dt \\
&= \left[12\,\frac{t^6}{6}+10\,\frac{t^5}{5}+12\,\frac{t^4}{4}+30\,\frac{t^3}{3}\right]_1^2 \\
&= 12\left[\frac{2^6-1}{6}\right]+10\left[\frac{2^5-1}{5}\right]+12\left[\frac{2^4-1}{4}\right] \\
&\quad +30\left[\frac{2^3-1}{3}\right] \\
&= 2(63)+2(31)+3(15)+10(7) \\
&= 303
\end{aligned}$$

Problem 1.9.3. *If* $\vec{F} = xyi + yzj + zxk$, *Evaluate* $\int_c \vec{F}\cdot\vec{dr}$. *Where C is the curve represented by* $x = t, y = t^2, z = t^3, \quad -1 \le t \le 1$ *(VTU Jan 2020)*

Solution::we have $\vec{F} = xyi + yzj + zxk$ and $\vec{r} = xi + yj + zk$ will give
$d\vec{r} = dxi + dyj + dzk$
$\vec{F}\cdot d\vec{r} = xydx + yzdy + zxdz$ since $x = t, y = t^2, z = t^3$,
$dx = dt, dy = 2tdt, dz = 3t^2dt$

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_C \vec{F}\cdot\frac{dr}{dt}dt \\
&= \int_C (xyi+yzj+zxk)\cdot\left(1i+2tj+3t^2k\right)dt \\
&= \int_C \left(xy+2yzt+3zxt^2\right)dt
\end{aligned}$$

$$\begin{aligned}\int_C \vec{F}\cdot dr &= \int_C \left(t(t^2)+2(t^2)(t^3)t+3(t^3)(t)t^2\right)dt\\ &= \int_{-1}^{1}\left(t^3+2t^6+3t^6\right)dt\\ &= \int_{-1}^{1}\left(t^3+5t^6\right)dt\\ &= \left[\frac{t^4}{4}+\frac{5t^7}{7}\right]_{-1}^{1}\\ &= \left[\left(\frac{1}{4}+\frac{5}{7}\right)-\left(\frac{1}{4}-\frac{5}{7}\right)\right]\\ &= \frac{27}{28}+\frac{13}{28}=\frac{40}{28}=\frac{10}{7}\end{aligned}$$

Problem 1.9.4. *If* $F = (3x^2+6y)\,i - 14yzj + 20xz^2k$. *Evaluate* $\int F\cdot dr$ *from* $(0,0,0)$ *to* $(1,1,1)$ *along the path* $x=t, y=t^2, z=t^3$.

Solution:: Given path is $x=t, y-t^2, z=t^3$ Therefore, $dx=dt, dy=2tdt$ and $dz=3t^2dt$

Also, (x,y,z) varies from $(0,0,0)$ to $(1,1,1) \Rightarrow t=0$ to $t=1$. Thus,

$$\begin{aligned}\int_C F\cdot dr &= \int_C (3x^2+6y)\,dx - 14yzdy + 20xz^2dz\\ &= \int_{i=0}^{1}\left(3t^2+6t^2\right)(dt) - 14(t^2)(t^3)(2tdt) + 20(t)(t^6)(3t^2dt)\\ &= \int_{t=0}^{1}\left(9t^2-28t^6+60t^9\right)dt\\ &= \left(3t^3-4t^7+6t^{10}\right)_0^1 = [3-4+6]=5\end{aligned}$$

Problem 1.9.5. *Find the work done by the force* $\vec{F}=(2y+3)i+xzj+(yz-x)k$ *to shift the particle from* $(0,0,0)$ *to* $(2,1,1)$ *along the curve* $x=2t^2, y=t, z=t^3$.

Solution:: $\vec{r}=xi+yj+zk$ and $d\vec{r}=dxi+dyj+dzk$

Work done is given by

$$W=\int_C F\cdot dr = \int_C (2y+3)dx + xzdy + (yz-x)dz \cdots (1)$$

Equation to the given curve is $x=2t^2, y=t, z=t^3$; so that $dx=4tdt, dy=dt$ and $dz=3t^2dt$

further $(x, y, z) = (0,0,0) \to (2,1,1) \Rightarrow t = 0 \to t = 1$. Substituting in (1), we get

$$\begin{aligned} W &= \int_0^1 (2t+3)4tdt + 2t^2t^3dt + \left(t(t^3) - 2t^2\right)3t^2dt \\ &= \int_0^1 \left(8t^2 + 12t + 2t^5 + 3t^6 - 6t^4\right)dt \\ &= \left(\frac{8t^3}{3} + 6t^2 + \frac{t^3}{3} + \frac{3t^7}{7} - \frac{6t^5}{5}\right)_0^1 \\ &= \left(\frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right) = \frac{288}{35} \end{aligned}$$

$\therefore$ work done $= \frac{288}{35}$.

Problem 1.9.6. *Find the total work done by the force represented by* $\vec{F} = 3xyi - yj + 2zxk$ *in moving a particle round the circle* $x^2 + y^2 = 4$ *(VTU-2010)*

Solution:: Total work done is

$$W = \int_C \vec{F} \cdot \vec{dr}$$

$x^2 + y^2 = 4$ can be represented in the parameter from $x = rcos\theta, y = r\sin\theta$

$x = 2\cos\theta \quad y = 2\sin\theta$ and $z = 0, 0 \leq \theta \leq 2\pi$

$dx = -2\sin\theta d\theta \quad dy = 2\cos\theta d\theta$

$$\text{work done } = \int_C \vec{F}\cdot d\vec{r} = \int_C 3xydx - ydy + 2zxdz$$
$$= \int_{\theta=0}^{2\pi} 3(4\cos\theta\sin\theta)(-2\sin\theta d\theta) - 2\sin\theta(2\cos\theta d\theta) + 0$$
$$= -24\int_{\theta=0}^{2\pi} \sin^2\theta\cos\theta d\theta - 4\int_{\theta=0}^{2\pi} \sin\theta\cos\theta d\theta$$
$$= -24\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} - 2\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi}$$

In the first term, We substitute $\sin\theta = t,$
$\Rightarrow \cos\theta d\theta = dt$
$\Rightarrow \int t^2dt = \frac{t^3}{3}$

$$= \frac{-24}{3}\left[\sin^3 2\pi - \sin^3 0\right] + \left[\cos 4\pi - \cos 0\right]$$
$$= 0 + (1-1) = 0$$

Problem 1.9.7. *Find the work done in moving a particle in the force field* $F = 3x^2i + (2xz - y)j + zk$, *along a) The straight line from (0,0,0) to (2,1,3) b) The curve defined by* $x^2 = 4y, 3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. *(VTU-2017)*

Solution:

$$\text{work done} = \int_C \vec{F}\cdot d\vec{r}$$
$$= \int_C \left(3x^2i + (2xz - y)j + zk\right)\cdot(dxi + dyj + dzk)$$
$$= \int_C \left(3x^2\right)dx + (2xz - y)dy + zdz \qquad \cdots(1)$$

a) The equations of the straight line from (0,0,0) to (2,1,3) are $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t($ say)

Then $x = 2t, y = t, z = 3t$ are its parametric equations.

and $dx = 2dt$, $dy = dt$ and $dz = 3dt$

The points $(0, 0, 0)$ to $(2, 1, 3)$ corresponds to $t = 0$ and $t = 1$ respectively.

$$\begin{aligned}
\text{workdone} &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_C \left(3x^2\right) dx + (2xz - y)dy + zdz \\
&= \int_0^1 \left[(3(2t)^2 \, (2dt) + [2(2t)(3t) - t] \, dt \right. \\
&\qquad \left. +(3t)(3dt)\right] \\
&= \int_0^1 \left(36t^2 + 8t\right) dt = 16
\end{aligned}$$

b) Let $x = t$ in $x^2 = 4y$ and $3x^3 = 8z$.

Then the parametric equations of C are

$$x = t, \quad y = \frac{t^2}{4}, \quad z = \frac{3t^3}{8} \text{ and}$$

$$dx = dt, \quad dy = \frac{2tdt}{4} = \frac{t}{2}dt, \quad dz = \frac{9t^2}{8}dt$$

and t varies from 0 to 2

$$\begin{aligned}
\text{Work done} &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_C \left(3x^2\right) dx + (2xz - y)dy + zdz \\
&= \int_0^2 3t^2 dt + \left(2(t)\,\frac{3t^3}{8} - \frac{t^2}{4}\right) \frac{tdt}{2} + \left(\frac{3t^3}{8}\right) \frac{9t^2}{8} dt \\
&= \int_0^2 \left[3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right] dt \\
&= t^3 - \frac{t^4}{32} + \frac{17}{128}t^6 \Big|_0^2 = 16
\end{aligned}$$

Problem 1.9.8. *A vector field is given by* $\vec{F} = sinyi + x(1 + cosy)j$. *Evaluate the line integral over a circular path given by* $x^2 + y^2 = a^2, z = 0$ *[VTU Jan 2018]*

Solution:: We have,

$$\text{Work done} = \int_C \vec{F} \cdot \vec{dr}$$
$$= \int_C [(\sin y)\hat{i} + x(1+\cos y)\hat{j}] \cdot \left[dx\hat{i} + dy\hat{j} + 0k\right]$$
$$(\because z = 0, \text{ hence } dz = 0)$$
$$= \int_C \sin y dx + x(1+\cos y)dy$$
$$\text{Work done} = \int_C (\sin y dx + x\cos y dy + x dy)$$

The parametric equations of given path $x^2 + y^2 = a^2$ are $x = a\cos\theta, y = a\sin\theta$,

$dx = -a\sin\theta d\theta$ and $dy = a\cos\theta d\theta$

and θ varies from 0 to 2π

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\sin y dx + x\cos y dy + x dy)$$
$$= \int_C d(x\sin y) + \int_C x dy$$
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} d[a\cos\theta \sin(a\sin\theta)]d\theta$$
$$+ \int_0^{2\pi} a\cos\theta \cdot a\cos\theta d\theta$$
$$= \int_0^{2\pi} d[a\cos\theta \sin(a\sin\theta)] + \int_0^{2\pi} a^2\cos^2\theta \, d\theta$$
$$= [a\cos\theta \sin(a\sin\theta)]_0^{2\pi} + \int_0^{2\pi} a^2\cos^2\theta d\theta$$
$$= 0 + a^2 \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$
$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$
$$= \frac{a^2}{2} \cdot 2\pi = \pi a^2$$

Problem 1.9.9. *If $F = 3xyi - y^2j$. Evaluate $\int F.dR$, where C is the curve in the xy-plane $y = 2x^2$ from (0,0) to $(1,2)$* *[VTU 2010]*

Solution: Since the particle moves in the xy -plane $(z = 0)$

We take $\vec{r} = x_i + y_j$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (3xyi - y^2 j) \cdot (dxi + dyj)$$

where C is the parabola $y = 2x^2$. Then $dy = 4xdx$ and x varies from 0 to 1.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (3xydx - y^2 dy) \qquad \cdots (1)$$

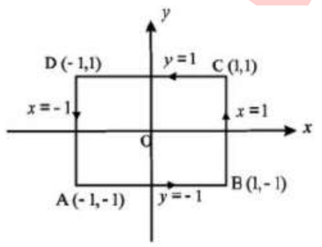
Substituting $y = 2x^2$, where x goes from 0 to 1

Therefore (1) becomes

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_{x=0}^{1} \left[3x\left(2x^2\right)dx - \left(2x^2\right)^2 (4xdx)\right] \\ &= \int_0^1 \left(6x^3 - 16x^5\right) dx \\ &= \frac{-7}{6}\end{aligned}$$

Problem 1.9.10. *Evaluate the line integral* $\int_C \left[\left(x^2 + xy\right) dx + \left(x^2 + y^2\right) dy\right]$ *where* $\mathbf{c}$ *is the square formed by the lines* $x = \pm 1$ *and* $y = \pm 1$.

Solution::



Here $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_C \left[\left(x^2 + xy\right) dx + \left(x^2 + y^2\right) dy\right]$

In the counter clockwise direction

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r} \quad \cdots (1)$$

Along AB:

Here $y = -1$. $\therefore dy = 0$

$$\therefore \int_{AB} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \left(x^2 - x\right) dx$$
$$= \int_{-1}^{1} x^2 dx - \int_{-1}^{1} x dx$$
$$= 2\int_{0}^{1} x^2 dx - 0 = 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3}$$

Along BC:

Here $x = 1. \ \therefore dx = 0$

$$\therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \left(1 + y^2\right) dy$$
$$= 2\int_{0}^{1} \left(1 + y^2\right) dy$$
$$= 2\left(y + \frac{y^3}{3}\right)_0^1$$
$$= 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}$$

Along CD :

Here $y = 1. \therefore dy = 0.$

$$\therefore \int_{CD} \vec{F} \cdot d\vec{r} = \int_{1}^{-1} \left(x^2 + x\right) dx$$
$$= (-1)\int_{-1}^{1} \left(x^2 + x\right) dx$$
$$= (-1)\left[2\int_{0}^{1} x^2 dx + 0\right]$$
$$= -\frac{2}{3}$$

Along DA:

Here $x = -1. \therefore dx = 0.$

$$\therefore \int_{DA} \vec{F}\cdot d\vec{r} = \int_{1}^{-1} \left(1+y^2\right) dy$$
$$= (-1)\int_{-1}^{1} \left(1+y^2\right) dy$$
$$= (-2)\int_{0}^{1} \left(1+y^2\right) dy$$
$$= (-2)\left(y+\frac{y^3}{3}\right)_0^1$$
$$= (-2)\left(1+\frac{1}{3}\right) = -\frac{8}{3}$$

Hence the required line integral in the counter clockwise direction is

$$\int_C \vec{F}\cdot d\vec{r} = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0, \text{ using } (1).$$

1.10 Green's theorem in a plane:

If R is a closed region of the $x-y$ plane bounds by a simple closed curve C and if M and N are two continuous function of x, y having continuous first order partial derivatives in the region R then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Problem 1.10.1. *Using Green's theorem evaluate* $\int_c \left[\left(xy+y^2\right)dx + x^2dy\right]$, *where C is rounded by* $y = x$ *and* $y = x^2$ *[VTU- July 2019, Jan10,Dec 11, June 17]*

Solution: We shall find the points of intersection of $y = x$ and $y = x^2$.

Equating the expressions of y from both equations

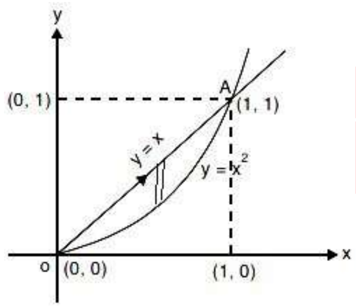
$$x = x^2 \Rightarrow x - x^2 = 0$$
$$x(1-x) = 0$$
$$x = 0,\ 1$$
$$\therefore y = 0,\ 1$$

Hence $(0,0),\ (1,1)$ are the points of intersection.

The given integral is in the form $\oint_C Mdx + Ndy$ where

$M = xy + y^2 \& N = x^2$

The region R bounded by the curves is as shown in the following figure.

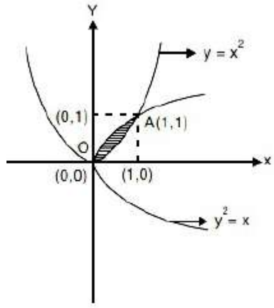


By Greens Theorem,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$
$$= \iint_R \frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(xy + y^2\right)$$
$$= \iint_R 2x - (x + 2y)$$
$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y) dy dx$$
$$= \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx$$
$$= \int_{x=0}^{1} \left[\left(x^2 - x^2\right) - \left(x^3 - x^4\right)\right] dx$$
$$= \int_{x=0}^{1} \left(x^4 - x^3\right) dx$$
$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$
$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

Problem 1.10.2. *Using green's theorem evaluate* $\oint_C \left(3x^2 - 8y^2\right) dx + (4y - 6x) dy$, *where* **c** *is the boundary of the region enclosed by* $y = \sqrt{x} \& y = x^2$

Solution: The region enclosed by $y = \sqrt{x} \& y = x^2$ is as shown in the figure.



We shall find the points of intersection of the parabola $y = \sqrt{x}$ & $y = x^2$

At the point of intersection,

$$\begin{aligned} y &= \sqrt{x} \& y = x^2 \\ \sqrt{x} &= x^2 \\ x &= x^4 \\ (x - x^4) &= 0 \\ x(1 - x^3) &= 0, \\ \Rightarrow x &= 0, x = 1 \end{aligned}$$

When $x = 0, y = x^2 \Rightarrow y = 0$

$x = 1, y = x^2 \Rightarrow y = 1$

Hence the points of interaction are $(0,0) \& (1,1)$ The given integral is in the form $\oint_C Mdx + Ndy$ where

$M = 3x^2 - 8y^2 \quad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$$

By green's theorem,

$$\begin{aligned} \oint_c Mdx + Ndy &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \\ &= \iint_R (-6y + 16y) dxdy \\ &= \int_0^1 \int_{y=x^2}^{\sqrt{x}} (-6y + 16y) dydx \end{aligned}$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (10y)dydx$$

$$= \int_0^1 \left(10\frac{y^2}{2}\right)\Big|_{x^2}^{\sqrt{x}} \, dx$$

$$= \int_0^1 (5y^2)_{x^2}^{\sqrt{x}} \, dx$$

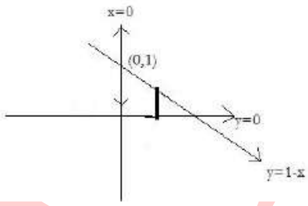
$$= \int_0^1 5\left(x - x^4\right) dx$$

$$= \left(\frac{5x^2}{2} - \frac{5x^5}{5}\right)_0^1$$

$$= \frac{5}{2} - 1 = \frac{3}{2}$$

Problem 1.10.3. *Using Green's theorem evaluate* $\oint_C \left(3x^2 - 8y^2\right) dx + (4y - 6xy)dy$ *where* C *is boundary of the region defined by* $x = 0, y = 0$ & $x + y = 1$

Solution: The region bounded by $x = 0, y = 0$ & $x + y = 1$ is given by



The given integral is in the form $\oint_C Mdx + Ndy$ where
$M = 3x^2 - 8y^2, \quad N = 4y - 6xy$
$\frac{\partial M}{\partial y} = -16y$ and $\frac{\partial N}{\partial x} = -6y$
By green's theorem,

$$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\int_C \left[\left(xy + y^2\right) dx + x^2 dy\right] = \int_{x=0}^1 \int_{y=0}^{1-x} (-6y) - (-16y)dxdy$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} 10ydxdy$$

$$= \int_{x=0}^1 10\left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= \int_{x=0}^{1} \frac{10}{2} (1-x)^2 \, dx$$

$$= 5 \left[\frac{(1-x)^3}{-3} \right]_{x=0}^{1}$$

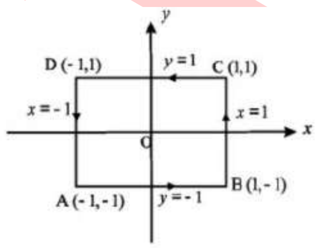
$$= \frac{-5}{3} \left[(1-1)^3 - (1-0)^3 \right]$$

$$= -5 \left[\frac{0-1}{3} \right]$$

$$= \frac{5}{3}$$

Problem 1.10.4. *Use green's theorem to evaluate* $\int_c \left(x^2+xy\right) dx + \left(x^2+y^2\right) dy$*, where* c *is the square formed by the lines* $y = \pm 1, x = \pm 1$

Solution:: The square formed by the lines $y = \pm 1, x = \pm 1$ is in the form :



The given integral is in the form $\oint_C M dx + N dy$ where

$M = x^2 + xy \quad N = x^2 + y^2$

By green's theorem, we have

$$\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

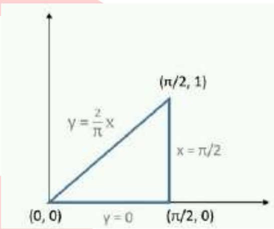
$$= \int_{-1}^{1} \int_{-1}^{1} \left[\frac{\partial}{\partial x} \left(x^2+y^2\right) - \frac{\partial}{\partial y} \left(x^2+xy\right) \right] dx dy$$

$$= \int_{-1}^{1} \int_{-1}^{1} (2x - x) dx dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x dx dy$$
$$= \int_{-1}^{1} x dx \int_{-1}^{1} dy$$
$$= \int_{-1}^{1} x dx \ [y]_{-1}^{1}$$
$$= \int_{-1}^{1} x dx \ (1+1)$$
$$= \int_{-1}^{1} 2x dx$$
$$= 2\left(\frac{x^2}{2}\right)_{-1}^{1} = 1 - 1 = 0$$

Problem 1.10.5. *Using Green's theorem, evaluate* $\int_C [(y - \sin x)dx + \cos x dy]$ *where* C *is the plane triangle enclosed by the lines* $y = 0, x = \pi/2$ *and* $y = \frac{2}{\pi}x$

Solution:



The given integral is in the form $\oint_C M dx + N dy$ where $M = (y - \sin x)$ and $N = \cos x$ By green's theorem, we have

$$\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$
$$\therefore \int_C [(y - \sin x)dx + \cos x dy]$$
$$= \int_{x=0}^{x=\frac{\pi}{2}} \int_{y=0}^{y=\frac{2x}{\pi}} (-\sin x - 1) dy dx$$
$$= -\int_0^{\frac{\pi}{2}} (\sin x + 1) \ \ [y]_0^{\frac{2x}{\pi}} \ dx$$
$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x(\sin x + 1) dx$$

$$= -\frac{2}{\pi}\left\{[x(-\cos x + x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (1) \times (-\cos x + x) dx\right\}$$

(Using Integration by parts)

$$= -\frac{2}{\pi}\left\{\left[\frac{\pi}{2}(-\cos\frac{\pi}{2} + \frac{\pi}{2}) - 0\right] - \int_0^{\frac{\pi}{2}} (1) \times (-\cos x + x) dx\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left[-\sin x + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}}\right\}$$

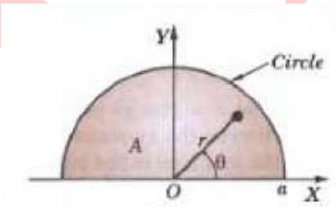
$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left[-\sin\frac{\pi}{2} + \frac{\left(\frac{\pi}{2}\right)^2}{2} - 0\right]\right\}$$

$$= -\frac{\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right)$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Problem 1.10.6. *Apply Green's theorem to evaluate* $\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$, *where* C *is the boundary of the area enclosed by the x-axis and the upper -half of the circle* $x^2 + y^2 = a^2$

Solution::



By Green's theorem

$$\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$$

$$= \iint_A \left[\frac{\partial}{\partial x}\left(x^2 + y^2\right) - \frac{\partial}{\partial y}\left(2x^2 - y^2\right)\right] dxdy$$

$$= 2\iint_A (x + y) dxdy, \quad \text{where } A \text{ is the region given in the figure}$$

Changing to polar coordinates (r, θ), we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$. Also r varies from 0 to a and θ varies from 0 to π

$$\begin{aligned}\text{Integral} &= 2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta)\cdot r d\theta dr \\ &= 2\int_0^a r^2 dr \times \int_0^\pi (\cos\theta + \sin\theta) d\theta \\ &= 2\left(\frac{r^3}{3}\right)_0^a \times [\sin\theta - \cos\theta]_0^\pi \\ &= 2\frac{a^3}{3} \times [(\sin\pi - \cos\pi) - (\sin 0 - \cos 0)] \\ &= 2\frac{a^3}{3} \times (1+1) = \frac{4a^3}{3}\end{aligned}$$

1.11 Area using Green's Theorem

Problem 1.11.1. *Apply Green's theorem to prove the area enclosed by a plane curve is* $\frac{1}{2}\int_C xdy - ydx$

Solution: We have Green's theorem that

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \qquad \cdots (1)$$

Now taking $M = -y$ and $N = x$, in equation (1), it gives

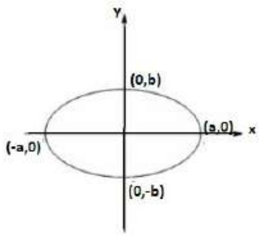
$$\begin{aligned}\int_C (-y)dx + (x)dy &= \iint_R [1 - (-1)] dxdy \\ &= 2\iint_R dxdy \\ \Rightarrow \frac{1}{2}\oint_C xdy - ydx &= \iint_R dxdy \qquad \cdots (2)\end{aligned}$$

The RHS of (2) represents the area of the region R bounded by the simple closed curve C.

$\therefore$ the area enclosed by a plane curve $= \boxed{\frac{1}{2}\int xdy - ydx}$.

Problem 1.11.2. *Using Green's theorem find the area of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: The given ellipse is in the form



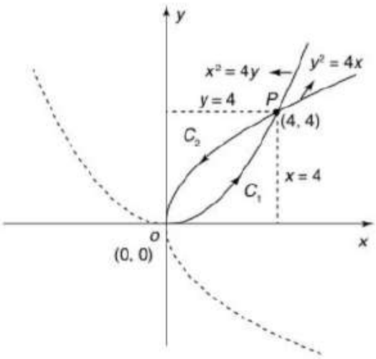
The parametric form of equation of the ellipse is $x = a\cos\theta$, $y = b\sin\theta$

Then $dx = -a\sin\theta d\theta$ and $dy = b\cos\theta d\theta$

$$\begin{aligned}A &= \frac{1}{2}\oint_c xdy - ydx\\&= \frac{1}{2}\int_0^{2\pi}(a\cos\theta)(b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)\\&= \frac{1}{2}ab\int_0^{2\pi}(cos^2\theta + \sin^2\theta)d\theta\\&= \frac{1}{2}ab\,(2\pi)\\&= \pi ab\end{aligned}$$

Problem 1.11.3. *Find the area between the parabolas* $y^2 = 4x \& x^2 = 4y$ *with help of green's theorem in a plane.* *(VTU Model 2018)*

Solution:



For intersection points, substitute

$y^2 = 4x$ or $y = 2\sqrt{x}$ in $x^2 = 4y$

$$\Rightarrow x^2 = 4 \times 2\sqrt{x}$$
$$\Rightarrow x^4 = 64x$$
$$\Rightarrow x\left(x^3 - 64\right) = 0$$
$$\Rightarrow x = 0, \quad \text{and } x = 4$$
$$\text{When } x = 0,\ y^2 = 4 \times 0 \Rightarrow y = 0$$
$$\text{When } x = 4,\ y^2 = 4 \times 4 \Rightarrow y = 4$$

Hence the Points of intersection are $(0, 0)$ and $(4, 4)$. On $C_1 : \quad x^2 = 4y$

$\therefore \quad 2xdx = 4dy \quad \Rightarrow \quad dy = \frac{1}{2}xdx$

and x varies from 0 to 4.

$$I_1 = \int_{C_1} xdy - ydx$$
$$= \int_0^4 x \cdot \frac{1}{2}xdx - \frac{x^2}{4}dx$$
$$= \int_0^4 \left(\frac{x^2}{2} - \frac{x^2}{4}\right) dx$$
$$= \int_0^4 \frac{x^2}{4}dx$$
$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$
$$= \frac{64}{4 \cdot 3} = \frac{16}{3}$$

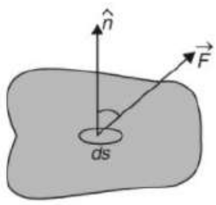
On $C_2 : y^2 = 4x \quad \therefore \quad 2ydy = 4dx \Rightarrow dx = \frac{1}{2}ydy$

and y varies from 4 to 0.

$$\begin{aligned}
\therefore I_2 &= \int_{C_2} xdy - ydx \\
&= \int_4^0 \frac{y^2}{4}dy - y \cdot \frac{1}{2}ydy \\
&= \int_4^0 \left(\frac{y^2}{4} - \frac{y^2}{2}\right) dy \\
&= \int_4^0 -\frac{y^2}{4}dy \\
&= \frac{1}{4}\int_0^4 y^2dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{16}{3}
\end{aligned}$$

$$\therefore \quad \text{Required area} = \frac{1}{2}\left[\frac{16}{3} + \frac{16}{3}\right] = \frac{16}{3}$$

1.12 Surface Integral



If (x, y, z) are the coordinates of a point in space, then the equation $z = f(x, y)$ or $f(x, y, z) = 0$ represents a surface S. If S has a unique normal at each of its points whose direction depends continuously on the points of S, then the surface S is called a smooth surface. Surface integral is a generalization of a double integral. In a surface integral the integrand is integrated along a curved surface.

Consider a vector function $\vec{F}$ defined over a surface S. Let P(x, y, z) be a point on the surface S and let $\hat{n}$ be the unit outward normal to the surface S at P. Then the normal surface integral of $\vec{F}$ over S is denoted by $\iint_S F.dS$ or $\iint_S F \cdot ndS$.

Other types of surface integrals are $\iint_s(\nabla \times F) \cdot \hat{n}dS$ and $\int fdS$

1.13 Evaluation of a Surface Integral:

A surface integral is evaluated by reducing it to a double integral by projecting the given surface S onto one of the coordinate planes.
Let R be the projection of S onto the xy -plane. Then,

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

Then,

$$\boxed{\iint_S \vec{F}\cdot\hat{n}dS = \iint_R \vec{F}\cdot\hat{n}\frac{dxdy}{|\hat{n}\cdot\hat{k}|}}$$

where $\hat{n}$ is unit outward drawn normal to S.
In a similar way the surface integral can be evaluated by projecting S onto the yz -plane as R_1 and xz -plane as R_2 as follows :

$$\iint_S \vec{F}\cdot\hat{n}dS = \iint_{R_1} \vec{F}\cdot\hat{n}\frac{dydz}{|\hat{n}\cdot\vec{i}|}$$

$$\iint_S \hat{F}\cdot\hat{n}dS = \iint_{R_2} \vec{F}\cdot\hat{n}\frac{dxdz}{|\hat{n}\cdot\hat{j}|}$$

Note : Suppose the velocity of a fluid in xyz space is described by the vector field $F(x, y, z)$. Let S be a surface in xyz space. The amount of the fluid flowing through the surface per unit time is called the **flux** of fluid through the surface. For this reason, the surface integral $= \iint_S(\vec{F}\cdot\hat{n})ds$ is often called the **flux integral**.
If $\iint_S(\vec{F}\cdot\hat{n})ds = 0$, then $\vec{F}$ is said to be a solenoidal vector point function.

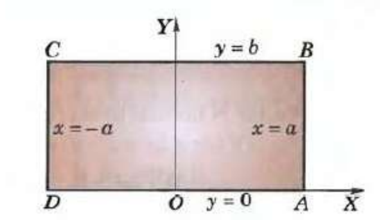
1.14 Stokes Theorem

If S is a surface bounded by a simple closed curve C and if $\vec{F}$ any continuously differentiable vector is function then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}ds = \iint_S \left(\nabla\times\vec{F}\right)\cdot\hat{n}ds}$$

Problem 1.14.1. *Using Stoke's theorem evaluate* $\int_C F\cdot dR$ *for* $\vec{F} = \left(x^2+y^2\right)i - 2xyj$ *taken round the rectangle bounded by the lines* $x = \pm a, y = 0 \& y = b$ *(VTU-June17)*

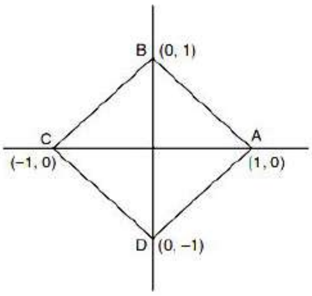
Solution:: Let $ABCD$ be the given rectangle as shown in figure.



$$\begin{aligned}
\text{CurF} &= \nabla \times \vec{F} \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} \\
&= i(0-0) - j(0-0) + k(-2y-2y) \\
&= -4yk \\
\hat{n} &= k \\
ds &= \frac{dxdy}{|\hat{n}\cdot k|} = dxdy \\
\int_s \text{CurF}\cdot\hat{n}\ \text{ds}) &= \iint (-4y)dxdy \\
&= -4\int_0^b\int_{-a}^{a} ydxdy \\
&= -\ 4\int_0^b x\Big|_{-a}^{a} ydy \\
&= -4(2a)\int_0^b ydy \\
&= -\left[8a\frac{y^2}{2}\right]_0^b \\
&= -4ab^2
\end{aligned}$$

Problem 1.14.2. *Evaluate* $\int_c xydx + xy^2dy$ *by stoke's theorem where* C *is the square in the* xy *plane with vertices* $(1,0)(-1,0),(0,1),(0,-1)$

Solution:



According to stoke's theorem we have

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}ds$$

Now,

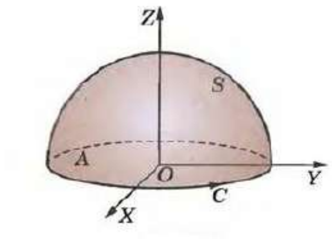
$$\begin{aligned}
\text{curl}\,\vec{F} &= \nabla\times\vec{F}\\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & xy^2 & 0\end{vmatrix}\\
&= \left(y^2-x\right)\hat{k},\\
\text{Further } ds &= \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = dxdy\\
\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}ds &= \iint \left(y^2-x\right)dxdy
\end{aligned}$$

It can be clearly seen from the figure that $-1\le x\le 1$ and $-1\le y\le 1$ Now,

$$\begin{aligned}
\iint_S \text{curl}\,\vec{F}\cdot\hat{n}ds &= \int_{x=-1}^{1}\int_{y=-1}^{1}\left(y^2-x\right)dydx\\
&= \int_{x=-1}^{1}\left[\frac{y^3}{3}-xy\right]_{y=-1}^{1}dx\\
&= \int_{x=-1}^{1}\left[\left(\frac{1}{3}+\frac{1}{3}\right)-x(1+1)\right]dx\\
&= \int_{x=-1}^{1}\left(\frac{2}{3}-2x\right)dx\\
&= \left[\frac{2}{3}x-x^2\right]_{x=-1}^{1}\\
&= \frac{2}{3}(1+1)-(1-1) = \frac{4}{3}
\end{aligned}$$

Problem 1.14.3. *Using Stoke's theorem Evaluate* $\int_C \boldsymbol{F} \cdot \boldsymbol{dR}$ *for the vector field* $\boldsymbol{F} = (2x-y)i + yz^2j - y^2zk$ *over the upper half of surface of* $x^2+y^2+z^2 = 1$ *bounded by its projection on the xy plane*

Solution::



$$\int_C \vec{F} \cdot \overrightarrow{dr} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} dS \text{ (Stoke's theorem)}$$

C is the circle: $x^2 + y^2 = 1, z = 0(xy\text{ -plane })$

Here unit normal vector to the surface of the sphere $\phi = x^2 + y^2 + z^2 = 1$ is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)i + \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)j + \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)k$$

$$= 2xi + 2yj + 2zk$$

$$|\nabla\phi| = \sqrt{4(x^2+y^2+z^2)}$$

$$= \sqrt{4} \qquad \left(\because (x^2+y^2+z^2) = 1\right)$$

$$= 2$$

$$\hat{n} = \frac{2xi + 2yj + 2zk}{2} = xi + yj + zk$$

$$|\hat{n} \cdot k| = z$$

$$ds = \frac{dxdy}{|\hat{n}\cdot k|} = \frac{dxdy}{z}$$

$$\text{Now} \quad \text{curl F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)i + 0j + k$$

$$= k$$

$$\therefore \quad \int \text{curl}\,\vec{F}\cdot\hat{n}ds = \iint_S z\cdot\frac{dxdy}{z}$$

$$= \iint_R dxdy$$

$$= \text{ area of circle } C : x^2+y^2 = 1$$

$$= \pi$$

Here R is the projection of S on xy -plane

1.15 Question Bank : Module 2-Vector Calculus

1.15.1 Question Bank :The Gradient, Divergence and Curl :

1. Find $grad\phi$ where $\phi = 3x^2y - y^3z^2$ at the point (1, -2, -1) (VTU Jan 2017)
2. If $\vec{F} = \nabla(xy^3z^2)$ find $divF$ and $curlF$ at (1,-1,1) (VTU Model 2018)
3. Find $divF$ and $curlF$ where $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$ Is $\vec{F}$ irrotational?

   (VTU Model 2022, Jan 2020, June 2019, July 2017, Jul 2015. Jul 2013, Jan 2008, Aug 2002)
4. If $F = (x+y+1)i + j - (x+y)k$, then show that $\vec{F}.curl\vec{F} = 0$ (VTU Jan 2014, July 2013)
5. If $\vec{V} = \vec{w}\times\vec{r}$, P.T. $w = \frac{1}{2}(\nabla\times\vec{V})$ or $curl\vec{V} = 2\vec{w}$ where $\vec{w}$ is a constant vector.

   (VTU Jan 2015)

6. If $\vec{r} = xi + yj + zk$, and $|\vec{r}| = r$, find $graddiv\left(\frac{\vec{r}}{r}\right)$ (VTU Jan 2015)

7. If $\phi = x^2 + y^2 + z^2$, and $\vec{F} = x^2 i + y^2 j + z^2 k$, then find $grad\phi$, $div\vec{F}$ and $curl\vec{F}$.

   (VTU July 2014) **Ans:** $2xi + 2yj + 2zk$, $2x + 2y + 2z$, $\vec{0}$.

8. If $\vec{F} = 2xy^3z^4 i + 3x^2y^2z^4 j + 4x^2y^3z^3 k$, find (i) $\nabla \cdot \vec{F}$ (ii) $\nabla \times \vec{F}$ (VTU Jan 2014)

9. $div\vec{F}$ and $curl F$ at the point (1,2,3) if $\vec{F} = grad(x^3y + y^3z + z^3x - x^2y^2z^2)$ (VTU 2007)

10. Find the divergence and curl of the vector $\vec{V} = xyzi + 3x^2yj + (xz^2 - y^2z)k$ at the point (2,-1,1)

    (VTU Jan 2017)

11. If $\vec{F} = x^2y\hat{\imath} + y^2z\hat{\jmath} + z^2x\hat{k}$, find $\text{Curl}(\text{Curl}\,\vec{F})$ (VTU Model 2022)

12. If $\Phi = x^3y^3z^3$, then find $\nabla\Phi$ at $(1, 2, 1)$

    Ans : $24i + 12j + 24k$

13. If $\vec{V} = 3xzi + 2xyj - yz^2k$, find $div\vec{V}$

    Ans : $3z + 2x - 2yz$.

14. If $\vec{V} = yzi + 3xzj + zk$, find $curl\vec{V}$

    Ans : $-3xi + yj + 2zk$

15. If $\Phi = x^2 - y^2 - z^2 - 2$, find $grad\Phi$ at (1,-1,2)

    Ans : $2i + 2j - 4k$

16. If $F = xy^2i + 2x^2yzj + 3yz^2k$, then find $div(curlF)$

    Ans : 0

17. If $F = (3x^2y - z)i + (xz^3 + y^4)j - 2x^3z^2k$, then find $grad(divF)$ at $(2, -1, 0)$

    Ans : $-6i + 24j - 32k$

18. If $\vec{F} = 3xyz^2i - 4x^2yj - xy^2zk$, then find $\nabla(\nabla \cdot F)$ at $(-1, 2, 1)$

    Ans : $-16i + 7j + 12k$

19. If $A = x^2yi - 2xzj + 2yzk$, find $curl(curlA)$.

    Ans : $2(x + 2)j$

20. If $A = x^2yi + xzj + 2yzk$, find $curl(curlA)$ at (1,1,1)

    Ans : $4j$

21. If $\phi = x^2 + y^2 + z^2$, and $\vec{F} = x^2i + y^2j + z^2k$, then find $grad\phi$, $div\vec{F}$ and $curl\vec{F}$. (VTU July 2014)

    Ans: $2xi + 2yj + 2zk$, $2x + 2y + 2z$, $\vec{0}$.

22. Find $div\vec{F}$ and $curlF$ at the point (1,2,3) if $\vec{F} = x^2yzi + xy^2zj + xyz^2k$

    Ans : 12, $5i - 16j + 9k$

23. $div\vec{F}$ and $curlF$ at the point (1,2,3) if $\vec{F} = 3x^2i + 5xy^2j + 5xyz^3k$

    Ans : 278, $5(27i - 54j + 8k)$

24. If $\vec{V} = \frac{\vec{r}}{r}$, show that $div(\vec{V}) = \frac{2}{r}$ and $curl(\vec{V}) = 0$

25. Find $curl(grad\ f)$ given $f(x, y, z) = x^2 + y^2 - z$

    Ans : 0

26. Find $curl(curl\vec{A})$, given $\vec{A} = x^2yi + y^2zj + z^2yk$ (VTU 2003)

    Ans : $2(x + z)j + 2yk$

27. prove that $\nabla\left(\frac{1}{r^2}\right) = \frac{-2\vec{r}}{r^4}$

28. prove that $\nabla r = \frac{1}{r}\vec{r} = \hat{r}$

29. Find $divF$ and $curlF$ where $\vec{F} = \nabla(xy^3z^2)$ (VTU Model 2018)

1.15.2 Question Bank :Angle between two surfaces :

1. Find the angle between two surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at the point $(2,-1,2)$
(VTU Model 2022, June 2019, July 2017, Dec 2010, Jul 2009)

2. Find the angle between two surfaces $x^2+y^2+z^2=4$ & $z=x^2+y^2-13$ at the point $(2,1,2)$
(VTU Model 2018)

3. Find the constants a & b so that the surface $ax^2-byz=(a+2)x$ is orthogonal to the surface $4x^2y+z^3=4$ at the point $(1,-1,2)$. (VTU Model 2018)

4. Find the angle between two surfaces $xlogz=y^2-1, x^2y=2-z$ at the point (1,1,1)

5. Find the unit vector normal to the surface $xy^3z^2=4$ at the point $(-1,-1,2)$
Ans : $\frac{1}{\sqrt{11}}(-i-3j+k)$

6. Find the unit vector normal to the surface $x^2y+y^2z+z^2x=5$ at the point $(1,-1,2)$
Ans : $\frac{2i-3j+5k}{\sqrt{38}}$

7. Find the unit vector normal to the surface $x^2+y^2+z^2=3$ at the point $(1,1,1)$ **Ans :** $\frac{i+j+k}{\sqrt{3}}$

8. Find a unit normal to the surface $xy^2z^3=1$ at the point $(1,1,1)$.
Ans : $\frac{i+2j+3k}{\sqrt{14}}$

9. Find the angle between two surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at the point $(2,-1,2)$ **(VTU Dec 2010, Jul 2009)**
Ans : $cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

10. Find the angle between the directions of the normal to the surface $x^2yz=1$ at the points $(-1,1,1)$ and $(1,-1,-1)$. **Ans :** $\theta=\pi$

11. Show that the surfaces $5x^2 - 2yz - 9x = 0$ and $4x^2y + z^3 - 4 = 0$ are orthogonal at the point $(1, -1, 2)$.

12. Find the unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$ **Ans :** $\frac{-i+3j+2k}{\sqrt{14}}$

13. Find the angle between two surfaces $xlogz = y^2 - 1, x^2y = 2 - z$ at the point (1,1,1) **Ans :** $cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$

14. Find the constants $a\&b$ so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$
Ans : $a = -6, b = -10$

1.15.3 Question Bank :Directional Derivative :

1. Find the directional derivative of $4xz^3 - 3x^2y^2z$ at (2,-1,2) along $2i - 3j + 6k$. (VTU Jan 2020, Model 2018, Jan 2015) **Ans :** $\frac{376}{7}$

2. Find the directional derivative of $xy^3 + yz^3$ at $(2, -1, 1)$ in the direction of the vector $i + 2j + 2k$
(VTU July 2017)

3. Find the directional derivative of $\Phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of the vector $2i - j - k$ (VTU Aug 2005, Aug 2004, Mar 2000)

4. Find the directional derivative of $f = xy^2 + yz^3$ at the point $(1, 2, -1)$ in the direction of the normal to the surface $xlogz - y^2 = -4$ at the point $(-1, 2, 1)$. (VTU Feb 2004)

5. In which direction the directional derivative of x^2yz^3 is maximum at $(2, 1, -1)$ and find the magnitude of this maximum. (VTU Jan 2009, Dec 2008) **Ans :** $-4i - 4j + 12k, 4\sqrt{11}$

6. Find the directional derivative of $\Phi = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of the vector $2i - j - 2k$ (VTU 2007) **Ans :** $\frac{37}{3}$

7. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$ (Model 2022)

8. If $\Phi = \frac{xz}{x^2+y^2}$, then find the directional derivative at $(1, -1, 1)$ in the direction of $\vec{a} = i - 2j + k$ **Ans :** $\frac{-1}{2\sqrt{6}}$

9. Find the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of $i + 2j + 2k$ **Ans :** $\frac{-11}{3}$

10. Show that the directional derivative of $\Phi = x^3y^2z$ is maximum along the direction of $9i + 3j + k$ at (1,2,3). Also find magnitude of maximum.
    Ans : $\nabla\Phi|_{(} 1, 2, 3) = 4(9i + 3j + k)$ and $|\nabla\Phi| = 4\sqrt{91}$

11. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.
    Ans : $96(i + 3j - 3k)$, $96\sqrt{19}$

12. Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1, 2, 3)$ in the direction of $2i + j - k$. Ans : $\frac{86}{\sqrt{6}}$

13. Find the directional derivative of $f = x^2 - y^2 + 2z^2$ in the direction of $4i - 2j + k$. Also calculate the magnitude of the maximum directional derivative.
    Ans: $\frac{28}{\sqrt{21}}$, $\sqrt{164}$

14. Find the directional derivative of $\nabla \cdot \nabla\Phi$ at the point (1, -2, 1) in the direction of normal to the surface $x^2yz = 3x + z^2$, where $\Phi = 2x^2y^2z^4$ **Ans :** $\frac{1724}{21}$

15. Find the directional derivative of $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ in the direction $2i - j - k$ Ans : $\frac{16}{\sqrt{6}}$

1.15.4 Question Bank :Solenoidal and irrotational vector fields

1. Find 'a' for which $\vec{F} = (x + 3y)i + (y - 2z)j + (x + az)k$ is solenoidal. (VTU Jan 2017)

2. Prove that $\frac{xi+yj}{x^2+y^2}$ is both solenoidal & irrotational. (VTU Model 2018, Model 2014)

3. Find the constants a, b, c such that the vector field $\vec{f} = (x+2y+az)i+(bx-3y-z)j+(4x+cy+2z)k$ is irrorational. (VTU July 2014)

4. Find the value of the constant a such that the vector field $\vec{F} = (axy+bz^3)i+(3x^2-cz)j+(3xz^2-y)k$ is a conservative force field. Hence find a scalar function Φ such that $\vec{F} = \nabla\Phi$ (VTU June 2019, Model 2018)

5. Find the constants a, b, c such that the vector field $\vec{f} = (x+y+az)i+(x+cy+2z)k+(bx+2y-z)j$ is irrorational. (VTU July 2017). Also find Φ such that $\vec{F} = \nabla\Phi$ (VTU Jul 2015)

6. Find the value of the constant a such that the vector field $\vec{F} = (axy-z^3)i+(a-2)x^2j+(1-a)xz^2k$ is irrotational. Hence find a scalar function Φ such that $\vec{F} = \nabla\Phi$ (VTU Jul 2014)

7. Prove that $\vec{F} = (y^2-z^2+3yz-2x)i+(3xz+2xy)j+(3xy-2xz+2z)k$ is both solenoidal and irrotational. (VTU Jan 2016)

8. Find the constants a and b such that $F = (axy+z^3)i+(3x^2-z)j+(bxz^2-y)k$ is irrotational. Find Φ such that $F = \nabla\Phi$ (VTU Jan 2020, July 2014, Jan 2014, Jun 2012, Feb 2005)

9. Show that $\vec{F} = (x^2-yz)i+(y^2-zx)j+(z^2-xy)k$ is irrotational and hence find its scalar potential. (VTU July 2015, Jan 2013)

10. If $\vec{u} = x^2i+y^2j+z^2k$ and $\vec{v} = yzi+zxj+xyk$ Show that $\vec{u}\times\vec{v}$ is a solenoidal vector.

    (VTU July 2017)

11. A vector field is given by $\vec{A} = (x^2+xy^2)i+(y^2+x^2y)j$, show that the field is irrotational and find the scalar potential. (VTU July 2017)

12. Show that $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational. (VTU Model 2022)

13. Define an irrotational vector. Find the constants a, b and c such that $\vec{F} = \left(axy - z^3\right)\hat{\imath}$ is irrotational. (VTU Model 2022)

14. Prove that $F = 3y^4z^2i + 4x^3z^2j - 3x^2y^2k$ is a solenoidal vector .

15. Find a for which $\vec{F} = (x + 3y)i + (y - 2z)j + (x + az)k$ is solenoidal. (VTU Jan 2017)

16. Find the value of m if $\vec{A} = (x + 2y)i + (my + 4y)j + (5z + 6x)k$ is a solenoidal vector. **Ans :** $m = -10$

17. Find the value of a if $\vec{F} = (ax + 3y + 4z)i + (x - 2y + 3z)j + (3x + 2y - z)k$ is a solenoidal vector. **Ans :** $a = 3$

18. Find the value of 'a' if the vector $\vec{F} = \left(2x^2y + yz\right)i + \left(xy^2 - xz^2\right)j + \left(axyz - 2x^2y^2\right)k$ is solenoidal. Ans: $a = -6$

19. Prove that $\vec{r}r^3$ is both solenoidal & irrotational .

20. Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational. (VTU Jan 2016)

21. Prove that $r^n r$ is solenoidal iff $n = -3$ and irrotational for all n.

22. Show that $\vec{F} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational. Find Φ such that $\vec{F} = \nabla\Phi$ **(VTU Aug 2003)** **Ans** $\Phi = 3x^2y + xz^3 - yz$

23. Show that $\vec{F} = (2x + yz)i + (4y + zx)j - (6z - xy)k$ is both solenoidal and irrotational and also find its scalar potential Φ **Ans :** $\Phi = x^2 + 2y^2 - 3z^2 + xyz$

24. Show that $\vec{F} = (y + z)i + (z + x)j + (x + y)k$ is conservative and also find its scalar potential. **Ans :** $\Phi = xy + yz + zx$

25. If $\nabla\Phi = (y^2 - 2xz^2 - 1)i + 2xyj + 2x^2zk$, find Φ **Ans :** $\Phi = xy^2 + x^2z^2 - x + c$

26. Show that $F = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ is a conservative force field and find its scalar potential.
Ans : $\Phi = x^2y^2 + y^2z^2 + xyz + c$)

27. Prove that $\vec{F} = (-x^2 + yz)i + (4y - z^2x)j + (2xz - 4z)k$ is solenoidal.

28. Show that $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right) i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is irrotational and solenoidal.

29. Determine the constants a and b such that the curl of vector
$\vec{A} = (2xy + 3yz)\hat{i} + \left(x^2 + axz - 4z^2\right)\hat{j} - (3xy + byz)\hat{k}$ is zero.
Ans : $a = 3, b = 8$

1.15.5 Question Bank :Line Integrals, workdone

1. Find the total work done in moving a particle in the force field $F = 3xyi - 5z] + 10xk$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$ (VTU June 2019)

2. If $\vec{F} = \text{xyi} + \text{yz}j + zxk$, Evaluate $\int_c \vec{F}.\overrightarrow{dr}$. Where c is the curve represented by x $=$ t, y $= t^2$, z $= t^3, -1 \leq t \leq 1$ (VTU Jan 2020)

3. Find the total work done by the force $\vec{F} = 3x^2i + (2xz - y)j + zk$, along $x = t, y = \frac{t^2}{4}, z = \frac{3t^3}{8}$ (VTU Model 2018)

4. Find the total work done by the force represented by $\vec{F} = 3xyi - yj + 2zxk$ in moving a particle round the circle $x^2 + y^2 = 4$ [VTU-2010]

5. Find the work done in moving a particle in the force field $\vec{F} = 3x^2i + (2xz - y)j + zk$, along [VTU-17]

   (i) The straight line from (0,0,0) to (2,1,3). (VTU Model 2022)

(ii) The curve defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$

6. If $\vec{F} = 3xyi - y^2 j$. Evaluate $\int \vec{F}.dR$ where C is the curve in the xy-plane $y = 2x^2$ from (0, 0) to (1, 2). [VTU 2010]

7. If $\vec{F} = (3x^2 + 6y)i - 14yzj + 20xz^2k$, evaluate $\int \vec{F}.d\vec{r}$ from (0,0,0) to (1,1,1) along the curve given by $x = t, y = t^2, z = t^3$ [VTU 2001]

8. A vector field is given by $\vec{F} = sinyi + x(1 + cosy)j$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2, z = 0$ [VTU Jan 2018]

9. If C is a simple closed curve in the xy-plane not enclosing the origin. Show that $\int_c \vec{F}.dR = 0$ where $\vec{F} = \frac{yi - xj}{(x^2+y^2)}$ [VTU Jan 2018]

10. If $\vec{F} = \left(5xy - 6x^2\right)\hat{\imath} + (2y - -4x)\hat{\jmath}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $C : y = x^3$ in the xy - plane from the point $(1, 1)$ to $(2, 8)$ (VTU Model 2022)

11. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left(x^2 - y^2\right)\mathrm{i} + (\mathrm{xy})\mathbf{j}$, where **c** is the arc of the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$ Ans: $\frac{824}{21}$

12. If $\vec{F} = x^2 i + xyj$, Evaluate $\int_C \vec{F} \cdot d\vec{r}$ from (0,0) to (1,1) along (i) the line $y = x$ (ii) the parabola $y = \sqrt{x}$ Ans: (i) $\frac{2}{3}$ (ii) $\frac{7}{12}$

1.15.6 Question Bank :Greens Theorem

1. Using Green's theorem, Evaluate $\oint_c (x^2 + y^2)dx + 3x^2 ydy$, where C is the circle $x^2 + y^2 = 4$ traced in the positive sense. (VTU Model 2018)

2. Using Green's theorem evaluate $\oint_c (xy + y^2)dx + x^2 dy$, where C is rounded by $y = x$ and $y = x^2$ [VTU- Model 2022, Jan 2020, Jan10,Dec 11, June 17]

3. Using Green's theorem evaluate $\oint_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region enclosed by $y = \sqrt{x}$ & $y = x^2$ [VTU July 2018]

4. Find the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ using Green's theorem in a plane. [VTU: – June 2018]

5. Find the area between the parabolas $y^2 = 4x$ & $x^2 = 4y$ with help of green's theorem in a plane. (VTU Model 2018)

6. Apply Green's theorem to evaluate $\int_C \left[\left(3x - 8y^2\right) dx + (4y - 6xy)dy\right]$, where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$ (VTU Model 2022)

7. Using green's theorem, evaluate $\int_c \left(x^2ydx + x^2dy\right)$, where c is the boundary described counter clockwise of the triangle with vertices $(0, 0), (1, 0), (1, 1)$. Ans: $\frac{5}{12}$

8. Employ Green's theorem in a plane to show that the area enclosed by a plane curve c is $\frac{1}{2}\oint_c xdy - ydx$ and hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Ans: πab sq.units

9. Find the area of the asteroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ by employing green's theorem. Ans : $\frac{3\pi a^2}{8}$ sq .units

10. Using Green's theorem, Evaluate $\int_C \left(xy - x^2\right) dx + x^2ydy$, where c is the closed curve formed by $y = 0, x = 1 \& y = x$ Ans :$\frac{-1}{12}$

1.15.7 Question Bank :Surface Integrals, Stokes Theorem

1. Using Stoke's theorem, evaluate $\int_c \vec{F}.d\vec{R}$, where $\vec{F} = yi + zj + xk$ and C is the boundary of upper half of the sphere $x^2 + y^2 + z^2 = 1$ (VTU Model 2018)

2. Evaluate $\int_s F.dS$ where $\vec{F} = 4xi - 2y^2j + z^2k$, s is the surface bounding the region $x^2 + y^2 = 4, z = 0 \& z = 3$ [VTU-June17]

3. Verify stoke's theorem for $\vec{F} = (x^2+y^2)i - 2xyj$ taken round the rectangle bounded by the lines $x = \pm a, y = 0 \& y = b$ [VTU-Model 2022, Jan 2020, June 2018, June 17]

4. If $\vec{F} = 2xyi + yz^2j + xzk$ and S is the rectangular parallelepiped bounded by $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3$ evaluate $\iint_s \vec{F}.nd\hat{s}$ [VTU-June 2018]

5. Use Stokes theorem to evaluate $\int_c \vec{F}.d\vec{R}$, where $\vec{F} = (2x-y)i - yz^2j - y^2zk$ over the upper half surface of $x^2+y^2+z^2 = 1$, bounded by its projection on the xy-plane. [VTU Jan 2018]

6. Using Stoke's theorem, evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \left(x^2+y^2\right) - 2xy$, taken around the rectangle whose vertices are $(a,0),(a,b),(-a,b),(-a,0)$. (VTU Model 2022)

7. Evaluate $\int_c xydx + xy^2dy$ by stoke's theorem where c is the square in the x -y plane with vertices (1,0)(-1,0)(0,1)(0,-1). Ans : $\frac{8}{3}$

8. Using Stoke's theorem Evaluate $\int_C^{\square} F.dR$ for the vector field $F = (2x - y)i + yz^2j - y^2zk$ over the upper half of surface of $x^2+y^2+z^2 = 1$ bounded by its projection on the xy plane Ans: π

9. Using Stoke's theorem Evaluate $\int_c (ydx + zdy + xdz)$ where c is the curve of interaction of $x^2+y^2+z^2 = a^2 \& x + z = a$

10. Using stoke's theorem evaluate $\int_c [(x+y)dx + (2x-z)dy + (y+z)dz]$, where C is the boundary of the triangle with vertices $(2,0,0)(0,3,0)\&(0,0,6)$ Ans : 21

Mathematics II for EE Stream - Lecture Notes

Subject Code : BMATE201

Module 2- Vector Spaces and Linear Transformation

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Mathematics II for EE Stream (BMATE201)

Module 2 - Vector Spaces and Linear Transformations

2.1 Vector Spaces

Let $\boldsymbol{F}$ be a set. A vector space over $\boldsymbol{F}$, is a non-empty set $\boldsymbol{V}$ together with two operations (called addition and scalar multiplication) such that for each $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ there is a unique element $\boldsymbol{u}+\boldsymbol{v} \in \boldsymbol{V}$ and for each $\boldsymbol{\alpha} \in \boldsymbol{F}$ and $\boldsymbol{u} \in \boldsymbol{V}$ there is $\boldsymbol{a}$ unique element $\boldsymbol{\alpha u} \in \boldsymbol{V}$, and it satisfies the following conditions:

$\boldsymbol{I}$. (i) $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$ (Associativity)

(ii) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$, for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ (commutativity)

(iii) There exists an element $\mathbf{0} \in \boldsymbol{V}$ such that $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}=\mathbf{0}+\boldsymbol{u}$, for all $\boldsymbol{u} \in$ V (Existence of identity element)

(iv) For each $\boldsymbol{u} \in \boldsymbol{V}$, there exists a unique element $-\boldsymbol{u} \in \boldsymbol{V}$ such that $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}=(-\boldsymbol{u})+\boldsymbol{u}$ (Existence of additive inverse)

II. There is an external composition in V over $\boldsymbol{F}$ called scalar multiplication. i.e $\forall \boldsymbol{\alpha} \in \boldsymbol{F}$ and $\boldsymbol{u} \in \boldsymbol{V} \Rightarrow \boldsymbol{\alpha} \cdot \boldsymbol{u} \notin \boldsymbol{V}$ In otherwords $\boldsymbol{V}$ is closed with respect to scalar multiplication. III. The two compositions, ie. vector addition and scalar multiplication satisfy the following postulates,

$\forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{F}$ and $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$

(i) $(\boldsymbol{a}+\boldsymbol{b}) \cdot \boldsymbol{v}=\boldsymbol{a} \cdot \boldsymbol{v}+\boldsymbol{b} \cdot \boldsymbol{v}$

(ii) $\boldsymbol{a} \cdot(\boldsymbol{b} \cdot \boldsymbol{v})=(\boldsymbol{a b}) \cdot \boldsymbol{v}$

(iii) $\boldsymbol{a} \cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{a} \cdot \boldsymbol{v}+\boldsymbol{a} \cdot \boldsymbol{w}$

(iv) $\mathbf{1} \cdot(\boldsymbol{v})=\boldsymbol{v}$.

2.1.1 Vectors in R^n : $-$

The set of all ordered triples (a, b, c) of real numbers is called Euclidean 3-space and is denoted by R^3 & the set of all n-tuples of real numbers, denoted by R^n, is called Eudidean n-space.
Eg:- The ordered pair $(2, -3)$ belongs to R^2; it is a 2-tuple of dimension two.
The ordered triple $(7, 3, 6)$ belongs to R^3; it is a 3-triple of dimension three.

2.1.2 Examples or Problems

1) Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addilion and scalar multiplication.
Sol:- Let V denotes the set of all coplanar vectors and R be the field of real numbers.
$\therefore$ The elements of V are the ordered pairs (x, y) where $x, y \in R$.

$$V = \{(x, y) : x, y \in R\}$$

I) $(V, +)$ is an abelian group. i) Associativity:- We know that for all $u, v, w \in V$

$$(u + v) + w = u + (v + w).$$

ii) Commutativity:- We know that for all $u, v \in V$

$$u + v = v + u$$

iii) Existence of additive identity for every vector $u \in V$ then exists a zero vector $0 \in V$ such that $u + 0 = 0 + u = u$.
iv) Existence of additive inverse: For every vector $u \in V$ there exists a vector $-u \in V$ such that $u + (-u) = (-u) + u = 0$.

II) Scalar multiplication in V. i) For $u, v \in V$ and $\alpha \in R$ we have

$$\alpha(u + v) = \alpha u + \alpha v$$

ii) For $u \in V$ and $a, b \in R$, we have

$$(a + b)u = au + bu.$$

ii $\boldsymbol{i}$ For $\boldsymbol{u} \in \boldsymbol{V}$ and $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}$, we have

$$\boldsymbol{a}(\boldsymbol{b}\boldsymbol{u}) = (\boldsymbol{a}\boldsymbol{b})\boldsymbol{u}$$

iv) For $\boldsymbol{u} \in \boldsymbol{V}$ and $\mathbf{1} \in \boldsymbol{R}$, we have

$$\mathbf{1}\boldsymbol{u} = \boldsymbol{u}.$$

Thees set $\boldsymbol{V}$ of coplanar vectors satisfies all the properties of vector addition and scalar multiplication.

$\therefore$ $\boldsymbol{V}$ is a vector space.

2) Prove that the set $\boldsymbol{C}$ of all complex numbers (the set of all ordered pairs of real numbers) is a vector space over the field $\boldsymbol{R}$ of all real numbers where vector addiction is defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \text{ for all } (x_1, x_2), (y_1, y_2)$$

and scalar multiplication is defined by

$$\alpha (x_1, x_2) = (dx_1, \alpha x_2), \text{ for all } \alpha \in \boldsymbol{R}.$$

Soln:- We observe that $\boldsymbol{C}$ is closed under vector addition and under scalar multiplication.

I. (i) Associativity: For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \boldsymbol{C}$, we have

$$\text{For all } (x_1, x_2), (y_1, y_2), (z_1, z_2)$$

$$\begin{aligned}(x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2) \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\ &= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2)\end{aligned}$$

(ii) Commutativity: For all $(x_1, x_2), (y_1, y_2) \in C_1'$, we have

$$\begin{aligned}(x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2)\end{aligned}$$

(iii) Existence of identity: For $(x_1, x_2) \in C$, there exists $(0,0) \in C$ such that

$$(x_1 + x_2) + (0,0) = (x_1 + 0, x_2 + 0)$$

$$= (x_1, x_2)$$

$$(0+0) + (x_1, x_2) = (0 + x_1, 0 + x_2) = (x_1, x_2)$$

$$\therefore \quad (x_1, x_2) + (0,0) = (x_1, x_2) = (0,0) + (x_1, x_2)$$

(iv) Existence of inverse: For any $(x_1, x_2) \in C$. there exists $(-x_1, -x_2) \in C$ such that

$$(x_1, x_2) + (-x_1, -x_2) = (0,0) = (-x_1, -x_2) + (x_1, x_2).$$

Thus C is an abelian group with respect to vector addiction.

II. Properties of Scalar multiplication in C. (i)

$$\alpha\left[(x_1, x_2) + (y_1, y_2)\right] = \alpha\left(x_1 + y_1, x_2 + y_2\right)$$

$$= \left(\alpha\left(x_1 + y_1\right), \alpha\left(x_2 + y_2\right)\right)$$

$$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2)$$

$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$$

$$= \alpha\left(x_1, x_2\right) + \alpha\left(y_1, y_2\right)$$

Thus $\alpha\left[(x_1, x_2) + (y_1, y_2)\right] = \alpha\left(x_1, x_2\right) + \alpha\left(y_1, y_2\right)$ for all $(x_1, x_2), (y_1 y_2) \in C, \alpha \in R$.

(ii)

$$(a+b)(x_1, x_2) = ((a+b)x_1, (a+b)x_2)$$

$$= (ax_1 + bx_1, ax_2 + bx_2)$$

$$= (ax_1, ax_2) + (bx_1, bx_2)$$

$$= a(x_1, x_2) + b(x_1, x_2)$$

Thus $(a+b)(x_1, x_2) = a(x_1, x_2) + b(x_1, x_2)$ for all

$$(x_1, x_2), (y_1, y_2) \in C \,\&\, a, b \in R.$$

$(ii$ i

$$a(b(x_1, x_2)) = a(bx_1, bx_2) = (abx_1, abx_2)$$

$$a(b(x_1, x_2)) = (ab)(x_1, x_2)$$

(iv) 1. $(x_1, x_2) = (x_1, x_2)$ for $all\ (x_1, x_2) \in C$.

Thus the set C satisfies all the properties of vector space.

Hence C is a vector space over R.

3) Show that the set V of all real valued contineeous functions of x defined on interval $[0, 1]$ is a vector space over the field R of real numbers with suspect to vector addition and scalar multiplication defined by

$$(f_1 + f_2)\, x = f_1(x) + f_2(x), \text{ for all } f_1, f_2 \in V$$

$$(af_1)\, x = \alpha f_1(x), \text{ for all } \alpha \in R, f_1 \in V.$$

Soln: - I. (i) Associativity: Let $f_1, f_2, f_3 \in V$ be arbitrary

$$\begin{aligned} [(f_1 + f_2) + f_3]\,(x) &= (f_1 + f_2)\,(x) + f_3(x) \\ &= [f_1(x) + f_2(x)] + f_3(x) \\ &= f_1(x) + [f_2(x) + f_3(x)] \\ &= f_1(x) + (f_2 + f_3)\,(x) \\ &= [f_1 + (f_2 + f_3)]\,(x) \\ (f_1 + f_2) + f_3 &= f_1 + (f_2 + f_3)\,. \end{aligned}$$

(ii) Commutativity: Let $f_1, f_2 \in V$ be arbitrary.

$$\begin{aligned} (f_1 + f_2)\,(x) &= f_1(x) + f_2(x) \\ &= f_2(x) + f_1(x) \\ &= (f_2 + f_1)\,(x), \text{ for all } x. \\ f_1 + f_2 &= f_2 + f_1. \end{aligned}$$

(iii) Existence of Identity: Define a function $\mathbf{0}$ such that $\mathbf{0}(x) = 0$ (real number), for all $x \in [0, 1]$.

Also, 0 is a continuous function and belongs to V

$$(0 + f_1)\,(x) = 0(x) + f_1(x) = 0 + f_1(x) = f_1(x)$$

$$(f_1 + 0)\,(x) = f_1(x) + 0(x) = f_1(x)$$

$$0 + f_1 = f_1 + 0 = f_1$$

Thus, the function 0 defined above is an additive identity element of V.

(iv) Existence of Inverse: For a function f, the function -fid defined by $(-f_1)\,(x) = -f_1(x)$, is called additive inverse, as $[f_1 + (-f_1)]\,(x) = f(x) + (-f_1)\,(x) = f_1(x) - f_1(x) = 0 = 0(x)$ $Similarly,$ $[-f_1 + f_1]\,(x) = o(x)$. Thus the set V is an abetian group under addition.

II. Properties of Scalar Multiplication in V.

(i) For $f_1, f_2 \in Y$ and $\alpha \in R$, we have

$$\begin{aligned}[\alpha\,(f1+f_2)]\,x &= \alpha\,[(f_1+f_2)\,(x)]\\ &= \alpha\,[f_1(x)+f_2(x)]\\ &= \alpha f_1(x)+\alpha f_2(x)\\ &= (\alpha f_1)\,x+(\alpha f_2)\,x\\ &= (\alpha f_1+\alpha f_2)\,x\\ \alpha\,(f_1+f_2) &= \alpha f_1+\alpha f_2\end{aligned}$$

(ii) For $f_1 \in V$ and $a, b \in R$.

$$\begin{aligned}[(a+b)f_1]\,(x) &= (a+b)f_1(x)\\ &= af_1(x)+bf_1(x)\\ &= (af_1)\,x+(bf_1)\,x\\ &= (af_1+bf_1)\,(x)\\ (a+b)f_1 &= af_1+bf_1\end{aligned}$$

(iii) For $f_1 \in V$ and $a, b \in R$,

$$\begin{aligned}[a\,(bf_1)]\,x &= a\,[(bf_1)\,x]\\ &= a\,[bf_1(x)]\\ &= (ab)\,f_1(x)\\ &= [(ab)\,f_1]\,x\\ a\,(bf_1) &= (ab)\,f_1\end{aligned}$$

(iv) For $f_1 \in V$ and $I \in R$, we have

$$(1, f_1)\,x = 1\cdot f_1(x) = f_1(x)$$

$$1f_1 = f_1.$$

Thus V satisfies all the properties of a vector space and hence V is a vector space.

4) Show that the set of all matrices of the type $m \times n$ where m and n are fixed positive integers is a vector space over R with suspect to matrix addition and multiplication of matrix by a scalar. (, a real number).

Soln: Let M denotes the set of all matrices of type mxn.

I. $(M, +)$ is an abelian group.

(i) Associativity: Let $A = [a_{ij}]_{m\times n}, B = [b_{ij}]_{m\times n}, C = [c_{ij}]_{m\times n}$ be three matrices belonging to set M.

$$\begin{aligned}(A+B)+C &= [[a_{ij}]+[b_{ij}])+[C_{ij}]\\ &= [a_{ij}+b_{ij}]+[c_{ij}]\\ &= [(a_{ij}+b_{ij})+(c_{ij})]\\ &= [a_{ij}+(b_{ij}+c_{ij})]\\ &= [a_{ij}]+[b_{ij}+c_{ij}]\\ &= [a_{ij}]+([b_{ij}]+[C_{ij}])\\ &= A+(B+C).\end{aligned}$$

(ii) Commutativity: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices belonging to set M.

$$\begin{aligned}A+B &= [a_{ij}]+[b_{ij}] = [a_{ij}+b_{ij}]\\ &= [b_{ij}+a_{ij}]\\ &= [b_{ij}]_{m\times n}+[a_{ij}]_{m\times n}\\ &= B+A.\end{aligned}$$

(iii) Existence of Identity: Let $A = [a_{ij}]_{m\times n} \in M$ be arbitrary. We know that a Zero matrix (null matrix) of type $m \times n$ also belongs to the set M and is denoted by $\mathbf{0}$. Now, $A + \mathbf{0} = [a_{ij}] + [0] = [a_{ij} + 0] = [a_{ij}] = A$.

Similarly, , $\quad \mathbf{0} + A = A$.

(iv) Existence of Inverse: If $A = [a_{ij}]_{m\times n}$ belongs to the set M, then $-A = [-a_{ij}]_{m\times n}$ also belongs to the set M. Now,

$$\begin{aligned}A+(-A) &= [a_{ij}]+[-a_{ij}] = [a_{ij}-a_{ij}]\\ &= [0] = \mathbf{0}\end{aligned}$$

Similarly, $\quad (-A) + (A) = \mathbf{0}$ Thus the set M is an abelian group under addition.

II. Properties of Scalar multiplication in M. (i) Let $A = [a_{ij}]_{m\times n}$ and $B : [b_{ij}]_{m\times n} \in M$ and

$\alpha \in R$, then

$$\begin{aligned}\alpha(A+B) &= \alpha\left([a_{ij}]+[b_{ij}]\right)\\ &= \alpha\left[a_{ij}+b_{ij}\right]\\ &= \left[\alpha\left[a_{ij}+b_{ij}\right)\right]\\ &= \left[\alpha\cdot a_{ij}+\alpha\cdot b_{ij}\right] [\because \text{ Multiplication is } \quad \text{distributive in } R]\\ &= \left[\alpha\cdot a_{ij}\right]+\left[\alpha\cdot b_{ij}\right] \quad \alpha A+\alpha B\end{aligned}$$

(ii) Let $A=[a_{ij}]_{m>n} \in M$ and $a, b \in R$, then

$$\begin{aligned}(a+b)A &= (a+b)\left[a_{ij}\right]\\ &= \left[(a+b)a_{ij}\right]\\ &= \left[a\cdot a_{ij}+b\cdot a_{ij}\right]\\ &= a\left[a_{ij}\right]+b\left[a_{ij}\right]\\ &= aA+bA.\end{aligned}$$

(iii) Let $A=[a_{ij}]_{m\times n} \in M$ and $a, b \in R$ then

$$\begin{aligned}a(bA) &= a\left(b\left[a_{ij}\right]\right)\\ &= a\left(\left[ba_{ij}\right]\right)\\ &= \left[(ab)a_{ij}\right]\\ &= (ab)\left[a_{ij}\right]\\ &= (ab)A\\ a(bA) &= (ab)A.\end{aligned}$$

(iv) Let $A=[a_{ij}]_{m\times n} \in M$, then

$$1.\ A = 1.\left[a_{ij}\right] = \left[1, a_{ij}\right] = \left[a_{ij}\right] = A.$$

Thus M satisfies all the properties of vector space and hence M is a vector space over R.

2.2 Subspaces

A non-empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V.

Eg:- Let

$$W_1 = \{(a, 0, 0) : a \in R\}$$

$$W_2 = \{(a, b, 0) : a, b \in R\}$$

Here W_1 is a subspace of W_2. Also W_1 and W_2 are subspace of R^3.

2.2.1 Necessary and Sufficient conditions for a subspace

Theorem 1: W is a subspace of $V(F)$ iff

i) W is non empty. ii) W is closed under vector addition.

$$\text{i.e } \forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W.$$

iii) W is closed under scalar multiplication. i.e $\forall a \in F$ and $w \in W \Rightarrow a \cdot w \in W$.

Theorem 2 : W is a subspace of $V(F)$ iff

i) W is non empty.

ii) $\forall a, b \in F$ and $v, w \in W \Rightarrow a \cdot v + b \cdot w \in W$.

2.2.2 Problems

1) Show that W is a subspace of $V(R)$ where $W : \{f : f(9) = 0\}$.

Soln:- Since $0 \in W$ as $0(9) = 0$.

So W is non empty set.

Let $f, g \in W$.

i.e. $f(9) = 0$ and $g(9) = 0$

then $\forall a, b \in R$

$$(a.f + b.g)9 = a.f(9) + b.g(9) = a.0 + b.0 = 0$$

Hence $a \cdot f + b \cdot g \in W$. By theorem (2), W is a subspace of V(R).

2) Show that W is a subspace of V(R) where $W = \{f : f(2) = f(1)\}$

Soln:- $0 \in W$ since $0(2) = 0 = 0(1)$. Hence W is a non empty set.

Let $f, g \in W$ then $f(2) = f(1)$ and $g(2) = g(1)$

then $\forall a, b \in R$.

$$\begin{aligned}(a \cdot f + b \cdot g)(2) &= a \cdot f(2) + b \cdot g(2) = a \cdot f(1) + b \cdot g(1)\\ &= (a \cdot f + b \cdot g)(1)\end{aligned}$$

Hence $a \cdot f + b \cdot g \in W$.

So by theorem 2, W is a subspace of $V(R)$.

3) Let $V = R^3$ be the Euclidean 3 space. Let $W = \{(x, y, z) : ax + by + cz = 0; x, y, z \in R\}, a, b, c$ being real numbers. Show that W is a subspace of V.

(OR)

Show that any plane passing through the origin is a subspace of R^3.

Soln:- Let $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$ be any two elements of W where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$.

Then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$.

For $\alpha \in R$,

$$\begin{aligned}\alpha u + v &= \alpha (x_1, y_1, z_1) + (x_2, y_2, z_2)\\ \alpha u + v &= (\alpha x_1, \alpha y_1, \alpha z_1) + (x_2, y_2, z_2)\\ &= (\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2) \rightarrow (1)\end{aligned}$$

where $\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + x_2 \in R$.

Now

$$\begin{aligned}&a(\alpha x_1 + x_2) + b(\alpha y_1 + y_2) + c(\alpha z_1 + z_2) =\\ &\alpha(ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2)\\ &\alpha(0) + 0 = 0 \rightarrow (2)\end{aligned}$$

From (1) and (2), we have

$\alpha u + v \in W$.

Thus, W is a subspace of V.

4). Let V be the vector space of all square matrices over R. Determine which of the following are subspaces of V.

(i) $W = \left\{\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}; x, y, z \in R\right\}$

(ii) $W = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in R\right\}$

(iii) $W = \{A : A \in V$ and A is singular $\}$

(iv) $W : \{A : A \in V, A^2 = A\}$

Soln : :- (i) Let $A = \begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix}, B = \begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$ be any two elements of W.

If $a, b \in R$ then

$$aA + bB = a\begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix} + b\begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} ax_1 & ay_1 \\ az_1 & 0 \end{bmatrix} + \begin{bmatrix} bx_2 & by_2 \\ bz_2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ az_1 + bz_2 & 0 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2 \in R.$$
$$\therefore aA + bB \in W.$$

Thus, W is a sub-space of V.

ii) Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$ be any two elements of W.

If $a, b \in R$, then

$$aA + bB = a\begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + b\begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$$
$$= \begin{bmatrix} ax_1 + bx_2 & 0 \\ 0 & ax_2 + by_2 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2 \in R.$$
$$aA + bB \in W.$$

(iii) Here W is the set of singular matrices.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two square matrices.

Since $|A| = 0$ and $|B| = 0$, therefore $A, B \in W$.

If $a, b \in R$ are non zero, then

$$aA + bB = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Also, $|aA_1 + bA_2| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \neq 0$, as none of a and b is zero.

$$\therefore aA_1 + bA_2 \notin W.$$

Thus, W is not a subspace of V.

(iv) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A \therefore A \in W$.

Now

$$A + A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(A + A)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= A + A$$

$\therefore (A + A)$ is not a member of the set W.

$\Rightarrow$ W not closed under addition.

Hence W is not a sub-space of V.

5) Let V be the vector space of all real valued continuous functions over R. Show that the set W of solutions of differential equations $5\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0$ is a subspace of V.

Soln: - Let $y_1, y_2 \in W$.

Then y_1, y_2 are solutions of differential equations.

$$5\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0$$

i.e $\quad 5\frac{d^2y_1}{dx^2} - 7\frac{dy_1}{dx} + 2y_1 = 0; 5\frac{d^2y_2}{dx^2} - 7\frac{dy_2}{dx} + 2y_2 = 0$

Let $a, b \in R$ be arbitrary.

$$\begin{aligned}
&5\frac{d^2}{dx^2}(ay_1+by_2)-7\frac{d}{dx}(ay_1+by_2)+2(ay_1+by_2)\\
&=5\frac{d^2}{dx^2}(ay_1)-7\frac{d}{dx}(ay_1)+2(ay_1)+5\frac{d^2}{dx^2}(by_2)-7\frac{d}{dx}(by_2)+2(by_2)\\
&=a\left[5\frac{d^2y_1}{dx^2}-\frac{fdy_1}{dx}+2y_1\right]+b\left[5\frac{d^2y_2}{dx^2}7\frac{dy_2}{dx}+2y_2\right]\\
&=a\cdot 0+b.0=0.
\end{aligned}$$

Thus ay_1+by_2 is a solution of (1) & so $ay_1+by_2 \in W$. $\therefore$ W is a subspace of V.

Linear Combination

Let V be a vector space over F and let $v_1, v_2 \dots v_n \in V$. Any vector of the form $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n$ in V, where $a_i \in F$ is called a linear combination of $v_1, v_2 \dots v_n$.

Linear Dependence

Let V be a vector space over F. The vectors $v_1, v_2 \dots v_n$ are said to be linearly dependent over F, if $\exists$ scalars $a_1, a_2 \dots a_n \in F$ not all zero but linear combination is zero.

$$\text{ie } a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0.$$

but all $a_i \neq 0$, where $i \in N$.

Linear Independence

Let V be a vector space over F. The vectors $v_1, v_2 \dots v_n$ are said to be linearly independent over F, if $\exists$ scalars $a_1, a_2 \dots a_n \in F$ such that

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0 \Rightarrow \text{ all } a_i = 0 \text{ where } i \in \mathbb{N}$$

Linear Span

Let S be a subset of the vector space V over F. The set of all linear combinations of vectors in S is called a linear span of S and is denoted by $L(S)$.

Basis or Base of Vector space V

Let V be a vector space over F. The set of vectors $\{v_1, v_2 \dots .v_n\}$ is called a basis of V, if

(i) $v_1, v_2 \dots v_n$ are linearly independent,

(ii) $v_1, v_2 \dots v_n$ span V. ie. each vector of V can be Uniquely expressed as linear combination of $v_1, v_2 \dots, v_n$.

Dimension of a vector space V : −

Number of elements in a basis of vector space V is called the dimension of V and is denoted by $\dim V$.

If V contains a basis with n elements then the $\dim V = n$.

Note: 1) The vector space $\{0\}$ is defined to have dim 0 , since empty set ϕ is independent and generates $\{0\}$.

$\therefore \dim\{0\}$ = Number of elements in $\phi = 0$ [Since no element is in ϕ]

2) When a vector space is not of finite dimension, it is said to be of infinite dimension.

Problems

1) Express the vector $V = (1, -2, 5)$ as a linear combination of the vectors $v_1 = (1, 1, 1), v_2 = (1, 2, 3), v_3 = (2, -1, 1)$ in the vector space $R^3(R)$.

Soln:- Let $v = a_1v_1 + a_2v_2 + a_3v_3; a_1, a_2, a_3 \in R$

$$(1, -2, 5) = a_1(1, 1, 1) + a_2(1, 2, 3) + a_3(2, -1, 1)$$

$$(1, -2, 5) = (a_1 + a_2 + 2a_3, a_1 + 2a_2 - a_3, a_1 + 3a_2 + a_3)$$

Equating the corresponding elements, we get

$$a_1 + a_2 + 2a_3 = 1; a_1 + 2a_2 - a_3 = -2; a_1 + 3a_2 + a_3 = 5$$

Solving above equations, we get

$$a_1 = -6, \quad a_2 = 3, a_3 = 2$$

Hence $(1, -2, 5) = -6(1, 1, 1) + 3(1, 2, 3) + 2(2, -1, 1)$.

2) Write the vector $v = (1, 3, 9)$ as a linear combination of the vectors $u_1 = (2, 1, 3), u_2 = (1, -1, 1), u_3 = (3, 1, 5)$ (Another method of solving)

Soln:- Let $y = xu_1 + yu_2 + zu_3$

$$(1,3,9) = x(2,1,3) + y(1,-1,1) + z(3,1,5)$$

$$\Rightarrow \quad 2x + y + 3z = 1; \quad x - y + z = 3; \quad 3y + y + 5z = 9$$

Consider $Ax = B$.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

Augmented matrix.

$$[A:B] = \begin{bmatrix} 2 & 1 & 3 & 1 & 1 \\ 1 & -1 & 1 & : & 3 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 1 & 3: & 1 \\ 3 & 1 & 5 & 9 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & :3 \\ 0 & 3 & 1: & -5 \\ 0 & 4 & 2:0 & \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & -5 \\ 0 & 1 & 1 & 5 \end{bmatrix}$$

$R_3 \Leftrightarrow R_4$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1: & 3 \\ 0 & 1 & 1: & +5 \\ 0 & 3 & 1: & -5 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2: & -20 \end{bmatrix}$$

$\therefore \text{Rank}[A] = \text{Rank}[A:B] = 3 =$ Number of unknowns.

$\therefore$ The system of linear equations is consistent and possesses a unique solution.

$$x + y + z = 3; \quad y + z = 5; -2z = -20$$

$$\therefore \quad z = 10, \quad y = -5, \quad x = -12$$

$$\therefore \quad (1, 3, 9) = -12(2, 1, 3) - 5(1, -1, 1) + 10(3, 1, 5).$$

3) Write the vector $v = (4, 2, 1)$ as a linear combination of the vectors $u_1 = (1, -3, 1), u_2 = (0, 1, 2), u_3 = (5, 1, 37)$.

Soln:- Let $v = xu_1 + yu_2 + zu_3$

$(4, 2, 1) = x(1, -3, 1) + y(0, 1, 2) + z(5, 1, 3)$

$\Rightarrow x + 0y + 5z = 4$

$-3x + y + z = 2$

$x + 2y + 37z = 1$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - R_1$

$$Ax = B \text{where } A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$[A : B] = \begin{bmatrix} 1 & 0 & 5: & 4 \\ 0 & 1 & 16: & 14 \\ 0 & 2 & 32: & -3 \end{bmatrix}$$

$$[A : B] = \begin{bmatrix} 1 & 0 & 5: & 4 \\ 0 & 1 & 16: & 14 \\ 0 & 0 & 0 & : -31 \end{bmatrix}$$

$\text{Rank}[A] = 2 \neq \text{Rank}[A : B] = 3$.

Since $\text{Rank}[A] \neq \text{Rank}[A : B]$.

Hence the system of linear equations is inconsistent.

i.e solution does not exist.

$\therefore$ v cannot be expressed as linear combination of the vectors u_1, u_2, u_3.

4) For what value of k (if any) the vector $v = (1, -2, k)$ can be expressed as a linear combination of vectors $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$ in $R^3(R)$.

Soln: - Since vector $v = (1, -2, k)$ is a linear combination of $v_1 = (3, 0, -2)$ and $v_2 =$

$(2, -1, -5)$; therefore there exist scalars a and b such that

$$v = av_1 + bv_2$$

$$(1, -2, k) = a(3, 0, -2) + b(2, -1, -5)$$

$$(1, -2, k) = (3a + 2b, -b, -2a - 5b)$$

Equating the corresponding elements,

$$3a + 2b = 1, \quad -b = -2, \quad -2a - 5b = k$$

$$3a + 4 = 1 \quad b = 2 \quad -2(-1) - 5(2) = k$$

$$a = -1 \quad 2 - 10 = k$$

$$\therefore k = -8.$$

5) Find a condition on a, b, c so that $w = (a, b, c)$ is a linear combination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$ in R^3 so that $w \in \text{span}(u, v)$.

Soln:-

$$\text{Let } w = xu + yv$$

$$(a, b, c) = x(1, -3, 2) + y(2, -1, 1)$$

$$1 + 2y = a; -3x - y = b; 2x + y = c$$

Let $AX = B$.

$$\text{Consider } [A : B] = \begin{bmatrix} 1 & 2 : & a \\ -3 & -1 : & b \\ 2 & 1 & : & c \end{bmatrix}$$

$$R_2 \to R_2 + 3R_1$$

$$R_3 \to R_3 - 2R_1$$

$$R_2 \rightarrow \frac{R_2}{5}$$

$$[A:B]: \begin{bmatrix} 1 & 2: & a \\ 0 & 1 & :\frac{b+3a}{5} \\ 0 & -3 & :c-2a \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$[A:B] = \begin{bmatrix} 1 & 2: a \\ 0 & 1: \frac{b+3a}{5} \\ 0 & 0: \frac{5c-a+3b}{5} \end{bmatrix}$$

Rank $[A] = 2$

The system of linear equations will be consistent if $\text{rank}[A:B] = 2$

So, $\frac{5c-a+3b}{5} = 0 \Rightarrow a - 3b - 5c = 0.$

i.e. w is linear combination of u and v if

$$a - 3b - 5c = 0.$$

6) Express the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of

$$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Soln: $Let A = a_1B + a_2C + a_3D; a_1, a_2, a_3 \in R \rightarrow (1)$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$$

Equating the corresponding elements.

$$a_1 + a_2 + a_3 = 3; \quad a_1 + a_2 - a_3 = -1; -a_2 = 1; -a_1 = -2$$

Solving the above equations,

$$a_1 = 2, a_2 = -1, a_3 = 2.$$

$$\therefore \quad (1) \Rightarrow$$

$$A = 2B - C - 2D.$$

(or)

$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is the required linear combination of A.

7) Express the matrix $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

Soln:-

$$\text{Let} \begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = a_1\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + a_3\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix};$$

$$a_1, a_2, a_3 \in R.$$

$$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 0+0+2a_3 & -3a_1+3a_3 \\ 2a_1+2a_2 & a_2+5a_3 \end{bmatrix}$$

$$2 = 2a_3 \Rightarrow a_3 = 1$$

$$\alpha = 2a_3$$

$$0 = -3a_1 + 3a_3 \rightarrow 2$$

$$0 = -3a_1 + 3a_3 \Rightarrow a_1 = 1$$

$$\text{of } = 2a_1 + 2a_2 \rightarrow (3)$$

$$4 = 2a_1 + 2a_2 \Rightarrow a_2 = 1$$

$$-5 = a_2 + 5a_3.$$

Equating the corresponding elements

$$2 = 2a_3 \Rightarrow a_3 = 1$$

$$2 = 2a_3 \Rightarrow a_3 = 1$$

But $a_1 = 1, a_2 = 1, a_3 = 1$ do not satisfy eq (4) Thus equations (1). (2): (3) have no solution.

Hence, the given matrix cannot be expressed as a linear combination of A, B, C.

Note: * Two vectors v_1 and v_2 are linearly dependent af one of them is a multiple of the other.

Eg:- Determine whether or not the vectors v_1, v_2 are linearly dependent.

i) $v_1 = (1, 3, 9) \quad v_2 = (2, 4, 1)$. $\rightarrow$ No vector is the multiple of the other hence $v_1 \& v_2$ are not linearly dependent.

ii) $v_1 = (3, 4) \quad v_2 = (6, 8)$. $\rightarrow v_2 = 2v_1$ hence v_1 and v_2 are linearly dependent vectors.

8) Determine whether the vectors $v_1 = (1, 2, 3), v_2 = (3, 1, 7)$ and $v_3 = (2, 5, 8)$ are linearly dependent or linearly independent.

Soln: $xv_1 + yv_2 + zv_3 = 0$

$$x(1, 2, 3) + y(3, 1, 7) + z(2, 5, 8) = (0, 0, 0)$$

$$x + 3y + 2z = 0; 2x + y + 5z = 0; 3x + 7y + 8z = 0.$$

Let $Ax = 0$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$R_2 \rightarrow -R_2$$

$$R_3 \leftrightarrow -\frac{R_3}{2}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix}_{R_3 \to R_3 - 5R_2}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Rank $[A] = 3 =$ Number of unknowns. $\Rightarrow$ Homogeneous system of linear equations possesses unique solution. $x = 0, y = 0, z = 0$. So vectors v_1, v_2, v_3 are linearly independent.

9) Determine whether the vectors $v_1 = (1, 9, 3) v_2 = (2, 5, 4)$ and $v_3 = (0, 0, 0)$ are linearly dependent or linearly independent.

Soln:- Vectors v_1, v_2, v_3 are linearly dependent as linearly independent set of vectors can not contain zero vector.

10) Determine whether the vectors $v_1 = (1, 4, 9) v_2 = (3, 1, 4)$ and $v_3 = (9, 3, 12)$ are linearly dependent or linearly independent.

Soln:-

$$xv_1 + yv_2 + zv_3 = 0$$

$$x(1, 4, 9) + y(3, 1, 4) + z(9, 3, 12) = (0, 0, 0)$$

$$x + 3y + 9z = 0; \quad 4x + y + 3x = 0; \quad 9x + 4y + 12z = 0$$

Consider $Ax = 0$. Where

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix} \quad R_2 \to R_2 - 4R_1, R_3 \to R_3 - 9R_1$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix} R_2 \to -\frac{R_2}{11}, R_3 \to -\frac{R_3}{23}$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} R_3 \to R_3 - R_2$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}[A] = 2 < 3$ (Number of unknowns).

So one unknown is constant and hence not always zero.

So the vectors are linearly dependent.

$[OR]$

$$v_3 = 3v_2$$

vectors v_1, v_2, v_3 are linearly dependent.

11) Determine whether or not each of the following forms a basis, $x_1 = (2, 2, 1), x_2 = (1, 3, 7), x_3 = (1, 2, 2)$ in R^3.

Soln: - Three vectors in R^3 form a basis iff they are linearly independent.

$$x(2, 2, 1) + y(1, 3, 7) + z(1, 2, 2) = (0, 0, 0)$$

$$2x + y + z = 0; \quad 2x + 3y + 2z = 0; \quad x + 7y + 2z = 0.$$

Let $AX = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{bmatrix}_{R_1 \leftrightarrow R_3}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix}_{R_2 \to R_2 - 2R_1}$$

$$R_3 \to R_3 - 2R_1$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -13 & -3 \end{bmatrix}$$

$$R_2 \to -R_2$$

$$R_3 \to -R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 13 & 3 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 1 & 1/2 \end{bmatrix}_{R_3 \leftrightarrow R_2} = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 11 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -7/2 \end{bmatrix}$$

Rank $[A] = 3$ = Number of unknowns. So unique solution and solution is

$$x = 0, y = 0, z = 0.$$

Hence vectors x_1, x_2, x_3 are linearly independent and hence form a basis.

12) Let W be the subspace of R^5, spanned by

$$x_1 = (1, 2, -1, 3, 4), x_2 = (2, 4, -2, 6, 8), x_3 = (1, 3, 2, 2, 6)$$

$$x_4 = (1, 4, 5, 1, 8), x_5 = (2, 7, 3, 3, 9).$$

Find a subset of vectors which forms a basis of W.

Soln:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$R_5 \to R_5 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$$R_4 \to R_4 - 2R_3, R_5 \to R_5 - 3R_3$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

Number of nonzero rows $= 3$.

$\therefore$ $\dim W = 3$ and $\{x_1, x_3, x_5\}$ forms a basis in W.

13) Let V is a vector space of polynomials over R. Find a basis and dimension of the subspace W of V, spanned by the polynomials.

$$x_1 = t^3 - 2t^2 + 4t + 1, x_2 = 2t^3 - 3t^2 + 9t - 1$$

$$x_3 = t^3 + 6t - 5, \quad x_4 = 2t^3 - 5t^2 + 7t + 5.$$

Soln:

$$A=\begin{bmatrix}1&-2&4&1\\2&-3&9&-1\\1&0&6&-5\\2&-5&7&5\end{bmatrix}=\begin{bmatrix}1&-2&4&1\\0&1&1&-3\\0&2&2&-6\\0&-1&-1&3\end{bmatrix}$$

$R_2\to R_2-2R_1\quad R_3\to R_3-2R_2$

$R_3\to R_3-R_1\quad R_4\to R_4+R_2$

$$A=\begin{bmatrix}1&-2&4&1\\0&1&1&-3\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$R_4\to R_4-2R_1$

Nonzero rows of Echelon matrix forms basis.

Number of non Zero rows $=2$

$\therefore$ dim $W=2$ and $\{x_1,x_2\}$ forms a basis in W.

2.3 Linear Transformations

Let V and W be any two subspaces over R. A mapping T from V to W is called a linear transformation if,

i) $v_1,v_2\in V$

$$T(v_1+v_2)=T(v_1)+T(v_2)$$

ii) $\forall a\in F$ and $\forall v\in V$

$$T(a\cdot v)=a\cdot T(v)$$

Problems

1) Show that the function $T:R^2\to R^3$ given by $T(x,y)=(x+y,x-y,y)$ is a linear transfomation.

Soln :- Let $u = (x_1, y_1), v = (x_2, y_2)$ be two vectors belonging to R^2.

$$\begin{aligned}
\therefore T(u+v) &= T(x_1 + x_2, y_1 + y_2) \\
&= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2, y_1 + y_2) \\
&= \{(x_1 + y_1) + (x_2 + y_2), (x_1 - y_1) + (x_2 - y_2), (y_1 + y_2)\} \\
&= (x_1 + y_1, x_1 - y_1, y_1) + (x_2 + y_2, x_2 - y_2, y_2) \\
&= T(u) + T(v)
\end{aligned}$$

For $a \in R$ and $u \in R^2$, we have

$$\begin{aligned}
T(au) &= T(ax_1, ay_1) \\
&= (ax_1 + ay_1, ax_1 - ay_1, ay_1) \\
&= a(x_1 + y_1, x_1 - y_1, y_1) \\
&= aT(u)
\end{aligned}$$

$\therefore T$ is a linear transformation.

2) Which of the following functions are linear transformation?

i) $T : R^3 \to R^3$ defined by $T(x, y, z) = (y, -x, -z)$

ii) $T : R^3 \to R^2$ defined by $T(x, y, z) = (2x - 3y, 7y + 2z)$.

Sol:- i) Let $u = (x_1, y_1, z_1), v = (x_2, y_1, z_2) \in R^3$ be arbitrary.

$$T(u) = T(x_1, y_1, z_1) = (y_1, -x_1, -z_1)$$

$$T(v) = T(x_2, y_2, z_2) = (y_2, -x_2, -z_2)$$

$$\begin{aligned}
T(u+v) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\
&= (y_1 + y_2, -x_1 - x_2, -z_1 - x_2) \\
&= (y_1, -x_1, -z_1) + (y_2, -x_2, -x_2) \\
&= T(u) + T(v)
\end{aligned}$$

Also, for any scalar $a \in R$,

$$\begin{aligned}
T(au) &= T(ax_1, ay_1, az_1) \\
&= (ay_1, -ax_1, -az_1) \\
&= a(y_1, -x_1, -z_1) \\
&= aT(u)
\end{aligned}$$

Hence, T is a linear transformation.

ii)

$$\text{Let } u=(x_1,y_1,z_1) \text{ and } v=(x_2,y_2,z_2)\in R^3 \text{ be arbitrary.}$$

$$T(u)=T(x_1,y_1,z_1)=(2x_1-3y_1,7y_1+2z_1)$$

$$T(v)=T(x_2,y_2,x_2)=(2x_2-3y_2,7y_2+2z_2)$$

$$\therefore T(u+v)=T(x_1+x_2,y_1+y_2,z_1+x_2)$$

$$=(2x_1+2x_2-3y_1-3y_2,7y_1+7y_2+2z_1+2z_2)$$

$$=\{(2x_1-3y_1)+(2x_2-3y_2),(7y_1+2z_1)+(7y_2+2z_2)\}$$

$$=(2x_1-3y_1,7y_1+2z_1)+(2x_2-3y_2,7y_2+2z_2)$$

$$=T(u)+T(v)$$

Also, for any scalar $a\in R$

$$T(au)=T(ax_1,ay_1,az_1)$$

$$=(2ax_1-3ay_1,7ay_1+2az_1)\quad =a(2x_1-3y_1,7y_1+2y_1)$$

$$=aT(u).$$

Hence, T is a linear transformation.

3) Let $I:V(R)\to V_2(R)$ be a mapping $f(x)=(3x,5x)$. Show that f is a linear transformation.

Soln:- i)

$$f(x_1+x_2)=(3(x_1+x_2),5(x_1+x_2))$$

$$=(3x_1+3x_2,5x_1+5x_2)$$

$$=(3x_1+5x_1)+(3x_2,5x_2)$$

$$=f(x_1)+f(x_2)$$

ii)

$$f(a\cdot x)=(3(a\cdot x),5(a\cdot x))$$

$$=(3ax,5ax)$$

$$=a(3x,5x)$$

$$=a\cdot f(x)$$

Hence f is a linear transfomation.

4) Show that the translation mapping $f:V_2(R)\to V_2(R)$ defined by $f(x,y)=(x+6,y+2)$

is not linear.

Soln:

$$f(x_1+x_2, y_1+y_2) = (x_1+x_2+6, y_1+y_2+2) \rightarrow (1)$$
$$f(x_1, y_1) + f(x_2, y_2) = (x_1+6, y_1+2) + (x_2+6, y_2+2)$$
$$= (x_1+x_2+12, y_1+y_2+4) \rightarrow (2)$$

Equation (1) $\neq$ Equation (2).

Hence f is not a linear transformation.

2.4 Kernel of T and Image of T

Let T be a linear transformation from V to W.

Kernel of T or $\ker(T) = \{v \in V : T(v) = 0, 0 \in W\}$.

Remark:- Kernel of T is also known as null space of T.

Image of T or $\operatorname{Img}(T) = \{w \in W; w = T(v)$ for some $v \in V\}$

2.4.1 Algebra of Linear Transformations

Sum of linear transformations

Let $T_1 : U \to V$ and $T_2 : U \to V$ be two transformations where U and V are two vector spaces over R. We define the sum of T_1 and T_2 by $T_1 + T_2 : U \to V$, such that $(T_1 + T_2)\, u = T_1(u) + T_2(u), u \in U$.

Scalar multiplication of Línear Transformations

If $a \in R$, the function aT, ie product of a linear transformation $T : U \to V$ with a, and defined by $(aT) : U \to V$, such that $(aT)u = a(T(u))$ for all $u \in U$ is a linear transformation from U into V.

Composition (Product) of two Linear transformations:

Let U, V, W be three vector spaces over R. We define T_2T_1, called the composition or product of T_2 and T_1, by $(T_2T_1)(u) = T_2(T_1(u))$.

T_2T_1 is also denoted by $(T_2 \circ T_1)$.

Problems:

1) Let the linear transformations $T_1 : R^3 \to R^2$ such that $T_1(x, y, z) = (4x, 3y - 2z)$ and $T_2 : R^2 \to R^2$ such that $T_2(x, y) = (-2x, y)$. Compute T_1T_2 and T_2T_1.

Soln: - As the range of T_2 is not contained in the domain of T_1, $\therefore$ T_1T_2 is not defined.

Now, T_2T_1 is defined as the range of T_1 is contained in the domain of T_2.

$$\begin{aligned} T_2T_1(x, y, z) &= T_2(T_1(x, y, z)) \\ &= T_2(4x, 3y - 2z) \\ &= (-8x, 3y - 2z). \end{aligned}$$

2) Illustrate with the help of an example that there exist linear transformations $T_1 : R^2 \to R^2$ and $T_2 : R^2 \to R^2$ such that $T_2T_1 = 0$ but $T_1T_2 \neq 0$.

Soln : - Let us define $T_1 : R^2 \to R^2$ by

$$T_1(x, y) = (0, 4x)$$

& $T_2 : R^2 \to R^2$ by $T_2(x, y) = (x, 0)$

$\therefore$ T_1T_2 and T_2T_1 both are defined.

$$(T_2T_1)(x, y) = T_2(T_1(x, y)) = T_2(0, 4x)$$

$$= (0, 0) = 0(x, y)$$

$$T_2T_1 = 0.$$

Also

$$\begin{aligned} (T_1, T_2)(x, y) &= T_1(T_2(x, y)) = T_1(x, 0) \\ &= (0, 4x) \neq 0(x, y) \end{aligned}$$

$$T_1T_2 \neq 0.$$

2.4.2 Matrix of a Linear transformation

Let $T : U \to V$ be the linear transformation, where U and V are vector spaces over R. Let $B = \{u_1, u_2 \ldots . u_n\}$ and $B' = \{v_1, v_2 \ldots v_m\}$ be ordered basis for the finite dimensional vector spaces U and V respectively. Since $T(u_1), T(u_2) \ldots T(u_n) \in V$ and $\{v_1, v_2 \ldots v_m\}$ spans V, each $T(u_i)$ can be expressed at a linear combination of the vectors $v_1, v_2 \ldots, v_m$.

Let $T(u_1) = a_{11}v_1 + a_{21}v_2 + \ldots . + + a_{m1}v_m$

$T(u_2) = a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m$

$T(u_n) = a_{1n}v_1 + a_{2n}v_2 + \ldots . + a_{mn}v_m$ where $a_{ij} \in R$.

The coefficient matrix of this system of equations is

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \ldots & a_{m2} \\ - & - & - & \cdots & - \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}$$

The transpose of this matrix is a matrix representation of T, called matrix of T with respect to ordered basis B and B'. It is denoted by $[T : B, B']$.

Problems

1) Find the matrix representing the transformation $T : R^2 \to R^3$ given by

$T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2)$ relative to the standard basis of R^2 and R^3.

Soln: - The ordered standard basis of R^2 is

$B = \{(1, 0), (0, 1)\}$ and that of R^3 is

$B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

$\therefore T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2)$

$T(1, 0) = (3, 2, 5)$ & $T(0, 1) = (-1, 4, -6)$

$T(1, 0) = (3, 2, 5) = 3(1, 0, 0) + 2(0, 1, 0) + 5(0, 0, 1)$

$T(0, 1) = (-1, 4, -6) = -1(1, 0, 0) + 4(0, 1, 0) + (-6)(0, 0, 1)$

Hence, matrix of the transformation is $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

2) Find the matrix representing the transformation $T : R^3 \to R^4$ defined by $T(x, y, z) =$

$(x + y + z, 2x + z, 2y - z, 6y)$ relative to the standard basis of R^3 and R^4.

Sol: - We know that ordered standard basis of R^3 is $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and R^4 is $B' = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

Since $T(x, y, z) = (x + y + z, 2x + z, 2y - z, 6y)$

$$T(1, 0, 0) = (1, 2, 0, 0)$$

$$T(0, 1, 0) = (1, 0, 2, 6)$$

$$T(0, 0, 1) = (1, 1, -1, 0)$$

Let $e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)$

$\therefore \quad T(1, 0, 0) = (1, 2, 0, 0) = 1e_1 + 2e_2 + 0e_3 + 0e_4$

$T(0, 1, 0) = (1, 0, 2, 6) = 1e_1 + 0e_2 + 2e_3 + 6e_4$

$T(0, 0, 1) = (1, 1, -1, 0) = 1e_1 + 1e_2 - 1e_3 + 0e_4$

Matrix of T relative to B and B' is transpose of matrix of coefficients in above system of equations

i.e, $[T : B, B'] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$.

2.4.3 Change of coordinates or change of Basis Matrix or Transition Matrix

Let $B_1 = \{u_1, u_2 \ldots, u_n\}$ be a basis of n-dimensional vector space V and $B_2 = \{v_1, v_2 \ldots v_n\}$ be another basis of V. Then, each element in B_2 can be expressed as a linear combination of the vectors in basis B_1.

Let

$$v_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$v_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$

$$\vdots$$

$$v_n = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & \cdots a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Then the transpose of the matrix of coefficients in above system of equations is said to be the transition matrix or change of basis matrix from the old basis B_1 to the new basis B_2. It is denoted by P and written as

$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

Since the vectors in B_2 are linearly independent, so the matrix P is invertible. Its inverse P' is the transition matrix from the basis B_2 to the basis B_1.

Problems:- 1) Consider the following basis of R^2 :

$$E = \{e_1, e_2\} = \{(1, 0), (0, 1)\} \text{ and } S = \{u_1, u_2\} = \{(1, 3), (1, 4)\}$$

(a) Find the change-of-basis matrix P from the usual basis E to S.

(b) Find the change - of -basis matrix Q from S back to E.

(c) Find the coordinate vector $[v]$ of $v = (5, -3)$ relative to S.

Soln: (a) Since E is the usual basis of R^2,

$$\therefore P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

(b) Find P^{-1}

$$P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

(c) Let $[v]_S = P^{-1}[v]_E$ we have $[v]_E = [5, -3]^\top$

$$[v]_S = P^{-1}[v]_E = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -18 \end{bmatrix}.$$

2) The vectors $u_1 = (1,2,0), u_2 = (1,3,2), u_3 = (0,1,3)$ form a basis S of R^3. Find:

a) The change of basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of R^3 to the basis S.

b) The change of basis matrix Q from the above basis S back to the usual basis E of R^3.

Soln: - (a) Since E is a usual basis of R^3.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

(b) Find P^{-1}

$$P^{-1} = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

3) Find the co-ordinates of vector $(1,1,1)$ relative to basis $(1,1,2), (2,2,1), (1,2,2)$.

Sol:-

$$\text{Let } (1,1,1) = a(1,1,2) + b(2,2,1) + c(1,2,2)$$

$$a + 2b + c = 1 \to (1)$$

$$a + 2b + 2c = 1 \to (2)$$

$$2a + b + 2c = 1 \to (3)$$

Subtracting (1) from (2), we have $c = 0$.

putting $c = 0$ in (2) and (3), we get

$$a + 2b = 1 \to (4)$$

$$2a + b = 1 \to (5)$$

Solving (4) and (5). we get $a = b = \frac{1}{3}$.

$\therefore$ Coordinates of $(1,1,1)$ relative to given basis are $\left(\frac{1}{3}, \frac{1}{3}, 0\right)$.

4) Find the co-ordinates of the following rectors relative to the basis $y_1 = (1,1,2), y_2 = (2,2,1), y_3 = (1,2,2)$.

$(i)(1,0,1)$ $\quad$ sol $n : -\left(\frac{4}{3}, \frac{1}{3}, -1\right)$

(ii) $(1,1,0)$ $\quad$ sol$^n - \left(-\frac{1}{3}, \frac{2}{3}, 0\right)$

2.5 Rank and nullity of a linear Operator

Rank of $T:-$ Dimension of the $\operatorname{Img}(T)$ is known as rank of T.

Nullity of T : - Dimension of the $\operatorname{Ker}(T)$ is known as nullity of T.

2.5.1 Rank-nullity theorem

Let V and W be vector spaces over R and let T be a linear transformation from V to W. If V is a finite dimensional then $\operatorname{rank}(T)+$ nullity of $(T)=\dim(V)$.

Problems:

1) For the linear transformation $T: R^2 \rightarrow R^3$ such that $T\left(x_1, x_2\right)=\left(x_1-x_2, x_2-x_1,-x_1\right)$, find a basis and dimension of its range space and its null space. Also verify that $\operatorname{rank}(T)+$ nullity $(T)=\dim R^2$.

Sol: - i) To find null space of T and its dimension:

By definition, null space $=N(T)=\left\{u \in R^2: T(u)=0 \in R^3\right\}$

Let $u=\left(x_1, x_2\right) \in R^2$ be an arbitrary element of null space.

$$T(u)=0$$

$$T\left(x_1, x_2\right)=0$$

$$\left(x_1-x_2, x_2-x_1,-x_1\right)=0$$

$$\Rightarrow \quad \begin{aligned} x_1-x_2&=0\\ x_2-x_1&=0\\ -x_1&=0\end{aligned}$$

On solving the above equations, We get

$$x_1=0, x_2=0.$$

Thus, the only member of null space is a zerovector $\in R$

$$\text{i.e } N(T)=\{0\}.$$

$\therefore$ Nullity of $T=\dim(N(T))=0$.

ii) To find range space of T and it ts dimension:

The range space consists of ordered triples $(x_1 - x_2, x_2 - x_1, -x_1)$ for $(x_1, x_2) \in R^2$ such that

$$T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$$

Let v be any element of $R(T)$.

Thus there exist $(x_1, x_2) \in R^2$ such that $v = T(x_1, x_2)$

$$\Rightarrow (x_1, x_2) = x_1(1,0) + x_2(0,1)$$
$$= x_1e_1 + x_2e_2 \text{ where } e_1 = (1,0), e_2 = (0,1)$$
$$T(x_1, x_2) = T(x_1e_1 + x_2e_2)$$
$$= x_1T(e_1) + x_2T(e_2) \to (2)$$

$$T(e_1) = T(1,0) = (1-0, 0-1, -1) = (1,-1,-1)$$
$$T(e_2) = T(0,1) = (0-1, 1-0, 0) = (-1,1,0)$$

Putting the values of $T(e_1), T(e_2)$ in (2)

$$T(x_1, x_2) = x_1(1,-1,-1) + x_2(-1,1,0)$$
$$v = x_1(1,-1,-1) + x_2(-1,1,0)$$

Since $v \in R(T)$ is arbitrary.

(iii) Let $a(1,-1,-1) + b(-1,1,b) = 0$ for $a, b \in R$.

$$\Rightarrow (a-b, -a+b, -a) = (0,0,0)$$
$$\Rightarrow a-b=0, b-a=0, -a=0$$
$$\Rightarrow a=b=0.$$

Thus $\{(1,-1,-1), (-1,1,0)\}$ is linearly independent.

$\therefore$ It is a basis set of $R(T)$.

$\therefore$ Dimension of $R(T) = 2$

Also dimension of $R^2 = 2$

$\therefore$ **Rank** $N(T)$ + **nullily**$(T) = 2 + 0 = 2 = \dim R^2$.

2) Find a linear transformation $T : R^3 \to R^3$ whose range is spanned by the vectors $(1,2,3), (4,5,6)$.

Soln:- We know that $\{e_1, e_2, e_3\}$ is a standard basis of R^3, where $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$

Since R^3 is a three dimensional vector space and $\{e_1, e_2, e_3\}$ is a basis of R^3, there exist a unique linear transformation T such that

$$T(e_1) = (1, 2, 3)$$

$$T(e_2) = (4, 5, 6)$$

$$T(e_3) = (0, 0, 0)$$

Also $R(T)$ is spanned by $\{(1, 2, 3), (4, 5, 6)\}$

i.e by $\{(1, 2, 3), (4, 5, 6), (0, 0, 0)\}$ or by $\{T(e_1), T(e_2), T(e_3)\}$

Now, for each $(x_1, x_2, x_3) \in R^3$.

$$(x_1, x_2, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$$

$$= x_1e_1 + x_2e_2 + x_3e_3$$

$$T(x_1, x_2, x_3) = x_1T(e_1) + x_2T(e_2) + x_3T(e_3)$$

$$= x_1(1, 2, 3) + x_2(4, 5, 6) + x_3(0, 0, 0)$$

$$= (x_1 + 4x_2, 2x_1 + 5x_2, 3x_1 + 6x_2)$$

which is the required linear transformation.

3) Verify the Rank-nullity theorem for the $T : R^3 \to R^3$ defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

Soln:- To find the basis of nullity of T

Let $v = (x, y, z) \in R^3$ such that Nullity of $T = \{v \in V \mid T(v) = 0\}$

$$T(x, y, z) = 0.$$

$$(x + 2y - z, y + z, x + y - 2z) = (0, 0, 0).$$

$$x + 2y - z = 0 \Rightarrow x = 3z$$

$$y + z = 0 \Rightarrow y = -z.$$

$$x + y - 2z = 0$$

On solving above equations we get

$$x = 3z, \quad y = -z$$

$$\therefore \quad \{(x, y, z)\} = \{3z, -z, z\} = (z\{3, -1, 1\})$$

Thus $\{(3, -1, 1)\}$ is a basis of nullity of T & Nullity $(T) = 1$.

To find the basis of range of T.

As $\{(1, 0, 0), (0, 10), (0, 0, 1)\}$ generates R^3

$\Rightarrow \{T(1,0,0), T(0,1,0), T(0,0,1)\}$ generates range

$\Rightarrow \{(1,0,1),(2,1,1),(-1,1,-2)\}$ generates range of T

To find the basis of range, consider a matrix.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1$$

$$R_3 \to R_3 - R_2$$

$$R_3 \to R_3 + R_2$$

Thus $\{(1,0,1),(0,1,-1)\}$ form a basis of Range of T & $\dim($ Range of $T) = 2$.

$$\therefore \text{Rank}(T) + \text{Nullity}(T) \ = 2 + 1 = 3 = \dim\left(R^3\right).$$

Hence rank-Nullity theorem verified.

4) Verify the Rank-Nullily theorem for the $T : R^3 \to R^3$ defined by $T(x,y,z) = (x+2y, y-z, x+2z)$.

2.6 Inner Product Spaces and Orthogonality

2.6.1 Inner Product Space

Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number, denoted by $\langle u, v\rangle$. This function is called inner product on V if it satisfies the following axioms:

(i) Linear Property: $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$

(ii) Symmetric Property: $\langle u, v\rangle = \langle v, u\rangle$

(iii) Positive Definite Property: $\langle u, u\rangle \geqslant 0$; and $\langle u, u\rangle = 0$ if and only $u = 0$.

Norm of a vector

An inner product, $\langle u, u\rangle$ is nonnegative for any vector u.

$$\|u\| = \sqrt{\langle u, u\rangle} \text{ or } \|u\|^2 = \langle u, u\rangle$$

This non negative number is called the norm or length of is $\boldsymbol{u}$.

* Every non- zero vector $\boldsymbol{v}$ in $\boldsymbol{V}$ can be multiplied by the reciprocal of its length to obtain the unit vector $\hat{v} = \frac{1}{\|v\|}\boldsymbol{v}$, which is a positive multiple. This process is called normalizing $\boldsymbol{v}$.

Orthogonalily

Let $\boldsymbol{V}$ be an inner product space. The vectors $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ are said to be orthogonal and $\boldsymbol{u}$ if said to be orthogonal to $\boldsymbol{v}$ if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$.

Problems

1) Consider vectors $\boldsymbol{u} = (1, 2, 4), \boldsymbol{v} = (2, -3, 5), \boldsymbol{w} = (4, 2, -3)$ in $\boldsymbol{R}^3$. Find

a) $\langle \boldsymbol{u} \cdot \boldsymbol{v} \rangle$

b) $\langle \boldsymbol{u} \cdot \boldsymbol{w} \rangle$

c) $\langle \boldsymbol{v} \cdot \boldsymbol{w} \rangle$

d) $\langle (\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{\omega} \rangle$

e) $\|\boldsymbol{u}\|$

f) $\|\boldsymbol{v}\|$

Soln: a) $\langle \boldsymbol{u} \cdot \boldsymbol{v} \rangle = 2 - 6 + 20 = 16$

b) $\langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle = 4 + 4 - 12 = -4$

c) $\langle \boldsymbol{v} \cdot \boldsymbol{\omega} \rangle = 8 - 6 - 15 = -13$

d) $\langle (\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} \rangle = (3, -1, 9) \cdot (4, 2, -3) = 12 - 2 - 27 = -17$.

e) $\|\boldsymbol{u}\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21}$.

f) $\|\boldsymbol{v}\| = \sqrt{4 + 9 + 25} = \sqrt{38}$.

2) Consider the vectors $\boldsymbol{u} = (1, 5)$ and $\boldsymbol{v} = (3, 4)$ in $\boldsymbol{R}^2$. Find:

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with respect to the usual inner product in $\boldsymbol{R}^2$.

b) $\|\boldsymbol{v}\|$ using the inner product in $\boldsymbol{R}^2$.

3) Consider the following polynomials in $\boldsymbol{P}(t)$ and inner product:

$$f(t) = t + 2, g(t) = 3t - 2, h(t) = t^2 - 2t - 3 \text{ and}$$

$$< f, g >= \int_0^1 f(t)g(t)dt.$$

(a) find $\langle f, g \rangle$ and $\langle f, h \rangle$

(b) find $\|f\|$ and $\|g\|$

(c) normalize f and g.

Soln: : (a)

$$< f,g\rangle = \int_0^1 (t+2)(3t-2)dt = \int_0^1 \left(3t^2+4t-4\right) dt$$
$$< f,g\rangle = t^3+2t^2-4t_0' = -1.$$
$$\langle f,h\rangle = \int_0^1 (t+2)\left(t^2-2t-3\right) dt = \frac{t^4}{4} - \frac{7t^2}{2} - 6t\Big]_0^1 = -\frac{37}{4}$$

(b)

$$\langle f,f\rangle = \int_0^1 (t+2)(t+2)dt = \frac{19}{3}; \|f\| = \frac{\sqrt{19}}{\sqrt{3}} = \frac{\sqrt{57}}{3}.$$
$$\langle g,g\rangle = \int_0^1 (3t-2)(3t-2)dt = 1; \|g\| = \sqrt{1} = 1$$

(c) Since $\|f\| = \frac{\sqrt{57}}{3}$ and g is already a unit vector,

$$\hat{f} = \frac{1}{\|f\|} f = \frac{3}{\sqrt{57}}(t+2)$$
$$\hat{g} = \frac{1}{\|g\|} g = 3t-2.$$

4) Let $M = M_{2,3}$ with inner product, $\langle A,B\rangle = tr\left\langle B^\top A\right\rangle$ and let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$ Find (a) $\langle A,B\rangle, \langle A,C\rangle, \langle B,C\rangle$ (b) $\langle 2A+3B, 4C\rangle$ (c) $\|A\|$ and $\|B\|$

$$\text{(a) } \langle A,B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}$$
$$\langle A,B\rangle = 9+16+21+24+25+24 = 119$$
$$\langle A,C\rangle = 27-40+14+6+0-16 = -9$$
$$\langle B,C\rangle = 3-10+6+4+0-24 = -21$$

(b)

$$2A+3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix} \quad 4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$$
$$\langle 2A+3B, 4C\rangle = 252-440+96+0-416 = -324$$

(c) $\|A\|^2 = \langle A, A\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$, the sum of the squares of all the elements of A.

$$\|A\|^2 = \langle A, A\rangle = 9^2 + 8^2 + 7^2 + 6^2 + 5^2 + 4^2 = 271 \Rightarrow \|A\| = \sqrt{271}$$

$$\|B\|^2 = \langle B, B\rangle = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91 \Rightarrow \|B\| = \sqrt{91}$$

5) Verify the vectors $u = (1, 1, 1), v = (1, 2, -3)$ and $w = (1, -4, 3)$ in R^3 are orthogonal or not.

Soln: $\langle u, v\rangle = 1 + 2 - 3 = 0, \langle u, w\rangle = 1 - 4 + 3 = 0, \langle v, w\rangle = 1 - 8 - 9 = -16$

Thus u is orthogonal to v and w, v and w are not orthogonal.

Lecture Notes

BMATE201

Mathematics-II for EEE stream

Module 3 : Laplace Transforms

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Module 3

Laplace Transforms

Let $f(t)$ be a real-valued function defined for all $t > 0$ and s be a parameter, real or complex. Suppose the integral $\int_0^\infty e^{-st} f(t)dt$ exists (converges). Then this integral is called the Laplace transform of $f(t)$ and is denoted by $L[f(t)]$.
Thus

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)dt} \tag{1}$$

We note that the value of the integral on the right hand side of (1) depends on s. Hence $L[f(t)]$ is a function of s denoted by $F(s)$ or $\overline{f}(s)$ Thus,

$$L[f(t)] = F(s) \tag{2}$$

Consider relation (2). Here $f(t)$ is called the Inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}[F(s)]$ Thus,

$$L^{-1}[F(s)] = f(t)$$

Suppose $f(t)$ is defined as follows :

$$f(t) = \begin{cases} f_1(t), & 0 < t \leq a \\ f_2(t), & a < t \leq b \\ f_3(t), & t > b \end{cases}$$

Note that $f(t)$ is piecewise continuous. Then Laplace transform of $f(t)$ is defined as

$$L[f(t)] = \int_0^a e^{-st} f(t)dt + \int_a^b e^{-st} f(t)dt + \int_b^\infty e^{-st} f(t)dt$$

NOTE: In a practical situation, the variable t represents the time and s represents frequency. Hence the Laplace transform converts a problem in the time domain into the frequency domain.This makes the problem much easier to solve.

3.1 Laplace Transform of Standard Functions

Sl. No.	Laplace Transform
1	$L\{1\} = 1$
2	$L\{e^{at}\} = \frac{1}{s-a}$
3	$L\{sinat\} = \frac{a}{s^2+a^2}$
4	$L\{cosat\} = \frac{s}{s^2+a^2}$
5	$L\{sinhat\} = \frac{a}{s^2-a^2}$
6	$L\{coshat\} = \frac{s}{s^2-a^2}$
7	$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$ where $\Gamma(n+1) = \begin{cases} n!, & \text{n is an integer} \\ n\ \Gamma(n), & \text{n is a non- integer} \end{cases}$

3.2 Basic Properties

The following are some basic properties of Laplace transforms:

3.3 Linearity property:

For any two functions $f(t)$ and $g(t)$ (whose Laplace transforms exist) and any two constants a and b, we have

$$\boxed{L[af(t) + b(t)] = aL[f(t)] + bL[(t)]}$$

Problem 3.3.1. *Find the Laplace Transforms of* $e^{2t} + 4t^3 - 2sin3t + 3cos3t$

Solution : Let $f(t) = e^{2t} + 4t^3 - 2sin3t + 3cos3t$ then

$$L\{f(t)\} = L\{e^{2t}\} + 4L\{t^3\} - 2L\{sin3t\} + 3L\{cos3t\}$$
$$= \frac{1}{s-2} + 4\frac{3!}{s^4} - 2\frac{3}{s^2+3^2} + 3\frac{s}{s^2+3^2}$$
$$= \frac{1}{s-2} + \frac{24}{s^4} + 3\frac{(s-2)}{s^+9}$$

Problem 3.3.2. *Find the Laplace Transforms of* $Sin2tsin3t$

Solution : Let $f(t) = Sin2tsin3t$

(Recall $sinAsinB = \frac{cos(A-B)-cos(A+B)}{2}$)

then

$$L\{f(t)\} = L\left\{\frac{cos(2t-3t) - cos(2t+3t)}{2}\right\}$$
$$= \frac{1}{2}L\left\{cos(-t) - cos(5t)\right\}$$
$$= \frac{1}{2}L\left\{cos(t) - cos(5t)\right\} \qquad (\because cos(-t) = cost)$$
$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

Problem 3.3.3. *Find the Laplace Transforms of* $\sin 5t \cos 3t$

Solution : Since,

$$\sin 5t \cos 3t = \frac{1}{2}[\sin(5t+3t) + \sin(5t-3t)]$$
$$= \frac{1}{2}[\sin 8t + \sin 2t]$$

Therefore,

$$L(\sin 5t \cos 3t) = L\left\{\frac{1}{2}[\sin 8t + \sin 2t]\right\}$$
$$= \frac{1}{2}\{L(\sin 8t) + L(\sin 2t)\}$$
$$= \frac{1}{2}\left[\frac{8}{s^2+8^2} + \frac{2}{s^2+2^2}\right]$$
$$= \frac{5\left(s^2+16\right)}{\left(s^2+64\right)\left(s^2+4\right)}$$

Problem 3.3.4. *Find the Laplace Transforms of* $\sin t \sin 2t \sin 3t$

Solution:

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

$$\begin{aligned}
\therefore f(t) &= \sin t \sin 2t \sin 3t \\
&= \sin t \ \frac{1}{2}[\cos t - \cos 5t] \\
&= \frac{1}{2}[\sin t \cos t - \sin t \cos 5t] \\
&= \frac{1}{4}[\sin 2t - \sin 6t + \sin 4t] \\
&= \frac{1}{4}[\sin 2t + \sin 4t - \sin 6t]
\end{aligned}$$

Therefore,

$$\begin{aligned}
L(\sin t \sin 2t \sin 3t) &= \frac{1}{4} \ L[\sin 2t + \sin 4t - \sin 6t] \\
&= \frac{1}{4}\left[\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2+36}\right]
\end{aligned}$$

Problem 3.3.5. *Find the Laplace Transforms of* $\cos t \cos 2t \cos 3t$

Solution : We have $\cos t \cos 2t \cos 3t$

$$\begin{aligned}
&= \frac{1}{2}\cos t(\cos 5t + \cos t) \\
&= \frac{1}{2}\left[\cos 5t \cos t + \cos^2 t\right] \\
&= \frac{1}{2}\left[\frac{1}{2}(\cos 6t + \cos 4t) + \frac{1}{2}(1 + \cos 2t)\right] \\
&= \frac{1}{4}(\cos 6t + \cos 4t) + \frac{1}{4}(1 + \cos 2t) \\
&= \frac{1}{4}[1 + \cos 2t + \cos 4t + \cos 6t]
\end{aligned}$$

Therefore,

$$\begin{aligned}
L(\cos t \cos 2t \cos 3t) &= \frac{1}{4}[L(1) + L(\cos 2t) + L(\cos 4t) + L(\cos 6t)] \\
&= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+2^2} + \frac{s}{s^2+4^2} + \frac{s}{s^2+6^2}\right].
\end{aligned}$$

Problem 3.3.6. *Find Laplace Transform of* $\sin^2 t$

Solution :

$$\begin{aligned} L\left(\sin^2 t\right) &= L\left(\frac{1-\cos 2t}{2}\right) \\ &= \frac{1}{2}L(1-\cos 2t) \\ &= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right) \end{aligned}$$

Problem 3.3.7. *Find* $L\left(\cos^3 t\right)$

Solution: we know that $\cos 3A = 4\cos^3 A - 3\cos A$

$$\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$$

$$\begin{aligned} \text{hence } L\left(\cos^3 t\right) &= \frac{1}{4}L(3\cos t + \cos 3t) \\ &= \frac{1}{4}\left(\frac{3s}{s^2+1}+\frac{s}{s^2+9}\right) \end{aligned}$$

Problem 3.3.8. *Find* $L(\sin(2t+3))$

Solution:

$$\begin{aligned} L(\sin(2t+3)) &= L(\sin 2t\cos 3 + \sin 3\cos 2t) \\ &= \cos 3L(\sin 2t) + \sin 3L(\cos 2t) \\ &= (\cos 3)\ \frac{2}{s^2+4} + (\sin 3)\ \frac{s}{s^2+4} \end{aligned}$$

3.4 Shifting Rule

Let a be any real constant. If $L[f(t)] = F(s)$ Then

$$\boxed{L[e^{at}f(t)] = F(s-a)}$$

Here we note that the Laplace transform of $e^{at}f(t)$ can be written down directly by changing s to $s-a$ in the Laplace transform of $f(t)$.

Problem 3.4.1. *Find* $L\left\{e^{-3t}(2cos5t - 3sin5t)\right\}$ *(VTU Jan 2017)*

Solution : Let $f(t) = 2cos5t - 3sin5t$ Then

$$\begin{aligned} F(s) &= L\{f(t)\} \\ &= L\{2cos5t - 3sin5t\} \\ &= 2L\{cos5t\} - 3L\{sin5t\} \\ &= 2\left[\frac{s}{s^2+25}\right] - 3\left[\frac{1}{s^2+25}\right] \end{aligned}$$

By shifting rule we have

$$\begin{aligned} L[e^{at}f(t)] &= F(s-a) \\ \therefore L\left\{e^{-3t}(2cos5t - 3sin5t)\right\} &= F(s-(-3)) \\ &= F(s+3) \\ &= 2\left[\frac{s+3}{(s+3)^2+25}\right] - 3\left[\frac{1}{(s+3)^2+25}\right] \end{aligned}$$

Problem 3.4.2. *Find L.T. of* $\mathrm{e^{2t}\cos^2 t}$

Solution : Let $\mathrm{f(t) = e^{2t}\cos^2 t = e^{2t}\cdot 1/2(1+\cos 2t)}$

$$\begin{aligned} L[f(t)] &= \frac{1}{2}\left[L\left(e^{2t}\right) + L\left(e^{2t}\cos 2t\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{s-2} + \{L(\cos 2t)\}_{s\to s-2}\right] \\ &= \frac{1}{2}\left[\frac{1}{s-2} + \left\{\frac{s}{s^2+4}\right\}_{s\to s-2}\right] \\ &= \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{s^2-4s+8}\right] \end{aligned}$$

Thus $L\left(e^{2t}\cos^2 t\right) = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{s^2-4s+8}\right]$

Problem 3.4.3. *Find* $L\left(e^{3t}\sin^2 4t\right)$

Solution :

$$\begin{aligned}
L\left(e^{3t}\sin^2 4t\right) &= L\left(\sin^2 4t\right)_{s\to s-3}\\
&= L\left(\frac{1-\cos 8t}{2}\right)_{s\to s-3}\\
&= \frac{1}{2}(L(1)-L(\cos 8t))_{s\to s-3}\\
&= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+64}\right)_{s\to s-3}\\
&= \frac{1}{2}\left(\frac{1}{s-3}-\frac{s-3}{(s-3)^2+64}\right)
\end{aligned}$$

Problem 3.4.4. *Find* $L(\cosh t\cos 2t)$

Solution :

$$\begin{aligned}
L(\cos ht\cos 2t) &= L\left(\left(\frac{e^t+e^{-t}}{2}\right)\cos 2t\right)\\
&= \frac{1}{2}L\left(e^t\cos 2t + e^{-t}\cos 2t\right)\\
&= \frac{1}{2}\left[L(\cos 2t)_{s\to s-1} + L(\cos 2t)_{s\to s+1}\right]\\
&= \frac{1}{2}\left[\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]\\
&= \frac{1}{2}\left(\frac{s-1}{(s-1)^2+4}+\frac{s+1}{(s+1)^2+4}\right)
\end{aligned}$$

Problem 3.4.5. *Find the Laplace transform of the following functions.*
(i) $e^{-t}\cos^2 3t$ *(ii)* $\mathrm{e}^{3t}\sin^3 2\mathrm{t}$ *(iii)* $\sqrt{t}e^t$

Sol : (i) Consider

$$\begin{aligned}
L\left(\cos^2 3t\right) &= L\left(\frac{1+\cos 6t}{2}\right)\\
&= \frac{1}{2}[L(1)+L(\cos 6t)]\\
&= \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+6^2}\right]\\
&= \frac{s^2+18}{s\left(s^2+36\right)}\\
\therefore L\left(e^{-t}\cos^2 3t\right) &= \frac{(s+1)^2+18}{(s+1)\left[(s+1)^2+36\right]}
\end{aligned}$$

(ii) We have

$$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

Hence

$$\sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$$
$$L\left(\sin^3 2t\right) = \frac{1}{4}[3L(\sin 2t) - L(\sin 6t)]$$
$$= \frac{1}{4}\left[3 \cdot \frac{2}{s^2+2^2} - \frac{6}{s^2+6^2}\right]$$
$$= \frac{48}{(s^2+4)\,(s^2+36)}$$

By using shifting Rule, we get, $s \to s-3$

$$L\left\{e^{3t}\sin^3 2t\right\} = \frac{48}{[(s-3)^2+4]\,[(s-3)^2+36]}$$

(iii) Now

$$L(\sqrt{t}) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$$
$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}$$
$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$
$$\text{Hence } L\left(\sqrt{t}e^t\right) = \frac{\sqrt{\pi}}{2(s-1)^{\frac{3}{2}}}$$

3.5 Multiplication by t^n

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)}$$

In particular

$$\boxed{L\{tf(t)\} = (-1)\frac{d}{ds}F(s)}$$
$$\boxed{L\{t^2 f(t)\} = (-1)^2\frac{d^2}{ds^2}F(s)}$$
$$\vdots$$

Problem 3.5.1. *Find* $L\{tsint\}$ *(VTU July 2014)*

Solution :

$$\text{Let } f(t) = sint$$
$$\text{Then } F(s) = \frac{1}{s^2+1}$$
$$\text{By using} L\{tf(t)\} = (-1)\frac{d}{ds}F(s)$$
$$L\{tsint\} = (-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$
$$= (-1)\frac{-1}{(s^2+1)^2}2s$$
$$\frac{2s}{(s^2+1)^2}$$

Problem 3.5.2. *Find* $L\left[e^{-t}t\cos t\right]$

Solution :

$$Let \;\; f(t) = t\cos t$$
$$F(s) = L\{f(t)\}$$
$$= L[t\cos t]$$
$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right] \qquad \left(\because L[\cos t] = \frac{s}{s^2+1}\right)$$
$$= -\left[\frac{s^2+1-s(2s)}{\left[s^2+1\right]^2}\right]$$
$$-\left[\frac{s^2+1-2s^2}{\left[s^2+1\right]^2}\right]$$
$$= \frac{s^2-1}{\left[s^2+1\right]^2}$$
$$L\left[e^{-t}t\cos t\right] = \frac{s^2-1}{\left[s^2+1\right]^2}\Big|_{s\to(s+1)}$$
$$= \frac{(s+1)^2-1}{\left[(s+1)^2+1\right]^2}$$

Problem 3.5.3. *Find* $L\left(te^{-2t}\sin 3t\right)$

Solution:

$$\begin{aligned}
L\left(e^{-2t}(t\sin 3t)\right) &= L(t\sin 3t)_{s\to s+2}\\
&= \left\{\frac{-d}{ds}(L(\sin 3t)\right\}_{s\to s+2}\\
&= \left\{\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right\}_{s\to s+2}\\
&= \left\{\frac{\left(s^2+9\right)0-3(2s)}{\left(s^2+9\right)^2}\right\}_{s\to s+2}\\
&= \frac{6(s+2)}{\left((s+2)^2+9\right)^2}
\end{aligned}$$

Problem 3.5.4. *Find* $\mathrm{L}\left\{\mathrm{e}^{-2t}t\cos 2\mathrm{t}\right\}$. *(VTU Sept 2020)*

SOl :

$$\begin{aligned}
F(t) &= \cos 2t\\
L[F(t)] &= L[\cos 2t]\\
F(s) &= \frac{s}{s^2+4}\\
L[t\cos 2t] &= -\frac{d}{ds}\left[\frac{s}{s^2+4}\right]\\
&= -\left[\frac{\left(s^2+4\right)(1)-s(2s)}{\left(s^2+4\right)^2}\right]\\
&= -\frac{\left[s^2+4-2s^2\right]}{\left(s^2+4\right)^2}\\
&= \frac{S^2-4}{\left(s^2+4\right)^2}\\
L\left[e^{-2t}t\cos 2t\right] &= \left.\frac{S^2-4}{\left(s^2+4\right)^2}\right|_{s\to s+2}\\
&= \frac{(s+2)^2-4}{\left((s+2)^2+4\right]^2}
\end{aligned}$$

3.6 Division by t

If $L\{f(t)\} = F(s)$ then

$$\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)ds}$$

Problem 3.6.1. *Find* $L\left[\frac{e^{at}-e^{-at}}{t}\right]$ *(VTU June 2015)*

Solution : Let $f(t) = e^{at} - e^{-at}$, Then

$$\begin{aligned} F(s) &= L\{e^{at} - e^{-at}\} \\ &= \frac{1}{s-a} - \frac{1}{s+a} \end{aligned}$$

$$\text{we have } L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)ds$$

$$\begin{aligned} \therefore L\left[\frac{e^{at} - e^{-at}}{t}\right] &= \int_s^{\infty}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]ds \\ &= [log(s-a) - log(s+a)]_s^{\infty} \\ &= log\left[\frac{s-a}{s+a}\right]_s^{\infty} \\ &= log\left[\frac{s\left(1-\frac{a}{s}\right)}{s\left(1+\frac{a}{s}\right)}\right]_s^{\infty} \\ &= log\left[\frac{\left(1-\frac{a}{s}\right)}{\left(1+\frac{a}{s}\right)}\right]_s^{\infty} \\ &= log\left[\frac{(1-0)}{(1+0)}\right] - log\left[\frac{(1-\frac{a}{s}}{(1+\frac{a}{s}}\right] \end{aligned}$$

$$= -log\left[\frac{(1-\frac{a}{s}}{(1+\frac{a}{s}}\right]$$

$$= log\left[\frac{(\frac{s-a}{s}}{(\frac{s+a}{s}}\right]$$

$$= log\left[\frac{s-a}{s+a}\right]$$

Problem 3.6.2. *Find* $L\left(\frac{cosat-cosbt}{t}\right)$ *(VTU August 2022, Jan 2017, Jan 2016, Jan 2013, Jan 2010, July 2008)*

SOlution : Let $f(t) = cosat - cosbt$

Then $F(s) = L(f(t)) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$\text{we have } L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$$

$$\begin{aligned}
\therefore L\left(\frac{cosat - cosbt}{t}\right) &= \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}ds \\
&= \left[\frac{1}{2}\log\left(s^2+a^2\right) - \frac{1}{2}\log\left(s^2+b^2\right)\right]_s^\infty \\
&= \left[\frac{1}{2}\log\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right]_s^\infty \\
&= \frac{1}{2}\left[\log\frac{s^2\left(1+a^2/s^2\right)}{s^2\left(1+b^2/s^2\right)}\right]_s^\infty \\
&= \frac{1}{2}\left[\log\frac{\left(1+a^2/s^2\right)}{\left(1+b^2/s^2\right)}\right]_s^\infty \\
&= \frac{1}{2}\left[\log 1 - \log\left(\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right)\right] \\
&= \frac{1}{2}\log\left(\frac{\left(s^2+b^2\right)}{\left(s^2+a^2\right)}\right]
\end{aligned}$$

Problem 3.6.3. *Find* $L\left\{\frac{e^{-t}sint}{t}\right\}$ *(VTU July 2013, 2009 S)*

Sol : Let

$$
\begin{aligned}
f(t) &= L\left[\frac{\sin t}{t}\right] \\
&= \int_s^\infty \frac{1}{s^2+1}ds \qquad \left(\because L(sint) = \frac{1}{s^2+1}\right) \\
&= \tan^{-1} s\big|_s^\infty \\
&= \left[\frac{\pi}{2} - \tan^{-1}\right] \\
&= cot^{-1}s
\end{aligned}
$$

$$
\therefore L\left\{\frac{e^{-t}sint}{t}\right\} = cot^{-1}s\,\big|_{s\to(s+1)}
$$

Problem 3.6.4. *Find the Laplace transform of (i)* $\sqrt{e^{4(t+3)}}+e^{-2t}\sin 3t$ *(ii)* $te^{-3t}\sin 4t$ *(iii)* $(1-\cos t)/t$ *(Model 2019)*

Sol: let

$$
\begin{aligned}
f(t) &= \sqrt{e^{4(t+3)}} + e^{-2t}\sin 3t \\
&= \left[e^{4(t+3)}\right]^{1/2} + e^{-2t}\sin 3t \\
&= e^{2(t+3)} + e^{-2t}\sin 3t \\
&= e^{2t+6} + e^{-2t}\sin 3t \\
F(t) &= e^{2t}\cdot e^{6} + e^{-2t}\sin 3t \\
L[F(t)] &= e^6 L\left[e^{2t}\right] + L\left[e^{-2t}\sin 3t\right] \\
&= e^6\frac{1}{s-2} + \left[\frac{3}{s^2+9}\right]_{s\to s+2} \\
&= \frac{e^6}{s-2} + \frac{3}{(s+2)^2+9} \\
&= \frac{e^6}{s-2} + \frac{3}{s^2+2s+4+9} \\
L[f(t)] &= \frac{e^6}{s-2} + \frac{3}{s^2+2s+13}
\end{aligned}
$$

$$L[t\sin 4t] = (-1)'\frac{d}{ds}\left[\frac{4}{s^2+16}\right]$$

$$= -\left[\frac{\left(s^2+16\right)(0)-4(2s)}{\left(s^2+16\right)^2}\right]$$

(ii)$L[\sin 4t] = \frac{4}{s^2+16}$

$$= -\left[\frac{-8s}{\left(s^2+16\right)^2}\right]$$

$$L[t\sin 4t] = \frac{8s}{\left(s^2+16\right)^2}$$

$$L\left[e^{-3t}t\sin 4t\right] = \left[\frac{8s}{\left(s^2+16\right)^2}\right]_{s\to s+3}$$

$= \frac{8(5+3)}{\left[(5+3)^2+16\right]^2}$

Problem 3.6.5. *Find* $L\left[\frac{1-\cos 3t}{t}\right]$

Sol : Let $\quad f(t) = 1-\cos 3t$

$$\therefore \quad \bar{f}(s) = L[f(t)] = \frac{1}{s} - \frac{s}{s^2+9}$$

We have the property: $L\frac{f(t)}{t} = \int_s^\infty \bar{f}(s)ds$

$$L\left[\frac{1-\cos 3t}{t}\right] = \int_s^\infty \left[\frac{1}{s}-\frac{s}{s^2+9}\right]ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+9\right)\right]_s^\infty$$

$$= \left[\log\frac{s}{\sqrt{s^2+9}}\right]_s^\infty$$

i.e.,

$$= \left[\log\frac{s}{s\sqrt{1+9/s^2}}\right]_{s=\infty}^{0} - \log\frac{s}{\sqrt{s^2+9}}$$

$$= \log 1 - \log\frac{s}{\sqrt{s^2+9}} = \log\frac{\sqrt{s^2+9}}{s}$$

$$L\left[\frac{1-\cos 3t}{t}\right] = \log\left[\frac{\sqrt{s^2+9}}{s}\right]$$

Problem 3.6.6. *) Find the Laplace transform of : (i)* $3^t+(4t+5)^3$ *(ii)* $te^{-4t}\sin 3t$ *(Model 2019)*

Sol :(i)

$$3^t + (4t+5)^3$$
$$f(t) = 3^t + (4t+5)^3$$
$$f(t) = e^{\log(3^t)} + (4t)^3 + 5^3 + 3(4t)^2 s + 3(4t)(5)^2$$
$$f(t) = e^{(\log 3)t} + 64t^3 + 125 + 240t^2 + 300t$$
$$f(t) = e^{(\log 3)t} + 64t^3 + 240t^2 + 300t + 125$$
$$L[f(t)] = L\left[e^{(\log 3)t}\right] + 64L\left[t^3\right] + 240L\left[t^2\right] + 300L[t] + 125L[1]$$
$$L[f(t)] = \frac{1}{s-\log 3} + 64.\frac{6}{s^4} + 240\frac{2}{s^3} + 300\cdot\frac{1}{s^2} + 125.\frac{1}{s}$$
$$F(s) = \frac{1}{s-\log 3} + \frac{384}{s^4} + \frac{480}{s^3} + \frac{300}{s^2} + \frac{125}{s}$$

ii)

$$L[\sin 3t] = \frac{3}{s^2+9}$$
$$L[t\sin 3t] = (-1)'\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$
$$= -\left[\frac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2}\right]$$
$$= -\left[\frac{-6s}{(s^2+9)^2}\right]$$
$$L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$
$$L\left[e^{-4t}t\sin 3t\right] = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+4}$$
$$= \frac{6(s+4)}{((s+4)^2+9)^2}$$
$$L\left[e^{-4t}t\sin 3t\right] = \frac{6(s+4)}{(s^2+8s+25)^2}$$

3.7 Periodic functions

A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t) = f(t+nT)$ where $n = 1, 2, 3, \cdots$ The graph of the periodic function repeats itself in equal intervals.

For example, $sint$, $cost$ are periodic functions of period 2π since $sin(t+2n\pi) = sint$, $cos(t+2n\pi) = cost$.

Laplace Transform of a periodic function:

If a function $f(t)$ is Periodic with period $T > 0$, so that $f(t+T) = f(t)$, then

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt}$$

Problem 3.7.1. *If $f(t) = t^2$ is a periodic function with period 2, then find $L\{f(t)\}$.*

Solution : Here $f(t)$ is a periodic function with the period $T = 2$.

Hence

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt \\
&= \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}t^2dt \\
&= \frac{1}{1-e^{-2s}}\int_0^2 t^2e^{-st}dt \\
&= \frac{1}{1-e^{-2s}}\left[t^2\frac{e^{-st}}{-s} - 2t\frac{e^{-st}}{s^2} + 2\frac{e^{-st}}{-s^3}\right]_0^2 \\
&= \frac{1}{1-e^{-2s}}\left[4\frac{e^{-2s}}{-s} - 2(2)\frac{e^{-2s}}{s^2} + 2\frac{e^{-2s}}{-s^3} - 2\frac{e^0}{-s^3}\right] \\
&= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3}\right]
\end{aligned}$$

Problem 3.7.2. *If $L\{f(t)\}$ when $f(t) = \begin{cases} E, & 0 \le t \le \frac{a}{2} \\ -E, & \frac{a}{2} \le t \le a \end{cases}$ and $f(t+a) = f(a)$ Show that $L\{f(t)\} = \frac{E}{S}tanh\left(\frac{as}{4}\right)$ (VTU July 2016, Model 2015, June 2014, Dec 2010, June 2011),* Sol : Given that $f(t+a) = f(t)$ Hence $f(t)$ is a periodic function with period $p = a$

$$
\begin{aligned}
L(f(t)) &= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t) dt \\
&= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} E dt + \int_{a/2}^{a} e^{-st} (-E) dt \right] \\
&= \frac{1}{1-e^{-as}} \left[E \int_0^{a/2} e^{-st} dt - E \int_{a/2}^{a} e^{-st} dt \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{E}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{a/2} - \left(\frac{e^{-st}}{-s} \right)_{a/2}^{a} \right] \\
&= \frac{E}{s\left(1-e^{-as}\right)} \left[\left(-e^{-sa/2} + 1 \right) + \left(e^{-sa} - e^{-sa/2} \right) \right] \\
&= \frac{E}{s\left(1-e^{-as}\right)} \left(1 - e^{-sa/2} - e^{sa/2} + e^{-sa} \right) \\
&= \frac{E}{s\left(1-e^{-as}\right)} \left(1 - e^{-2sa/2} + e^{-sa} \right) \\
&= \frac{E}{s\left(1-e^{-\frac{as}{2}}\right)\left(1+e^{-sa/2}\right)} \left(1 - e^{\frac{-sa}{2}} \right)^2 \\
&= \frac{E\left(1-e^{\frac{-sa}{2}}\right)}{s\left(1+e^{-sa/2}\right)} \\
&= \frac{E\left(1-e^{\frac{-sa}{2}}\right) e^{\frac{-as}{4}}}{s\left(1+e^{-sa/2}\right) e^{\frac{-as}{4}}} \\
&= \frac{E}{s} \cdot \frac{e^{\frac{as}{4}} - e^{\frac{-as}{4}}}{e^{\frac{as}{4}} + e^{\frac{-as}{4}}} \\
&= \frac{E}{s} \tan h \left(\frac{sa}{4} \right)
\end{aligned}
$$

Problem 3.7.3. *If* $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a \end{cases}$ *, where* $f(t+2a) = f(t)$*, show that* $L\{f(t)\} = \frac{1}{s^2} tanh\left(\frac{as}{2}\right)$ *(VTU Model 2022, Mar 2022, Jan 2021, Jan 2017, July 2016, June 2015, Dec 2014, July 2014, Dec 2011, July 2008)*

Solution : The given function is a periodic function with period **2 a**

$$\begin{aligned}
\therefore L(f(t)) &= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt \\
&= \frac{1}{1-e^{-2as}}\int_0^{a} e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt \\
&= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st} tdt + \int_a^{2a} e^{-st}(2a-t)dt\right] \\
&= \frac{1}{1-e^{-2as}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\right.\right.\right. \\
&= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a}{s}e^{-as}\frac{-e^{-as}}{s^2}\right] \\
&= \frac{1}{1-e^{-2as}}\left(\frac{1-2e^{-as}+e^{-2as}}{s^2}\right) \\
&= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})}\left.\left.\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\left.\right\} \\
&= \frac{1-e^{-as}}{s^2(1+e^{-as})} \\
&= \frac{1}{s^2}\frac{(1-e^{-as})\,e^{\frac{-as}{2}}}{(1+e^{-as})\,e^{\frac{-as}{2}}} \\
&= \frac{1}{s^2}\cdot\frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{e^{\frac{as}{2}} + e^{\frac{-as}{2}}} \\
&= \frac{1}{s^2}\tan h\left(\frac{as}{2}\right)
\end{aligned}$$

Problem 3.7.4. *Find the Laplace transform of the square-wave function of period a given by*

$$f(t) = \begin{cases} 1, & 0 < t < a/2 \\ -1, & a/2 < t < a \end{cases}$$

(VTU Model 2022)

Solution: Period of the function, $\quad T = a$

$$\begin{aligned}
L[f(t)] &= \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt \\
&= \left[\frac{1}{1-e^{-as}}\right]\left[\int_0^{\frac{a}{2}} e^{-st}dt - \int_{\frac{a}{2}}^{a} e^{-st}dt\right] \\
&= \left[\frac{1}{1-e^{-as}}\right]\left[\left.\frac{e^{-st}}{-s}\right|_0^{\frac{a}{2}} - \left.\frac{e^{-st}}{-s}\right|_{\frac{a}{2}}^{a}\right] \\
&= \left[\frac{1}{1-e^{-as}}\right]\left[-\frac{e^{-\frac{a}{2}s}}{s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-\frac{a}{2}s}}{s}\right] \\
&= \frac{1}{s}\left[\frac{1-2e^{-\frac{a}{2}s}+e^{-as}}{1-e^{-as}}\right] \\
&= \frac{1}{s}\left[\frac{\left(1-e^{-\frac{a}{2}s}\right)^2}{\left(1+e^{-\frac{a}{2}s}\right)\left(1-e^{-\frac{a}{2}s}\right)}\right] \\
&= \frac{1}{s}\left[\frac{\left(1-e^{-\frac{a}{2}s}\right)}{\left(1+e^{-\frac{a}{2}s}\right)}\right] \\
&= \frac{1}{s}\tan h\left(\frac{as}{2}\right)
\end{aligned}$$

Problem 3.7.5. *Find the Laplace transform of full wave rectifier $f(t) = Esin\omega t, 0 < t \leq \frac{\pi}{\omega}$ having the period $\frac{\pi}{\omega}$.* *(VTU August 2022, July 2019, Jan 2019, Jan 2018, July 2017,Dec 2011)*

Solution : Solution: $T=\frac{\pi}{\omega}$

$$
\begin{aligned}
L[f(t)] &= \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt \\
&= \left[\frac{E}{1-e^{-\frac{\pi s}{\omega}}}\right]\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t dt \\
&= \left[\frac{E}{1-e^{-\frac{\pi s}{\omega}}}\right]\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t]_0^{\frac{\pi}{\omega}}\right. \\
&\qquad \left(\because \int e^{at}\sin bt dt = \frac{e^{at}}{a^2+b^2}[a\sin bt - b\cos bt]\right) \\
&= \left[\frac{E}{1-e^{-\frac{\pi s}{\omega}}}\right]\left[\frac{\omega e^{-\frac{\pi s}{\omega}}+\omega}{s^2+\omega^2}\right] \\
&= \left[\frac{E\omega}{s^2+\omega^2}\right]\left[\frac{1+e^{-\frac{\pi s}{\omega}}}{1-e^{-\frac{\pi s}{\omega}}}\right]
\end{aligned}
$$

Problem 3.7.6. *If* $f(t)=t^2; 0<t<2$ *and* $f(t+2)=f(t)$ *for* $t>2$, *find* $f\{f(t)\}$.

Solution: Here, $f(t)=t^2; 0<t<2$ is periodic function with period 2 .

$$
\begin{aligned}
\therefore \quad \{f(t)\} &= \frac{1}{(1-e^{-2s})}\int_0^2 e^{-st} f(t)dt \\
&= \frac{1}{(1-e^{-2s})}\int_0^2 t^2 e^{-st}dt \\
&= \frac{1}{(1-e^{-2s})}\left[t^2\cdot\left(\frac{e^{-st}}{-s}\right)-2t\left(\frac{e^{-st}}{s^2}\right)+2\cdot\left(\frac{-e^{-st}}{s^3}\right)\right]_0^2 \\
&= \frac{1}{(1-e^{-2s})}\left[\frac{-4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}-\frac{2e^{-2s}}{s^3}+\frac{2}{s^3}\right] \\
&= \frac{-2e^{-2s}}{s^3(1-e^{-2s})}\left(2s^2+2s+1-e^{2s}\right) \\
&= \frac{2}{s^3(1-e^{2s})}\left(2s^2+2s+1-e^{2s}\right); s>0
\end{aligned}
$$

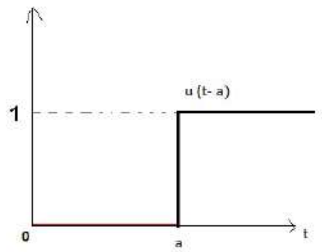
3.8 Unit Step functions

In many Engineering applications, we deal with an important function $H(t-a)$ defined as follows:

$$H(t-a) \text{ or } u(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a \end{cases}$$

where a is a non-negative constant.

This function is known as the **unit step function** or the **Heaviside function**. The function is named after the British electrical engineer Oliver Heaviside. The function is also denoted by $u(t-a)$. The graph of the function is shown below:



Note that the value of the function suddenly jumps from value zero to the value 1 as $t \to a$ from the left and retains the value 1 for all $t \geq a$. Hence the function $H(t-a)$ is called the unit step function.

Transform of step function

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

Proof :

$$
\begin{aligned}
L\{u(t-a)\} &= \int_0^\infty e^{-st}u(t-a)dt \\
&= \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}1dt \\
&= \left[\frac{e^{-st}}{-s}\right]_a^\infty \\
&= \left[0 - \frac{e^{-as}}{-s}\right] \\
&= \frac{e^{-as}}{s}
\end{aligned}
$$

Second Shifting Property :

$$\boxed{L[f(t-a)H(t-a)] = e^{-as}L\{f(t)\}}$$

Problem 3.8.1. *Evaluate* $L\left\{e^{t-1}u(t-1)\right\}$

Solution : *Let the given function be in the form* $f(t-a)u(t-a)$, *where* $f(t-a) = e^{t-1}$ *and* $u(t-a) = u(t-1)$

Then $a = 1$, $f(t-a) = f(t-1) = e^{t-1} \implies f(t) = e^t$

we have $L[f(t-a)H(t-a)] = e^{-as}L\{f(t)\}$

$$
\begin{aligned}
\therefore L\left\{e^{t-1}u(t-1)\right\} &= e^{-s}L\{f(t)\} \\
&= e^{-s}L\{t\} \\
&= \frac{e^{-s}}{s-1}
\end{aligned}
$$

Problem 3.8.2. *Express* $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$ *in terms Heaviside's unit step function and hence find its Laplace transform.* *(VTU Model 2019)*

Sol :
$$f(t) = 1 + (t-1)u(t-1)$$
$$\Rightarrow L[f(t)] = L[1] + L[(t-1)u(t-1)]$$

let $g(t-1) = t-1 \quad L[1] = \frac{1}{s}$

$$g(t) = t$$
$$L[g(t)] = L[t]$$
$$\bar{g}(s) = \frac{1}{s^2}$$
$$\therefore L[g(t-1)u(t-1)] = e^{-s}\bar{g}(s)$$
$$L[(t-1)u(t-1)] = e^{-s}\frac{1}{s^2}$$
$$\therefore \quad (1) \Rightarrow F(s) = \frac{1}{s} + \frac{e^{-s}}{s^2}$$

Problem 3.8.3. *Express $f(t)$ in terms of unit step function and hence find the Laplace transform.* $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4. \end{cases}$ *(VTU July 2017, June 2014, June 2015, June 2011, Jan 2010)*

Solution: $\quad f(t) = t^2 + \left(4t - t^2\right)u(t-2) + (8-4t)u(t-4).$

$$\therefore \quad L[f(t)] = L\left[t^2\right] + e^{-2s}L\left[4(t+2) - (t+2)^2\right] + e^{-4s}L[8 - 4(t+4)]$$
$$= L\left[t^2\right] + e^{-2s}L\left[4 - t^2\right] - 4e^{-4s}L[t+2]$$
$$= \frac{2}{s^3} + e^{-2s}\left[\frac{4}{s} - \frac{2}{s^3}\right] - 4e^{-4s}\left[\frac{1}{s^2} + \frac{2}{s}\right].$$

Problem 3.8.4. *Express* $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2t, & \text{if } 1 < t \le 2 \\ 3t^2, & t > 2 \end{cases}$ *in terms of unit step function and hence find its Laplace transform.* *(VTU July 2016)*

Solution: $f(t) = 1 + (2t-1)u(t-1) + \left(3t^2 - 2t\right) u(t-2)$.

$$\therefore \quad L[f(t)] = L[1] + e^{-s}L[2(t+1)-1] + e^{-2s}L\left[3(t+2)^2 - 2(t+2)\right]$$

$$= L[1] + e^{-s}L[2t+1] + e^{-2s}L\left[3t^2 + 10t + 8\right]$$

$$= \frac{1}{s} + e^{-s}\left[\frac{2}{s^2} + \frac{1}{s}\right] + e^{-2s}\left[\frac{6}{s^3} + \frac{10}{s^2} + \frac{8}{s}\right].$$

Problem 3.8.5. *Express* $f(t) = \begin{cases} \sin t, & 0 < t \leq \pi/2 \\ \cos t & t > \pi/2 \end{cases}$ *in terms Heaviside's unit step function and hence find its Laplace transform.* *(VTU Model 2019)*

SOl:

$$f(t) = \sin t + [\cos t - \sin t]u(t - \pi/2)$$

$$L[f(t)] = L[\sin t] + L[\cos t - \sin t]u(t - \pi/2) \longrightarrow (1)$$

$$\text{let } g(t - \pi/2) = \cos t - \sin t$$

$$g(t) = \cos(t + \pi/2) - \sin(t + \pi/2)$$

$$= -\sin t - \cos t$$

$$L[g(t)] = -L[\sin t] - L[\cos t]$$

$$\bar{g}(s) = -\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

$$\therefore L[g(t-\pi/2)u(t-\pi/2)] = e^{-\pi/2s} \cdot \bar{g}(s)$$

$$= e^{-\pi/2s} - \left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

$$(1) \Rightarrow F(s) = \frac{1}{s^2+1} - e^{-\pi/2s}\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

Problem 3.8.6. *Using unit step function, find the Laplace transform of* $f(t) = \begin{cases} sint, 0 < t < \pi \\ sin2t, \pi \leq t \leq 2\pi \\ sin3t, t \geq 2\pi \end{cases}$ *(VTU Mar 2022, Jan 2017, 2004)*

Solution: $f(t) = \sin t + (\sin 2t - \sin t)u(t-\pi) + (\sin 3t - \sin 2t)u(t-2\pi)$.

$$\begin{aligned} \therefore \quad L[f(t)] =& L[\sin t] + e^{-\pi s}L[\sin 2(t+\pi) - \sin(t+\pi)] + e^{-2\pi s}L[\sin 3(t+2\pi) \\ =& L[\sin t] + e^{-\pi s}L[\sin 2t + \sin t] + e^{-2\pi s}L[\sin 3t - \sin 2t] \\ =& \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]. \end{aligned}$$

Problem 3.8.7. *Express* $f(t) = \begin{cases} cost, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ sint, & t > 2\pi \end{cases}$ *in terms of unit step function and hence find* $L(f(t))$.

(VTU August 2022, July 2018, Jan 2016,July 2013, Jan 2008, July 2008, 2007)

Solution: $f(t) = \cos t + (1 - \cos t)u(t-\pi) + (\sin t - 1)u(t-2\pi)$

$$\begin{aligned} \therefore \quad L[f(t)] =& L[\cos t] + e^{-\pi s}L[1 - \cos(t+\pi)] \\ & + e^{-2\pi s}L[\sin(t+2\pi) - 1] \\ =& L[\cos t] + e^{-\pi s}L[1 + \cos t] + e^{-2\pi s}L[\sin t - 1] \\ =& \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{1}{s} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s}\right]. \end{aligned}$$

3.9 Inverse Laplace Transforms

Let $L[f(t)] = F(s)$. Then $f(t)$ is defined as the inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}[F(s)]$. Thus

$$\boxed{L^{-1}[F(s)] = f(t)}$$

. **Linearity Property** :

Let $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$ and a and b be any two constants. Then

$$\boxed{L^{-1}[aF(s) + bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]}$$

Table of Inverse Laplace Transfirms :

Sl. No.	Inverse Laplace Transform
1	$L^{-1}\left\{\frac{1}{s}\right\} = 1$
2	$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
3	$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$
4	$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = sinat$
5	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{sinat}{a}$
6	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = cosat$
7	$L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = sinhat$
8	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{sinhat}{a}$
9	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = coshat$
10	$L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$
11	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$ when n is an integer
12	$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma(n)}$
13	$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$ when n is an integer

Problem 3.9.1. *Find the inverse Laplace transform of* $\frac{s^2-3s+4}{s^3}$

Solution :

$$\begin{aligned} \text{Let } F(s) &= \frac{s^2-3s+4}{s^3} \\ &= \frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3} \\ &= \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3} \\ \therefore L^{-1}\{F(s)\} &= L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{}{s^3}\right\} \\ &= 1 - 3t + 2t^2 \end{aligned}$$

3.10 Evaluation of $L^{-1}\{F(s-a)\}$

We have, if $L[f(t)] = F(s)$, then $L[e^{at}f(t)] = F(s-a)$, and so

$$\boxed{L^{-1}[F(s-a)] = e^{at}f(t) = e^{at}L^{-1}[F(s)]}$$

Sl. No.	Inverse Laplace Transform
1	$L^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}sinbt$
2	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}cosbt$
3	$L^{-1}\left\{\frac{b}{(s-a)^2-b^2}\right\} = e^{at}sinhbt$
4	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2-b^2}\right\} = e^{at}coshbt$
5	$L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at}\frac{t^{n-1}}{(n-1)!}$

Note : To find inverse Laplace transform of a given function $F(s)$, we have to do some algebra to get $F(s)$ into a form suitable for the direct use of the above table. For this, we use a method called completing the square.

Problem 3.10.1. *Find the inverse Laplace transform of* $\frac{s+2}{s^2-4s+13}$ *(VTU 2008)*

Solution :

$$\begin{aligned}
\text{Let } F(s) &= \frac{s+2}{s^2-4s+13} \\
&= \frac{s+2}{(s^2-4s+4-4)+13} \quad \text{(by adding and subtracting square of half of the} \\
&= \frac{s+2}{(s-2)^2+9} \\
&= \frac{(s-2+2)+2}{(s-2)^2+9} \text{adjusting the numerator to get the form that are in the Inverse} \\
&= \frac{(s-2)+4}{(s-2)^2+9} \\
&= \frac{(s-2)}{(s-2)^2+3^2} + \frac{4}{(s-2)^2+3^2} \\
&= e^{2t}cos3t + \frac{4}{3}e^{2t}sin3t
\end{aligned}$$

Problem 3.10.2. *Find the inverse Laplace transform of* $(2\,s-1)/\left(s^2+2\,s+17\right)$

Sol :

$(2\,\mathrm{s}-1)/\left(\mathrm{s}^2+2\,\mathrm{s}+17\right)=(2\,\mathrm{s}-1)/\left((\mathrm{s}+1)^2+4^2\right)=(2(\,\mathrm{s}+1)-3)/\left((\mathrm{s}+1\right.$

$$L^{-1}\left[\frac{2s-1}{s^2+2s+17}\right]=L^{-1}\left[\frac{2(s+1)-3}{(s+1)^2+4^2}\right]$$

$$=e^{-t}L^{-1}\left[\frac{2s-3}{s^2+4}\right]$$

$$=e^{-t}\left\{2L^{-1}\left[\frac{s}{s^2+4}\right]-3L^{-1}\left(\frac{1}{s^2+4^2}\right)\right\}$$

Thus $L^{-1}\left[\dfrac{2s-1}{s^2+2s+17}\right]$

$$=e^{-t}\left(2\cos 4t-\frac{3}{4}\sin 4t\right).$$

Problem 3.10.3. *Find* $L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right]$ *(VTU July 2014, June 2012, July 2009)*

Solution: $\mathrm{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}=\mathrm{L}^{-1}\left\{\frac{3(s-1)+3+7}{(s-1)^2-1-3}\right\}$

$$=\mathrm{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^2-2^2}+\frac{10}{(s-1)^2-2^2}\right\}$$

$$=\mathrm{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^2-2^2}\right\}+10\,\mathrm{L}^{-1}\left\{\frac{1}{(s-1)^2-2^2}\right\}$$

$$=3\mathrm{e}^{\mathrm{t}}\cosh 2\mathrm{t}+5\,\mathrm{Sinh}\,2\mathrm{t}e^{t}$$

3.11 Inverse Laplace Transform by Partial Fraction Expansion

This technique uses Partial Fraction Expansion to split up a complicated fraction into forms that are in the Laplace Transform table.

Problem 3.11.1. *Find inverse Laplace Transform of* $\frac{4s+5}{(s-1)^2(s+2)}$ *(VTU Jan 2016)*

Solution :

$$\text{Let } F(s) = \frac{4s+5}{(s-1)^2(s+2)}$$

By using Partial Fractions

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s+1)}$$

$$\therefore 4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

$$s = 1 \implies B = 3$$

$$s = -2 \implies C = -\frac{1}{3}$$

$$s = 0 \implies 5 = 2A + B + C \implies 5 = 2A + 3 + \frac{1}{3} \implies$$

$$\therefore F(s) = \frac{\frac{1}{3}}{s-1} + \frac{3}{(s-1)^2} + \frac{\frac{1}{3}}{(s+1)}$$

$$L^{-1}\{F(s)\} = \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)}\right]$$

$$= \frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}$$

Problem 3.11.2. *Find the Inverse Laplace transform of* $\frac{s+1}{(s-1)^2(s+2)}$

SOl : Let $\frac{s+1}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s+2)}$ or $s+1 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$ Put $s = 1 \quad \therefore \quad B = \frac{2}{3}$ Put $s = -2 \quad \therefore \quad C = \frac{-1}{9}$

Equating the coefficient of s^2 on both sides

We have $0 = A + C \quad \therefore \quad A = \frac{1}{9}$ Now,

$$L^{-1}\left[\frac{s+1}{(s-1)^2(s+2)}\right]$$

$$= \frac{1}{9}L^{-1}\left[\frac{1}{s-1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{9}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{9}e^t + \frac{2}{3}e^tL^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9}e^{-2t}$$

$$= \frac{1}{9}e^t + \frac{2}{3}e^t \cdot t - \frac{1}{9}e^{-2t}$$

Thus

$$L^{-1}\left[\frac{s+1}{(s-1)^2(s+2)}\right] = \frac{1}{9}e^t + \frac{2}{3}e^t \cdot t - \frac{1}{9}e^{-2t}$$

Problem 3.11.3. *Find* $L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$ *(VTU Model 2022, July 2017, Dec 2011, June 2011, 2007)*

SOl : $s^3-6s^2+11s-6=(s-1)(s-2)(s-3)$

So by partial fractions $\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$

on solving We get $\mathrm{A}=1/2,\ \mathrm{B}=-1, \mathrm{C}=5/2$ (Get these values yourself)

$$\mathrm{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}=L^{-1}\left\{\frac{1}{2(s-1)}\right\}-L^{-1}\left\{\frac{1}{(s-2)}\right\}+L^{-1}\left\{\frac{5}{2(s-3)}\right\}$$

$$=1/2e^{t}-e^{-2t}+5/2e^{3t}$$

3.12 Inverse Laplace Transforms using the formula $L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}F(s)ds$

From Laplace transforms, we have $L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}F(s)ds$. This formula can be used to find the Inverse Laplace Transform $f(t)$ provided $\int_s^{\infty}F(s)ds$ can be evaluated easily and its inverse Laplace transform can be conveniently calculated.

Problem 3.12.1. *Evaluate* $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$$
$$= \int_s^\infty \frac{s}{(s^2+a^2)^2}$$
$$= \left[\frac{-1}{(s^2+a^2)}\right]_s^\infty \qquad (\because \int \frac{x}{(x^2+a^2)^2}dx = -\frac{1}{2}\frac{1}{(x^2+a^2)}$$
$$= \left[0 - \frac{1}{2}\frac{-1}{(s^2+a^2)}\right]$$
$$= \frac{1}{2}\frac{1}{(s^2+a^2)}$$

taking inverse l.T. on both sides

$$\left\{\frac{f(t)}{t}\right\} = L^{-1}\left\{\frac{1}{2}\frac{1}{(s^2+a^2)}\right)\}$$
$$= \frac{1}{2}\frac{sinat}{a}$$
$$\therefore f(t) = t\frac{sinat}{2a}$$

3.13 Inverse Laplace Transforms of logarithmic, tan^{-1} and cot^{-1} functions

We have

$$L\{tf(t)\} = -\frac{d}{ds}F(s)$$
$$tf(t) = L^{-1}\left\{-\frac{d}{ds}F(s)\right\}$$
$$\therefore \boxed{f(t) = \frac{L^{-1}\left\{-\frac{d}{ds}F(s)\right\}}{t}}$$

Problem 3.13.1. *Find* $L^{-1}log\left(\frac{s+1}{s-1}\right)$ *(VTU July 2013)*

Solution :

$$\text{let } F(s) = log\left(\frac{s+1}{s-1}\right)$$
$$= log(s+1) - log(s-1)$$
$$\frac{d}{ds}F(s) = \frac{d}{ds}\left(log(s+1) - log(s-1)\right)$$
$$= \frac{1}{s+1} - \frac{1}{s-1}$$
$$-\frac{d}{ds}F(s) = \frac{1}{s-1} - \frac{1}{s+1}$$

Taking inverse L.T. on both sides

$$L^{-1}\left\{-\frac{d}{ds}F(s)\right\} = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right]$$
$$= e^{t} - e^{-t}$$
$$\therefore f(t) = \frac{L^{-1}\left\{-\frac{d}{ds}F(s)\right\}}{t}$$
$$= \frac{e^{t} - e^{-t}}{t}$$
$$= \frac{2sinht}{t}$$

Problem 3.13.2. *Find the inverse Laplace transform of (i)* $\left\{\frac{1}{s(s+1)}\right\}$ *(ii)* $\{(s+1)/\left(s^2+6s\right.$
$[\log\{(s+a)/(s+b)\}]$ *(VTU Model 2019)*

Sol :

$$f(s) = \frac{1}{s(s+1)}$$
$$\text{let } \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \rightarrow (1)$$
$$1 = A(s+1) + Bs \rightarrow \text{ (2)}$$

when $s = 0$ when $s = -1$

$$1 = A(0+1) \quad 1 = B(-1)$$
$$A = 1 \qquad B = -1$$

$$\Rightarrow \frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

$$L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}[f(s)] = 1 - e^{-t}$$

$$f(t) = 1 - e^{-t}$$

(ii)

$$F(s) = \frac{s+1}{s^2+6s+9}$$

$$= \frac{s+1}{(s+3)^2}$$

$$= \frac{s+3-3+1}{(s+3)^2}$$

$$f(s) = \frac{(s+3)-2}{(s+3)^2}$$

$$L^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\frac{(s+3)-2}{(s+3)^2}\right]$$

$$L^{-1}[f(s)] = e^{-3t}L^{-1}\left[\frac{s-2}{s^2}\right]$$

$$= e^{-3tt^{-1}}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s^2}\right]\Bigg\}$$

$$f(t) = e^{-3t}[1-2t]$$

(iii)

$$\text{let} \quad f(s) = \log\left(\frac{s+a}{s+b}\right)$$

$$f(s) = \log(s+a) - \log(s+b)$$

differentiate w.r.to s,

$$f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$x'y = \text{ ve on both sides}$$

$$-f'(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$-f'(s) = \frac{1}{s+b} - \frac{1}{s+a}$$

$$L^{-1}\left[-f'(s)\right] = L^{-1}\left[\frac{1}{s+b}\right] - L^{-1}\left[\frac{1}{s+a}\right]$$

$$tf(t) = e^{-bt} - e^{-at}$$
$$f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

Problem 3.13.3. *Find (i)* $L^{-1}\left\{\frac{3s+2}{s^2-s-2}\right\}$ *(ii)* $L^{-1}\left\{(s+5)/\left(s^2-6s+13\right)\right\}$ *(iii)* $L^{-1}\left[\cot^{-1}\{s/a\}\right]$

Sol :

$$\text{let } f(s) = \frac{3s+2}{s^2-s-2} = \frac{3s+2}{(s-2)(s+1)}$$
$$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \longrightarrow (1)$$
$$3s+2 = A(s+1) + B(s-2) \rightarrow (2)$$

when $s = 2$ when $s = -1$

$$\Rightarrow \quad 3(2) + 2 = A(2+1) \qquad \Rightarrow 3(-1) + 2 = B(-1-2)$$
$$-1 = B(-3)$$
$$8 = 3A \Rightarrow A = \frac{8}{3} \qquad B = 1/3$$

$$(1) \Rightarrow \frac{3s+2}{(s-2)(s+1)} = \frac{8/3}{s-2} + \frac{1/3}{s+1}$$
$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3}L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]$$
$$L^{-1}[f(s)] = \frac{8}{3}e^{2t} + \frac{1}{3}e^{-t}$$
$$f(t) = \frac{1}{3}\left(8e^{2t} + e^{-t}\right)$$

(ii)

$$
\begin{aligned}
f(s) &= \frac{s+5}{s^2-6s+13} \\
&= \frac{s+5}{s^2-2(s)(3)+3^2-3^2+13} \\
&= \frac{s+5}{(s-3)^2+4} \\
&= \frac{(s-3)+3+5}{(s-3)^2+4} \\
f(s) &= \frac{(s-3)+8}{(s-3)^2+4} \\
L^{-1}[f(s)] &= L^{-1}\left[\frac{(s-3)+8}{(s-3)^2+4}\right] \\
&= e^{3t}L^{-1}\left[\frac{s+8}{s^2+4}\right] \\
&= e^{3t}L^{-1}\left[\frac{s}{s^2+4}+\frac{8}{s^2+4}\right] \\
& e^{3t}\left\{L^{-1}\left[\frac{s}{s^2+2^2}\right]+8L^{-1}\left[\frac{1}{s^2+2^2}\right]\right\} \\
& e^{3t}\left(\cos 2t+8\cdot\frac{1}{2}\sin 2t\right) \\
&= e^{3t}(\cos 2t+4\sin 2t)
\end{aligned}
$$

(iii)

$$f(s) = \cot^{-1}(s/a)$$

diff writ S

$$f'(s) = \frac{-1}{1+(s/a)^2}\cdot\frac{1}{a}$$

multiply with -1

$$-f'(s) = \frac{a}{a^2+s^2}$$

Taking inverse laplace on both sides

$$L^{-1}\left[-f'(s)\right] = L^{-1}\left[\frac{a}{s^2+a^2}\right] \quad L[tf(t)] = -f'(s)$$

$$tf(t) = \sin at \qquad L^{-1}\left[-f'(s)\right] = tf(t)$$

$$\therefore \quad f(t) = \frac{\sin at}{t}$$

3.14 Evaluation of $L^{-1}[e^{-as}F(s)]$ using Second Shifting Theorem

We have, if $L[f(t)] = F(s)$, then $L[f(t-a)H(t-a) = e^{-as}F(s)$, and so

$$\boxed{L^{-1}[e^{-as}F(s)] = f(t-a)H(t-a)}$$

Problem 3.14.1. *Find the inverse Laplace transform of* $\frac{e^{-2s}}{(s-3)^2}$

SOl : $L^{-1}\left\{\frac{1}{(s-3)^2}\right\} = e^{3t}L^{-1}\left(\frac{1}{s^2}\right) = e^{3t}(t) = f(t)$

Now, $L^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) = f(t-2)u(t-2)$

Thus, $L^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) = \left\{e^{3(t-2)}(t-2)\right\}u(t-2)$.

Problem 3.14.2. *Find the inverse Laplace transform of* $L^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right]$

This is in the form $e^{-as}F(s)$ whose inverse transform is $f(t-a)u(t-a)$

Here $a = \pi$ and $F(s) = \frac{1}{s^2+4}$

$\therefore f(t) = L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{\sin 2t}{2}$

$$\begin{aligned} L^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right] &= f(t-\pi)u(t-\pi) \\ &= \frac{1}{2}\sin 2(t-\pi)u(t-\pi) \\ &= \frac{1}{2}\sin 2tu(t-\pi) \end{aligned}$$

3.15 Convolution Theorem

If $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$, then

$$\boxed{L^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)du}$$

Problem 3.15.1. *Using convolution theorem evaluate* $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$ *(VTU Model 2019, Dec 2014, June 2012, Dec 2011, June 2010, July 2007)*

Solution : Let $F(s) = \frac{s}{(s^2+a^2)}$ and $G(s) = \frac{s}{(s^2+b^2)}$

Then $f(t) = l^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} = cosat$ and $g(t) = l^{-1}\left\{\frac{s}{(s^2+b^2)}\right\} = cosbt$

From convolution theorem, we have

$$
\begin{aligned}
L^{-1}\{F(s)G(s)\} &= \int_0^t f(u)g(t-u)du \\
\therefore L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) & \\
&= \int_0^t cosau\ cosb(t-u)du \\
&= \int_0^t cosau\ cos(bt-bu)du \\
&= \int_0^t \frac{cos(au+bt-bu)+cos(au-bt+bu)}{2}du \\
&\quad \left(\text{using } cosAcosB = \frac{cos(A+B)+cos(A-B)}{2}\right) \\
&= \int_0^t \frac{cos[(a-b)u+bt]+cos[(a+b)u-bt]}{2}du \\
&= \frac{1}{2}\left[\frac{sin[(a-b)u+bt]}{(a-b)} + \frac{sin[(a+b)u-bt]}{(a+b)}\right]_0^t \\
&= \frac{1}{2}\left[\frac{sin[(a-b)t+bt]}{(a-b)} + \frac{sin[(a+b)t-bt]}{(a+b)} - \frac{sinbt}{a-b} - \frac{sin(-}{(a+} \right. \\
&= \frac{1}{2}\left[\frac{sinat-sinbt}{a-b} + \frac{sinat+sinbt}{a+b}\right] \\
&= \frac{1}{2}\left[\frac{(sinat-sinbt)(a+b)+(sinat+sinbt)(a-b)}{(a-b)(a+b)}\right] \\
&= \frac{1}{2}\left[\frac{asinat-asinbt+bsinat-bsinbt+asinat-bsinat}{a^2-b^2}\right. \\
&= \frac{1}{2}\left[\frac{2asinat-2bsinbt}{a^2-b^2}\right] \\
&= \left[\frac{asinat-bsinbt}{a^2-b^2}\right]
\end{aligned}
$$

Problem 3.15.2. *Find* $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Sol : We have $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$, $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

$\therefore$ By convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$
$$= \int_0^t \frac{1}{a}\sin au\cos a(t-u)du$$
$$= \frac{1}{a}\int_0^t \sin au\cos(at-au)du$$
$$= \frac{1}{2a}\int_0^t [\sin at + \sin(2au-at)]du$$
$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos(2au-at)\right]_0^t$$
$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\cos at + \frac{1}{2a}\cos at\right] = \frac{1}{2}t\sin at$$

Problem 3.15.3.

Sol : We have $t^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$ $\therefore$ By convolution theorem

$$f^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \left(\frac{1}{a}\sin at\right) * \left(\frac{1}{a}\sin at\right) = \frac{1}{a^2}(\sin at * \sin at)$$
$$= \frac{1}{a^2}\int_0^t \sin au\sin a(t-u)du$$
$$= \frac{1}{2a^2}\int_0^t 2\sin au\sin(at-au)du$$
$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at) - \cos at]du$$
$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin(2au-at) - u\cos at\right]_0^t$$
$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin at + \frac{1}{2a}\sin at - t\cos at\right]$$
$$= \frac{1}{2a^3}[\sin at - at\cos at]$$

Problem 3.15.4. $L^{-1}\left(\frac{1}{(s-1)(s^2+1)}\right)$ *(VTU Jan 2018,Jan 2015, Dec 2010)*

Let $F(s) = \frac{1}{(s^2+1)}$ and $G(s) = \frac{1}{(s-1)}$

Then $f(t) = f(t) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

and $g(t) = L^{-1}\left[\frac{1}{s-1}\right] = e^t$

From convolution theorem, we have

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)du$$

$$\therefore L^{-1}\left(\frac{1}{(s-1)(s^2+1)}\right) = \int_0^t \sin u e^{t-u} du$$

$$= e^t \int_0^t e^{-u} \sin u du$$

$$= e^t \left[\frac{e^{-u}}{2}(-\sin u - \cos u)\right]_0^t$$

$$\left(\because \int e^{at} \sin bt dt = \frac{e^{at}}{a^2+b^2}[a \sin bt - b \cos bt]\right)$$

$$= \frac{e^t}{2}\left[e^{-t}(-\sin t - \cos t) + 1\right]$$

$$= \frac{1}{2}\left(e^t - \sin t - \cos t\right)$$

Problem 3.15.5. *Find* $L^{-1}\left(\frac{s}{(s-1)(s^2+4)}\right)$ *by using Convolution Theorem. (VTU July 2017, July 2016, Dec 2014, Jan 2014, Jun 2011)*

SOl :

$$Let\ \ F(s) = \frac{s}{s^2+4}, \quad G(s) = \frac{1}{s-1}$$

$$f(t) = L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$$

$$g(t) = L^{-1}\left[\frac{1}{s-1}\right] = e^t$$

From convolution theorem, we have

$$L^{-1}\left\{F(s)G(s)\right\} = \int_0^t f(u)g(t-u)du$$

$$L^{-1}\left(\frac{s}{(s-1)(s^2+4)}\right) = \int_0^t \cos 2u e^{t+u} du$$

$$= e^t \int_0^t e^u \cos 2u du$$

$$\left(\because \int e^{at}\cos bt dt = \frac{e^{at}}{a^2+b^2}[a\cos bt + b\sin bt]\right)$$

$$= e^t\left[\frac{e^u}{5}(\cos 2u + 2\sin 2u)\right]_0^t$$

$$= \frac{e^t}{5}\left[e^t(\cos 2t + 2\sin 2t) - 1\right]$$

Problem 3.15.6. *Find the Laplace transform of* $\frac{4}{(s^2+2s+5)^2}$*, using convolution theorem.* *(VTU Model 2019)*

$$F(s) = \frac{2}{(s+1)^2+4}, G(s) = \frac{2}{(s+1)^2+4}$$

Sol : $\Rightarrow L^{-1}[F(s)] = L^{-1}\left[\frac{2}{(s+1)^2+4}\right] = e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right] = e^{-t}\sin 2t$

$$= f(t)$$

$$L^{-1}[G(s)] = L^{-1}\left[\frac{2}{(s+1)^2+4}\right] = e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right] = e^{-t}\sin 2t = g(t)$$

$\therefore t(t) = e^{-t}\sin 2t \quad g(t) = e^{-t}\sin 2t$

$$\text{WKT } L^{-1}[\bar{f}(s)\cdot \bar{g}(s)] = f(t) * g(t)$$

$$L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right] = \int_{u=0}^t f(u)g(t-u)du$$

$$= \int_{u=0}^t e^{-u}\sin 2u e^{-(t-u)}\sin(2t-2u)du$$

$$
\begin{aligned}
&= \int_{u=0}^{t} e^{tu} \sin 2u e^{-t} \cdot e^{u} \sin(2t-2u) du \\
&= e^{-t} \int_{u=0}^{t} \sin 2u \cdot \sin(2t-2u) du \\
&= \frac{e^{-t}}{2} \int_{u=0}^{t} 2\sin 2u \sin(2t-2u) du \\
&= \frac{e^{-t}}{2} \int_{u=0}^{t} \cos(2u-2t+2u) - \cos(2u+2t-2u) du \\
&= \frac{e^{-t}}{2} \int_{u=0}^{t} [\cos(4u-2t) - \cos 2t] du \\
&= \frac{e^{-t}}{2} \left\{ \int_{u=0}^{t} \cos(4u-2t) du - \cos 2t \int_{0}^{t} 1 du \right\} \\
&= \frac{e^{-t}}{2} \left\{ \left[\frac{\sin(4u-2t)}{4} \right]_{u=0}^{t} - t\cos 2t \right\} \\
&= \frac{e^{-t}}{2} \left\{ \frac{\sin 2t}{4} + \frac{\sin 2t}{4} - t\cos 2t \right\} \\
&= \frac{e^{-t}}{2} \left[\frac{\sin 2t}{2} - t\cos 2t \right]
\end{aligned}
$$

3.16 Solution of Differential Equations using Laplace Transforms method

Laplace transform technique is employed to solve initial-value problems. The solution of such a problem is obtained by using the Laplace Transform of the derivatives of function and then the inverse Laplace Transform. The following are the expressions for the Laplace transform of derivatives.

$$\boxed{L[f'(t)] = sL\{f(t)\} - f(0)}$$

$$\boxed{L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)}$$

$$\boxed{L\{f'''(t\} = s^3L\{f(t)\} - s^2f(0) - sf'(0) - f''(0)}$$

$$\vdots$$

$$\boxed{L\{f^{(n)}(t\} = s^nL\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots f^{(n-1)}(0)}$$

Problem 3.16.1. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}, y(0) = 0, y'(0) = 0$ *(VTU July 2016, June 2014, Model 2015, Dec 2011, June 2010)*

Solution : Given,

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$$

Taking Lalace transform on both sides

$$L\{y''(t)\} + 4L\{y'(t)\} + 4L\{y(t)\} = L\{e^{-t}\}$$

$$\left(s^2L\{y(t)\} - sy(0) - y'(0)\right) + 4\left(sL\{y(t)\} - y(0)\right) + 4L\{y(t)\} = \frac{1}{s+1}$$

Substituting $y(0) = 0, y'(0) = 0$

$$\left(s^2L\{y(t)\}\right) + 4\left(sL\{y(t)\}\right) + 4L\{y(t)\} = \frac{1}{s+1}$$

$$(s^2 + 4s + 4)L\{y(t)\} = \frac{1}{s+1}$$

$$L\{y(t)\} = \frac{1}{(s+1)(s^2+4s+4)}$$

$$= \frac{1}{(s+1)(s+2)^2}$$

by using partial fractions we get

$$L\{y(t)\} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

Taking Inverse Lalace transform on both sides

$$y = e^{-t} - e^{-2t} - te^{-2t}$$

Problem 3.16.2. *SOlve* $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 1$ *(VTU 2013, 2011, 2008)*

SOl : SOl : Taking Laplace transform on both sides of the given equation, we have

$$L\left[y''(t)\right] + 4L\left[y'(t)\right] + 3L[y(t)] = L\left[e^{-t}\right]$$

$$\text{i.e.,} \quad \left\{s^2Ly(t) - sy(0) - y'(0)\right\}$$

$$+ 4\{sLy(t) - y(0)\} + 3Ly(t) = \frac{1}{s+1}$$

Using the given initial condition, we obtain

$$\left(s^2+4s+3\right)Ly(t)-s-1-4=\frac{1}{s+1}$$

i.e.,

$$\left(s^2+4s+3\right)Ly(t)=(s+5)+\frac{1}{(s+1)}$$

$$(s+1)(s+3)Ly(t)=\frac{s^2+6s+6}{s+1}$$

$$\therefore L[y(t)]=\frac{s^2+6s+6}{(s+1)^2(s+3)}$$

$$\therefore \quad y(t)=L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right]$$

Let $\frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+3)}$ Multiplying by $(s+1)(s+3)$, we get

$$s^2+6s+6=A(s+1)(s+3)$$
$$+B(s+3)+C(s+1)^2$$

Equating the coefficient of s^2 on both sides of (1), we get

$$1=A+C \quad \left(\because A=\frac{7}{4}\right)$$

Hence,

$$L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right]$$
$$=\frac{7}{4}L^{-1}\left(\frac{1}{s+1}\right)+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right]-\frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right)$$
$$\therefore y(t)=\frac{7}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{3}{4}e^{-3t}.$$

Problem 3.16.3. *Solve* $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=sint$, $y=\frac{dy}{dt}=0$ *when* $t=0$ *(VTU Jan 2017, July 2016)*

SOl :Given $y''+2y'-3y=\sin t$,

Taking L.T. on both sides We get,

$\left(s^2L\{y(t)\}-sy(0)-y'(0)\right)+2\left(sL\{y(t)\}-y(0)\right)-3L\{y(t)\}=\frac{1}{s^2+1}$

$$
\begin{aligned}
i.e.\ s^2 Y(s) + 2sY(s) - 3Y(s) &= \frac{1}{s^2+1} \\
\Rightarrow \left(s^2 + 2s - 3\right) Y(s) &= \frac{1}{s^2+1} \\
\Rightarrow Y(s) &= \frac{1}{(s^2+1)(s-1)(s+3)}
\end{aligned}
$$

Using Partial fractions,

$$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{As+B}{(s^2+1)} + \frac{C}{s-1} + \frac{D}{s+3}$$

$$1 = (As+B)(s-1)(s+3) + C\left(s^2+1\right)(s+3) + D\left(s^2\right.$$

$$\Rightarrow A = -\frac{1}{10},\ \ B = -\frac{1}{5},\ \ C = \frac{1}{8},\ \ D = -\frac{1}{40}\ \text{(try it)}$$

$$\therefore Y(s) = \frac{-\frac{1}{10}s - \frac{1}{5}}{(s^2+1)} + \frac{1}{8}\frac{1}{s-1} - \frac{1}{40}\frac{1}{s+3}$$

$$y(t) = L^{-1}[Y(s)] = -\frac{1}{10}\cos t - \frac{1}{5}\sin t + \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t}.$$

Question Bank-Module 1- Laplace Transforms

Laplace transform of Elementary functions

Find the Laplace Transforms of the following functions.

1) $e^{at} + 2t^n - 3sin3t + 4cosh2t$ (VTU June 2010)

2) $\left(\frac{4t+5}{e^{2t}}\right)^2$ (VTU Jan 2020)

3) $\left(\frac{\sin 2t}{\sqrt{t}}\right)^2$ (VTU Jan 2020)

4) 3^t (July 2014)

5) $(sint - cost)^2$ (VTU Jan 2016)

6) $Cos^2 2t$ (VTU June 2011)

7) $sint\ sin2t\ sin3t$ (VTU Jan 2017, Jan 2013)

8) Evaluate $L\{\cos t \cos 2t \cos 3t\}$ (VTU Jan 2021)

Shifting Rule

1. Find $L\left\{e^{-3t}(2cos5t - 3sin5t)\right\}$ (VTU Jan 2017)
2. Find the Laplace transform, (i) $\mathrm{e}^{-2t}(2\cos 5\mathrm{t} - \sin 5\mathrm{t})$ (ii) $\cosh^2 3\mathrm{t}$ (VTU August 2022)

3. Find $L\left\{e^{-3t}t^4\right\}$ (VTU July 2016)
4. Find $L\left\{e^{3t}t^4\right\}$ (VTU Jan 2017)
5. Find $L\left\{e^{-3t}sin5tsin3t\right\}$ (VTU Dec 2011, 2006)
6. Find $L\{e^{-2t}sin3t + e^t tcost\}$ (VTU Jan 2016) Ans : $\frac{3}{(s+2)^2+9} + \frac{s(s-2)}{(s^2-2s+2)^2}$
7. Find the Laplace Transform of $sin2t\ cos3t$
8. Find $L\left\{e^{2t}cos^2t\right\}$ (VTU 2012, 2006)
9. Find the Laplace Transform of $e^{-t}cos^2 3t$ (VTU Jan 2010)

Multiplication by t, t^2,...

1. Find $L\{tsint\}$ (VTU July 2014)
2. Find $L\{tsinat\}$ (VTU Jan 2010)
3. Find $L\{tcosat\}$ (VTU Jan 2019, Jan 2020)
4. Find $L\left\{e^{-2t}t\cos 2t\right\}$ (VTU Sept 2020)
5. Find $L\left\{t^2 sinat\right\}$ (VTU Dec 2010)
6. Find $L\left\{te^{-2t}sin4t\right\}$ (VTU Feb 2022, July 2016, Jan 2008)
7. Find $L\left\{t^2e^{-3t}sin2t\right\}$ (VTU July 2016)
8. Find $L\{te^{-t}sin3t\}$ (VTU June 2011)
9. Find $L\left(t^2e^{-2t}sint\right)$ (VTU Model 2015)
10. Find $L\left(te^{-4t}sin3t\right)$ (VTU June 2015)
11. Find $L\left(te^{-t}sin4t\right)$ (VTU July 2008)
12. Find $L\left(t^2e^{4t}cosh3t\right)$ (VTU Jan 2015)

13. Find $L\left(e^{-t}sin6t + tcos3t\right)$ (VTU July 2017)

14. Find $L\left(te^{-t}sin^2 3t\right)$ (VTU July 2017)

Division by t

1) Find the Laplace Transform of $\frac{e^{at}-e^{-at}}{t}$ (VTU June 2015)

2) Find $L\left(\frac{e^{-at}-e^{-bt}}{t}\right)$ (VTU June 2012)

3) Find $L\left\{\frac{e^{t}sint}{t}\right\}$ (VTU 2009, 2013)

4) Find $L\left(\frac{sin^2 t}{t}\right)$ (VTU Model 2015)

5) Find $L\left(\frac{sinat}{t}\right)$ (VTU Jan 2019)

6) Evaluate $L\left(\frac{cos2t-cos3t}{t}\right)$ (VTU Mar 2022, July 2018, Jan 2017, July 2016)

7) Find $L\left(\frac{cosat-cosbt}{t}\right)$ (VTU August 2022, Jan 2017, Jan 2016, Jan 2013, Jan 2010, July 2008)

8) Find $L\left\{\frac{e^{-t}sint}{t}\right\}$ (VTU July 2013, 2009 S)

9) Find $L\left\{\frac{1-cos3t}{t}\right\}$ (VTU 2006)

10) Find the Laplace transform of $te^{-t}\sin 2t + \frac{\cos 2t-\cos 3t}{t}$ (VTU Model 2022)

11) Find $L\left\{2^t + \frac{cos2t-cos3t}{t} + tsint\right\}$ (July 2019, July 2017, VTU 2004)

12) Find $L\left\{te^{2t} + \frac{cos2t-cos3t}{t} + tsint\right\}$ (VTU Jan 2018)

13) Find $L\left\{\frac{1-cost}{t}\right\}$ (VTU July 2016)

14) Find $L\left\{tcos2t + \frac{1-e^{3t}}{t}\right\}$ (VTU July 2019)

15) Find the Laplace transform of (i) $e^{-3t}\sin 5t\cos 3t$ (ii) $\frac{1-e^{t}}{t}$ (VTU Model 2022)

Laplace Transform of a periodic function:

1) Find $L\{f(t)\}$ when $f(t) = \begin{cases} E, & 0 \leq t \leq a \\ -E, & a \leq t \leq 2a \end{cases}$ where the period is $2a$
(VTU Sept. 2020, July 2017,June 2015, June 2014, June 2011)

2) If $L\{f(t)\}$ when $f(t) = \begin{cases} E, & 0 \leq t \leq \frac{a}{2} \\ -E, & \frac{a}{2} \leq t \leq a \end{cases}$ and $f(t+a) = f(a)$ Show that $L\{f(t)\} = \frac{E}{S} tanh\left(\frac{as}{4}\right)$ (VTU July 2016, Model 2015, June 2014, Dec 2010, June 2011),

3) Find Laplace transform of the function $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$ where $f(t+2\pi) = f(t)$ (VTU July 2018, June 2012)

4) Find the Laplace transform of full wave rectifier $f(t) = Esin\omega t, 0 < t \leq \frac{\pi}{\omega}$ having the period $\frac{\pi}{\omega}$. (VTU August 2022, July 2019, Jan 2019, Jan 2018, July 2017,Dec 2011)

5) If $f(t) = \begin{cases} t, & 0 \leq t \leq a \\ 2a - t, & a \leq t \leq 2a \end{cases}$, where $f(t+2a) = f(t)$, show that $L\{f(t)\} = \frac{1}{s^2} tanh\left(\frac{as}{2}\right)$ (VTU Model 2022, Mar 2022, Jan 2021, Jan 2017, July 2016, June 2015, Dec 2014, July 2014, Dec 2011, July 2008)

6) If $f(t) = \begin{cases} t, & 0 \leq t \leq \pi \\ 2\pi - t, & \pi \leq t \leq 2\pi \end{cases}$, where $f(t+2\pi) = f(t)$, show that $L\{f(t)\} = \frac{1}{s^2} tanh\left(\frac{\pi s}{2}\right)$ (VTU July 2019)

7) A periodic function of period 2 is given by $f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \end{cases}$. Find $L\{f(t)\}$. (June 2015)

8) A periodic function of period $\frac{2\pi}{\omega}$ is defined by $f(t) = \begin{cases} Esin\omega t, & 0 < t \leq \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < 2\frac{\pi}{\omega} \end{cases}$ where E and ω are positive constants. Find $L\{f(t)\}$ (VTU Jan 2017, July 2016, Jan 2013)

9) If $f(t) = \begin{cases} 1, & t \in [0, a) \\ -1 & t \in [a, 2a) \end{cases}$ where the period is 2a. Then Show that $L\{f(t)\} = \frac{1}{s} tanh\left(\frac{as}{2}\right)$ (Jan 2020)

10) Find the Laplace transform of the square-wave function of period a given by

$$f(t) = \begin{cases} 1, & 0 < t < a/2 \\ -1, & a/2 < t < a \end{cases}$$

(VTU Model 2022)

Unit Step functions

1) Find $L\left[\left(e^{t-1} - sin(t-1)\right) \, u(t-1)\right]$ (VTU Dec 2010)

2) Express $f(t) = \begin{cases} \pi - t, & 0 < t \leq \pi \\ sint & t > \pi \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU July 2018)

3) Express $f(t) = \begin{cases} 1, & 0 < t < 1 \\ t, & \text{if } 1 < t \leq 2 \\ t^2, & t > 2 \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU Jan 2021,Sept 2020, July 2019, July 2016, June 2015, Model 2015, Dec 2014, Jan 2014, Jan 2013, Dec 2011)

4) Express $f(t)$ in terms of unit step function and hence find the Laplace transform.

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4. \end{cases}$$ (VTU July 2017, June 2014, June 2015, June 2011, Jan 2010)

5) Express $f(t) = \begin{cases} t^2, & 0 \leq t \leq 3 \\ 4, & t > 3 \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU Dec 2011)

6) Express $f(t) = \begin{cases} 2t, & 0 \leq t \leq \pi \\ 1, & t > \pi \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU Jan 2013)

7) Express $f(t) = \begin{cases} sint, & 0 \leq t \leq \frac{\pi}{2} \\ cost, & t > \frac{\pi}{2} \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU July 2017, June 2010, June 2011)

8) Express $f(t) = \begin{cases} cost, & 0 \leq t \leq \pi \\ sint, & t > \pi \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU Jan 2019)

9) Express $f(t) = \begin{cases} cost, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ sint, & t > 2\pi \end{cases}$ in terms of unit step function and hence find $L(f(t))$.

(VTU August 2022, July 2018, Jan 2016,July 2013, Jan 2008, July 2008, 2007)

10) Express $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2t, & \text{if } 1 < t \leq 2 \\ 3t^2, & t > 2 \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU July 2016)

11) Express $f(t) = \begin{cases} t-1, & \text{if } 1 < t < 2 \\ 3-t, & \text{if } 2 < t < 3 \\ 0, & \text{otherwise.} \end{cases}$ in terms of unit step function and hence find $L(f(t))$. (VTU July 2013)

12) Using unit step function, find the Laplace transform of $f(t) = \begin{cases} sint, 0 < t < \pi \\ sin2t, \pi \leq t \leq 2\pi \\ sin3t, t \geq 2\pi \end{cases}$

(VTU Mar 2022, Jan 2017, 2004)

13) Express the following function in terms of unit step function and find its Laplace transform

$$f(t) = \begin{cases} cost, 0 < t < \pi \\ cos2t, \pi < t < 2\pi \\ cos3t, t > 2\pi \end{cases}$$

(VTU Model 2022, July 2014, Dec 2008)

14) Express the following function in terms of unit step function and find its Laplace transform

$$f(t) = \begin{cases} sint, 0 < t < \frac{\pi}{2} \\ cost, \frac{\pi}{2} < t < \pi \\ 1, t > \pi \end{cases}$$

(VTU July 2019)

15) Express the following function in terms of unit step function and find its Laplace transform

$$f(t) = \begin{cases} t^2, 1 < t < 2 \\ 4t, t > 2 \end{cases}$$

(VTU Jan 2017, July 2016)

16) Express $f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{t^2}{2} & \text{if } 1 < t < \frac{\pi}{2} \\ \cos t & t > \frac{\pi}{2} \end{cases}$ in terms of unit step function and hence find its Laplace transformation. (VTU Jan 2020)

17) Express the following function in terms of unit step function and hence find its Laplace transform

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2t, & 1 < t < 2 \\ 3t, & 2 < t < 3 \end{cases}$$

(VTU Model 2022)

Inverse Laplace Transforms

1) Find the inverse Laplace transform of (i) $\frac{(s^2-1)^2}{s^5}$ (ii) $\frac{s}{s^2+6s+13}$ (VTU Model 2022)

2) Find $L^{-1}\left(\frac{2s+1}{s^2+6s+13}\right)$ (VTU Jan 2021)

3) Find $L^{-1}\left(\frac{s+2}{s^2(s+3)}\right)$ (VTU Jan 2019)

4) Find $L^{-1}\left(\frac{7s+4}{4s^2+4S+9}\right)$ (VTU Jan 2018)

5) Find the inverse Laplace transform of $\frac{s^2-2s+1}{s^5}$ (VTU Dec 2010)

6) Find $L^{-1}\left(\frac{s^2-2s+1}{s^3}\right)$ (VTU June 2012)

7) Find $L^{-1}\left(\frac{s^2-3s+4}{s^3}\right)$ (VTU June 2020)

8) Find inverse Laplace Transform of $\frac{4s+5}{(s+1)^2(s+2)}$ (VTU Mar 2022, Jan 2016)

9) Find $L^{-1}\left(\frac{s}{(s+2)(s+3)}\right)$ (VTU Jan 2020)

10) Evaluate $L^{-1}\left(\frac{2s-1}{s^2+2s+17}\right)$ (VTU Jan 2013, Dec 2011)

11) Find $L^{-1}\left[\frac{3s+2}{s^2-s-2}\right]$ (VTU Jan 2013, june 2010)

12) Find $L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\}$ (VTU Sept 2020, Jan 2017, Dec 2008)

13) Find $L^{-1}\left\{\frac{s+5}{s^2-6s+13}\right\}$ (VTU July 2019)

14) Find $L^{-1}\left[\frac{s+5}{s^2-4s+13}\right]$ (VTU Model 2015)

15) Find $L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right]$ (VTU July 2014, June 2012, July 2009)

16) Find $L^{-1}\left\{\frac{7s+4}{4s^2+4s+9}+\frac{1}{(s+3)^4}\right\}$ (VTU July 2016)

17) Find $L^{-1}\left\{\frac{s-2}{s^2+7s+12}\right\}$ (VTU July 2008)

18) Find $L^{-1}\left(\frac{s+1}{(s^2+6s+9)}\right)$ (VTU July 2009)

19) Find $L^{-1}\left(\frac{s+2}{(s+1)^4}\right)$ (VTU June 2011)

20) Find $L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right)$ (VTU Jan 2017, July 2016)

21) Find $L^{-1}\left(\frac{1}{s(s+1)(s+2)(s+3)}\right)$ (VTU Dec 2010)

22) Find inverse Laplace Transform of $\frac{4s+5}{(s+1)^2(s+2)}$ (VTU Jul 2016, Dec 2010)

23) Find $L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$ (VTU Model 2022, July 2017, Dec 2011, June 2011, 2007)

24) Find $L^{-1}\left\{\frac{s}{(2s-1)(3s-1)}\right\}$ (VTU Jan 2010)

25) $L^{-1}\left(\{\frac{s^2+4}{s(s+4)(s-4)}\right\}$ (VTU August 2022, Jan 2015)

26) Find the inverse Laplace transform of $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$ (VTU Jan 2014)

27) Find $L^{-1}\left\{\frac{s^2}{(s+1)^3}\right\}$ (VTU Jan 2008)

28) Find $L^{-1}\left\{\frac{s+2}{s^2(s+1)(s-2)}\right\}$ (VTU 2003)

29) Evaluate $L^{-1}\left[log\left(\frac{s^2+1}{s(s+1)}\right)\right]$ (VTU Sept 2020, July 2017, June 2011, 2008)

30) Evaluate $L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right]$ (VTU July 2009)

31) Evaluate $L^{-1}\left[cot^{-1}\left(\frac{s}{2}\right)\right]$ (VTU Jan 2020)

32) Evaluate $L^{-1}\left[log\left(\frac{s+1}{s-1}\right)\right]$ (VTU July 2013)

33) Find $L^{-1}[\frac{s+3}{s^2-4s+13}+log(\frac{s+1}{s-1})]$ (VTU July 2018)

34) Evaluate $L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right]$ (VTU Dec 2008)

35) Evaluate $L^{-1}\left[\frac{1}{2}log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]$ (VTU July 2016, 2008)

36) Evaluate $L^{-1}\left[\frac{1}{3}log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]$ (VTU Jan 2021)

37) Evaluate $L^{-1}\left[log\left(1+\frac{a^2}{s^2}\right)\right]$ (VTU Jan 2008)

38) Evaluate $L^{-1}\left[cot^{-1}(s)\right]$ (VTU Jan 2005)

39) Evaluate $L^{-1}\left[\frac{e^{-6s}}{(s-4)^2}\right]$ (VTU July 2008)

40) Evaluate $L^{-1}\left[\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]$ (VTU June 2015, 2000)

41) Evaluate $L^{-1}[\frac{s+1}{s^2+2s+2}+log(\frac{s+a}{s+b})]$ (VTU July 2019)

Convolution Theorem

Using convolution theorem evaluate

1) $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$ (VTU Dec 2014, June 2012, Dec 2011, June 2010, July 2007)

2) $L^{-1}\left(\frac{1}{(s^2+1)(s^2+9)}\right)$ (VTU Model 2022)

3) $L^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right)$ (VTU Jan 2021, Sept 2020, July 2019, Jan 2013, Model 2015, 2010)

4) $L^{-1}\left(\frac{s}{(s-1)(s^2+4)}\right)$ (VTU July 2017, July 2016, Dec 2014, Jan 2014, Jun 2011)

5) $L^{-1}\left(\frac{s}{(s-1)(s^2+1)}\right)$ (VTU July 2019)

6) $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$ (VTU Jan 2019, July 2016, June 2011)

7) $L^{-1}\left[\frac{1}{4s^2-9}\right]$ (VTU July 2013)

8) $L^{-1}\left(\frac{1}{(s+1)(s^2+9)}\right)$ (VTU Jun 2015)

9) $L^{-1}\left(\frac{1}{(s+1)(s^2+4)}\right)$ (VTU July 2014)

10) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ (VTU Model 2022, Mar 2022, Jan 2017, Jan 2013, Jan 2010, July 2009, Dec 2008)

11) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ (VTU July 2016, June 2015)

12) $L^{-1}\left(\frac{1}{(s-1)(s^2+1)}\right)$ (VTU Jan 2018,Jan 2015, Dec 2010)

13) $L^{-1}\left(\frac{s}{(s+2)(s^2+9)}\right)$ (VTU July 2018, July 2008)

14) $L^{-1}\left(\frac{1}{s(s^2+1)}\right)$ (VTU Jan 2020)

Solution of Differential Equations using Laplace Transforms method

1) Solve by using Laplace transform method $y'''(t)+2y''(t)-y'(t)-2y(t)=0$. $y(0)=y'(0)=0$ and $y''(0)=6$ (VTU August 2022)

2) $y''+k^2y=0$ where $y(0)=2, y'(0)=0$ (VTU Jan 2019)

3) $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=e^{-t}, y(0)=0, y'(0)=0$
(VTU Feb 2022, July 2019, July 2016, June 2014, Model 2015, Dec 2011, June 2010)

4) $y''-3y'+2y=0, y(0)=1, y'(0)=0$ (VTU Sept. 2020)

5) $y''-y'=0, y(0)=y'(0)=3$ (VTU Jan 2020)

6) $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+8y=0$ with $y(0)=2, y'(0)=0$ (VTU July 2016)

7) $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$ with $y(0)=2, y'(0)=1$ (VTU July 2014)

8) $\frac{d^y}{dt^2}+2\frac{dy}{dt}+y=te^{-t}$ with $y(0)=1, y'(0)=-2$ (VTU Jun 2015)

9) $(D^3 - 3D^2 + 3D - 1)y = 2t^2e^t, y(0) = 1, y'(0) = 0, y''(0) = -2$ (VTU June 2012)

10) $y''' + 2y'' - y' - 2y = 0$ where $y = 1, \frac{dy}{dt} = 2 = \frac{d^2y}{dt^2}$ at $t = 0$ (VTU July 2017, July 2013)

11) $y'' + 6y' + 9y = 12t^2e^{-3t}$ with $y(0) = 0, y'(0) = 0$ (VTU Jan 2016, Dec 2010)

12) $y'' - 6y' + 9y = t^2e^{3t}$ with $y(0) = 2, y'(0) = 6$ (VTU July 2016)

13) $y'' - 3y' + 2y = 4t + e^{3t}, y(0) = 1, y'(0) = -1.$ (VTU Jan 2013)

14) $y'' - 3y' + 2y = 4t + 12e^{-t}, y(0) = 6, y'(0) = -1.$ (VTU July 2017)

15) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t}$, given that $x(0) = 0$ and $x'(0) = -1$ (VTU July 2017)

16) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, given that $x = 2, \frac{dx}{dt} = -1$ at $t = 0$ (VTU Model 2022, Jan 2016, June 2011)

17) $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 1$ (VTU 2013, 2011, 2008)

18) $y'' - 3y' + 2y = e^{3t}, y(0) = 1, y'(0) = -1$ (VTU Model 2022)

19) $y'' - 3y' + 2y = e^{3t}, y(0) = 1, y'(0) = 0$ (VTU July 2018, July 2010)

20) $y'' - 3y' + 2y = 2e^{3t}, y(0) = y'(0) = 0$ (VTU July 2019, July 2018, July 2010)

21) $y'' - 3y' + 2y = e^{3t}, y(0) = 1, y'(0) = 0$ (VTU Jan 2010)

22) $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 0$ (VTU Jan 2021)

23) $y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$ (VTU Jan 2017, June 2014, Dec 2011)

24) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = sint, y = \frac{dy}{dt} = 0$ when $t = 0$ (VTU Jan 2017, July 2016)

25) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2t^2 + 2t + 2$ under the conditions $y(0) = 2, y'(0) = 0$ (VTU June 2015)

26) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 5e^{2t}\ y(0) = 2, y'(0) = 1$ (VTU July 2014)

Lecture Notes

BMATE201

Mathematics-II for EEE stream

Module 4 - Numerical Methods-I

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Table of Contents

Module 4

Numerical Methods I

Syllabus

Importance of numerical methods for discrete data in the field of Mechanical Engineering

Solution of algebraic and transcendental equations: Regula-Falsi and Newton-Raphson methods (only formulae). Problems. Finite differences, Interpolation using Newton's forward and backward difference formulae, Newton's divided difference formula and Lagrange's interpolation formula (All formulae without proof). Problems. Numerical integration: Trapezoidal, Simpson's (1/3)rd and (3/8)th rules(without proof). Problems.

Self-Study: Bisection method, Lagrange's inverse Interpolation.

Applications: Finding approximate solutions to solve mechanical engineering problems involving numerical data. (RBT Levels: L1, L2 and L3)

4.1 Forward and Backward Differences

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots\ldots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ are respectively, called the first forward differences where Δ is the forward difference operator.

Thus the first forward differences are

$$\Delta y_r = y_{r+1} - y_r, \quad r = 0, 1, 2, 3, \ldots$$

The difference of first forward differences is called second forward differences. i.e,

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0, 1, 2, 3, \ldots$$

Similarly, the other higher order differences namely the third; fourth, etc. are obtained and tabulated. Such a tabular arrangement is called forward difference table:

In general,

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r$$

defines the n^{th} forward differences.

Following table shows how the forward differences of all orders can be formed.

The Forward Difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

The differences $y_1 - y_0, y_2 - y_1, \ldots\ldots., y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots\ldots. \nabla y_n$ respectively are called first backward differences where ∇ is the backward difference operator.

Similarly, we define higher order backward differences as,

$$\nabla y_r = y_r - y_{r-1}$$

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

Backward Difference Table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

4.2 Interpolation:

If $y_0, y_1, y_2, \ldots, y_n$ be the set of values of an unknown function $y = f(x)$ corresponding to the values of x : $\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$ the process of finding the values of y for any given value of x between x_0 and x_n is called interpolation. Also the process of finding the values of y outside the given range of x is called extrapolation.

4.3 Newton's Forward Interpolation Formula [NFIF]

Let $y = f(x)$ be a function which takes values $y_0 = f(x_0), y_1 = f(x_0 + h), y_2 = f(x_0 + 2h), \ldots, y_n$ corresponding to various equi-spaced values of x with spacing h, say $\mathrm{x}_0, x_1 = \mathrm{x}_0 + \mathrm{h}, x_2 = \mathrm{x}_0 + 2\,\mathrm{h}, \ldots, x_n = x_0 + nh$.
Suppose, we wish to evaluate the function $y = f(x)$ for a value $x_p = x_0 + ph$ where p is any real number, then for any real number p,

$$\boxed{\begin{aligned} y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \\ + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots \end{aligned}}$$

This formula is mainly used for interpolating the values of y **near the beginning** of a set of tabular values.

4.4 Newton's Backward Interpolation Formula [NBIF]

The value of $y = f(x)$ at $x_p = x_n + ph$ is approximately given by

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \cdots$$

This formula is mainly used for interpolating the values of y **near the end** of a set of tabular values (second half).

4.5 Step by step working procedure for the problems :

Step1: We construct the difference table in accordance with the interpolation formula.
Step 2: We compute the value of p where $p = \frac{x_p - x_0}{h}$ in case of forward interpolation formula, x_0 being the first value of x, h being the step length, and
$p = \frac{x_p - x_n}{h}$ in case of backward interpolation formula, x_n being the last value of x and h being the step length.
Step 3 : The value of p along with the value of the finite differences is substituted in the Interpolation formula which results in the value of y at the desired value of x.

Problem 4.5.1. *The population of a town in the decimal census was as given below. Estimate the population for the year* 1895

year	*1891*	*1901*	*1911*	*1921*	*1931*
Population(in thousands)	*46*	*66*	*81*	*93*	*101*

Solution:: Here $x_0 = 1891, h = 10, \quad x_p = 1895$

$\Rightarrow \quad p = \frac{x_p - x_0}{h} = \frac{1895-1891}{10} = 0.4$

The difference table is given by

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		−5		
		15		2	
1911	81		−3		−3
		12		−1	
1921	93		−4		
		8			
1931	101				

$$y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2}(-5) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(2) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

$$= 54.8528 \text{ thousands}$$

Problem 4.5.2. *Given* $f(40) = 184, f(50) = 204, f(60) = 226, f(70) = 250, f(80) = 276,\ f(90) = 304,$ *find* $f(85)$ *using suitable interpolation formula.* *[VTU: June12/Jan16]*

Solution:: Here we have to find y at x $= 85$ since the value x $= 85$ is in the second half of the table near x $= 90$, NBIF is appropriate and the backward differences are tabulated as below.

x	y					
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

We have Newton's backward difference interpolation formula

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

Here, $x_n = 90, h = 10, \nabla y_n = 28, \nabla^2 y_n = 2, x_p = 85$

$$p = \frac{x_p - x_n}{h}$$

$$p = \frac{85 - 90}{10} = -0.5$$

$$\text{y} = 304 + (-0.5)28 + \frac{(-0.5)((-0.5)+1)}{2!}2 = 289.75$$

Thus $f(85) = 289.75$

Problem 4.5.3. *In the following table, values of y are consecutive terms of a series of which* **23.6** *is the* **6***th term.*

x :	*3*	*4*	*5*	*6*	*7*	*8*	*9*
y:	*4.8*	*8.4*	*14.5*	*23.6*	*36.2*	*52.8*	*73.9*

Find the first and tenth terms of the series.

Solution:: To find first term (i.e. y when $x = 1$), let us use Newton's forward interpolation formula.

Here, $x_0 = 3, \quad h = 1, \quad x_p = 1$

$\therefore \quad p = \frac{x_p - x_0}{h} = -2$

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
3	4.8	3.6	2.5	0.5	0
4	8.4	6.1	3	0.5	0
5	14.5	9.1	3.5	0.5	0
6	23.6	12.6	4	0.5	
7	36.2	16.6	4.5		
8	52.8	21.1			
9	73.9				

$$y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

$$Y(1) = 4.8 + (-2)\times 3.6 + \frac{(-2)(-3)}{2}(2.5) + \frac{(-2)(-3)(-4)}{6}(0.5)$$
$$= 3.1$$

To obtain tenth term, we use Newton's Backward interpolation formula

$$x_n = 9, h = 1, x_p = x_n + ph = 10$$
$$10 = 9 + p \Rightarrow p = 1$$
$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$
$$y_{10} = 73.9 + 1(21.1) + \frac{1(2)}{2!}(4.5) + \frac{1(2)(3)}{3!}(0.5)$$
$$= 73.9 + 21.1 + 4.5 + 0.5 = 100$$

Problem 4.5.4. *From the following table find the number of students who have obtained (a) less than 45 marks (b) between 40 and 45 marks. [VTU- July15/Jan17]*

marks	$30-40$	$40-50$	$50-60$	$60-70$	$70-80$
No. of students	31	42	51	35	31

Solution: : Let us first prepare a new table with the following data :

x: marks	40	50	60	70	80
y: No. of students with marks < x	31	31+42=73	73+51=124	124+35=159	190

Now We shall first find y_{45}, number of students with marks less than 45.

$$x_0 = 40, h = 10, x_p = 45$$
$$x_0 + hp = 40 + 10p = 45 \Rightarrow p = .5$$

Marks less than (x)	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		−25	
60	124		−16		37
		35		12	
70	159		−4		
		31			
80	190				

By Newton's forward difference formula,

$$\begin{aligned} y(x_p) &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 \\ &+ \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \\ &= 31 + (.5)(42) + \frac{(.5)(.5-1)}{2}(9) + \frac{(.5)(.5-1)(.5-2)}{6}(-25) \\ &+ \frac{(.5)(.5-1)(.5-2)(.5-3)}{24}(37) \\ &= 47.8672 \approx 48 \end{aligned}$$

Hence no. of students getting marks less than $45 = 48$

By given data, no. of students getting marks less than $40 = 31$

Hence no. of students getting marks between 40 and $45 = 48 - 31 = 17$

Problem 4.5.5. *Find the cubic polynomial which takes the following values:*

$$\begin{array}{lcccc} x: & 0 & 1 & 2 & 3 \\ f(x): & 1 & 2 & 1 & 10 \end{array}$$

Here, $h = 1$. Hence using the formula, $x_p = x_0 + ph$, and choosing $x_0 = 0$, we get $p = \frac{x_p - x_0}{h} = \frac{x-0}{1} = x$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		−2	
		−1		12
2	1		10	
		9		
3	10			

$\therefore$ By Newton's forward difference formula,

$$\begin{aligned} y &= y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 \\ &= 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12) \\ &= 2x^3 - 7x^2 + 6x + 1 \end{aligned}$$

Problem 4.5.6. *Using newtons backward interpolation formula find the interpolating polynomial from the following table and hence find* $f(12.5)$.

x	*10*	*11*	*12*	*13*
y	*22*	*24*	*28*	*34*

Solution::The Backward difference table is :

x	$f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
10	22			
		2		
11	24		2	
		4		0
12	28		2	
		6		
13	34			

$P = \frac{x-x_n}{h} = \frac{x-13}{1} = x - 13$

By Newton's backward interpolation formula,

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$y = 34 + (x-13)6 + \frac{(x-13)(x-13+1)2}{2}$$

$$y = 34 + 6x - 78 + (x-13)(x-12)$$

$$y = 34 + 6x - 78 + x^2 - 25x + 156$$

$$y = x^2 - 19x + 112$$

is the interpolating polynomial for the data. Now

$$f(12.5) = (12.5)^2 - 19(12.5) + 112$$

$$= 268.25 - 237.5$$

$$\therefore f(12.5) = 30.75$$

4.6 Lagrange's interpolation formula and inverse interpolation formula

Let $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of x :

$x_0, x_1, x_2, \ldots, x_n$ at unequal intervals then

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}y_0$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2+\ldots$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)\ldots(x_3-x_n)}y_3+\ldots$$
$$+\vdots$$
$$+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n$$

Problem 4.6.1. *Use Lagrange's interpolation formula to find y at $x = 10$ given[VTU-Jan17]*

x	5	6	9	11
y	12	13	14	16

Solution:

$$\text{Let } x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$$

$$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$$

Lagrange's interpolation formula :

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

When $x = 10$

$$y = \frac{4(1)(-1)12}{(-1)(-4)(-6)} + \frac{5(1)(-1)13}{(1)(-3)(-5)} + \frac{5(4)(-1)14}{(4)(3)(-2)} + \frac{5(4)(1)16}{(6)(5)(2)} = 14.666$$

Thus y at $x = 10$ is 14.67

Problem 4.6.2. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1492	2366	5202

Evaluate $f(9)$, *using Lagrange's formula.*

Solution:

$$x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$$

$$y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202$$

Lagrange's interpolation formula :

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

Putting $x = 9$ and substituting the given values in Lagrange's formula, we get

$$\begin{aligned} f(9) =& \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\ &+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\ &+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\ &+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\ &+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 \\ &= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810 \end{aligned}$$

Problem 4.6.3. *Find the polynomial f(x) by using Lagrange's formula and hence find $f(3)$ for*

x :	0	1	2	5
$f(x)$:	2	3	12	147

Solution:

$$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 5$$

$$y_0 = 2,\ y_1 = 3,\ y_2 = 12,\ y_3 = 147$$

Lagrange's interpolation formula :

$$\begin{aligned} y(x) =& \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 \\ &+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 \\ &+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 \end{aligned}$$

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$
$$f(x) = x^3 + x^2 - x + 2 \text{ (on simplification)}$$
$$f(3) = 27 + 9 - 3 + 2 = 35$$

Problem 4.6.4. *Using Lagrange formula, calculate $f(3)$ from the following table.*

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solution:: Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$
$y_0 = f(x_0) = 1$, $y_1 = f(x_1) = 14$, $y_2 = f(x_2) = 15$, $y_3 = f(x_3) = 5$,
$y_4 = f(x_4) = 6$, $y_5 = f(x_5) = 19$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5$$

Here $x = 3$ then

$$f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1$$
$$+\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14$$
$$+\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15$$
$$+\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5$$
$$+\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6$$
$$+\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$
$$= \frac{12}{240} - \frac{18}{60} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{40} \times 19$$
$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95 = 10$$
$$f(3) = 10$$

4.7 Divided Differences :

Let $\mathrm{f}(\mathrm{x}_0), \mathrm{f}(\mathrm{x}_1), \mathrm{f}(\mathrm{x}_2), \ldots, \mathrm{f}(\mathrm{x}_n)$ be the values of an unknown function $\mathrm{y} = \mathrm{f(x)}$ corresponding to the values of $\mathrm{x} : \mathrm{x_0}, \mathrm{x_1}, \mathrm{x_2}, \ldots, \mathrm{x_n}$ at unequal intervals.

The first order divided differences are defined as follows

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0};$$
$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1};$$
$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}; \ldots$$

In general $f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

The second order divided differences are defined as follows

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$
$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$
$$\vdots$$

In general $\mathrm{f}(\mathrm{x_{n-2}}, \mathrm{x_{n-1}}, \mathrm{x_n}) = \dfrac{\mathrm{f}(\mathrm{x_{n-1}}, \mathrm{x_n}) - \mathrm{f}(\mathrm{x_{n-2}}, \mathrm{x_{n-1}})}{\mathrm{x_n} - \mathrm{x_{n-2}}}$

Similarly the other higher order divided differences are defined.

4.8 Newton's divided difference formula :

Newton's divided difference formula :

$$\begin{aligned} y(x) = f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) \\ &+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \cdots \\ &\vdots \\ &+ (x - x_0)(x - x_1) \ldots (x - x_{n-1}) f(x_0, x_1, x_2, \ldots, x_n) \end{aligned}$$

Problem 4.8.1. *Use Newton's divided difference formula to find* f(4) *given the data*

x	0	2	3	6
f(x)	-4	2	14	158

Solution: Here,

$$x_0 = 0, \ \ x_1 = 2, \ \ x_2 = 3, \ \ x_3 = 6$$

$$f(x_0) = -4, \ \ f(x_1) = 2, \ \ f(x_2) = 14, \ \ f(x_3) = 158$$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2 + 4}{2 - 0} = 3$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{14 - 2}{3 - 2} = 12$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{158 - 14}{6 - 3} = 48$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{12 - 3}{3 - 0} = 3$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{48 - 12}{6 - 2} = 9$$

Third order Divided Differences are give by,

$$\begin{aligned} f(x_0, x_1, x_2, x_3) &= \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \\ &= \frac{9 - 3}{6 - 0} = 1 \end{aligned}$$

Newton's divided difference formula is given by:

$$y(x) = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$y(4) = -4 + (4-0)3 + (4-0)(4-2)3 + (4-0)(4-2)(4-3)1 = 40$$

Problem 4.8.2. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Calculate $f(9)$, using Newton's divided difference formula.

Solution: : Here $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$f(x_0) = 150$, $f(x_1) = 392$, $f(x_2) = 1452$, $f(x_3) = 2366$, $f(x_4) = 5202$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{392 - 150}{7 - 5} = 121$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1452 - 392}{11 - 7} = 265$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{2366 - 1452}{13 - 11} = 457$$

$$f(x_3, x_4) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{5202 - 2366}{17 - 13} = 709$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{265 - 121}{11 - 5} = 24$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{457 - 265}{13 - 7} = 32$$

$$f(x_2, x_3, x_4) = \frac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2} = \frac{709 - 457}{17 - 11} = 42$$

Third order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3) = \frac{32 - 24}{13 - 5} = 1$$

$$f(x_1, x_2, x_3, x_4) = \frac{42 - 32}{17 - 7} = 1$$

Newton's divided difference formula is given by:

$$y(x) = f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$
$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4)$$

Taking $x = 9$ in the above divided difference formula, we obtain

$$f(9) = 150 + (9-5)\times 121 + (9-5)(9-7)\times 24$$
$$+ (9-5)(9-7)(9-11)\times 1 = 150 + 484 + 192 - 16 = 810$$

Problem 4.8.3. *Determine* $f(x)$ *as a polynomial in* x *for the following data:*

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

Solution: Here

$$x_0 = -4, \quad f(x_0) = 1245$$
$$x_1 = -1, \quad f(x_1) = 33$$
$$x_2 = 0, \quad f(x_2) = 5$$
$$x_3 = 2, \quad f(x_3) = 9$$
$$x_4 = 5, \quad f(x_4) = 1335$$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{33 - 1245}{-1 + 4} = -404$$
$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{5 - 33}{0 + 1} = -28$$
$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{9 - 5}{2 - 0} = 2$$
$$f(x_3, x_4) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{1335 - 9}{5 - 2} = 442$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{-28 - (-404)}{0 - (-4)} = 94$$
$$f(x_1, x_2, x_3) = \frac{2 - (-28)}{2 - (-1)} = 10$$
$$f(x_2, x_3, x_4) = \frac{442 - 2}{5 - 0} = 88$$

Third order Divided Differences are give by,

$$f\left(x_0, x_1, x_2, x_3\right) = \frac{10 - 94}{2 - (-4)} = -14$$
$$f\left(x_1, x_2, x_3, x_4\right) = \frac{88 - 10}{5 - 0} = 13$$

4th order Divided Difference is given by,

$$f\left(x_0, x_1, x_2, x_3, x_4\right) = \frac{13 + 14}{5 - (-4)} = 3$$

Applying Newton's divided difference formula,

$$\begin{aligned} y(x) = f(x) &= f\left(x_0\right) + \left(x - x_0\right) f\left(x_0, x_1\right) \\ &+ \left(x - x_0\right)\left(x - x_1\right) f\left(x_0, x_1, x_2\right) \\ &+ \left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right) f\left(x_0, x_1, x_2, x_3\right) \\ &+ \left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right)\left(x - x_3\right) f\left(x_0, x_1, x_2, x_3, x_4\right) \end{aligned}$$

$$\begin{aligned} f(x) =& 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + \\ &+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3) \\ &= 1245 - 404x - 1616 + (94)\left[x^2 + 5x + 4\right] \\ &- 14x\left[x^2 + 5x + 4\right] + 3x\left[\left(x^2 + 5x + 4\right)(x - 2)\right] \end{aligned}$$

$$\begin{aligned} f(x) &= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 \\ &- 70x^2 - 56x + 3x\left[x^3 - 2x^2 + 5x^2 - 10x + 4x - 8\right] \\ f(x) &= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 \\ &- 70x^2 - 56x + 3x^4 - 6x^3 + 15x^3 - 30x^2 + 12x^2 - 24x \\ &= 3x^4 - 5x^3 + 6x^2 - 14x + 5 \end{aligned}$$

Problem 4.8.4. *By means of Newton's divided difference formula, find f(8) from the following data-*

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: : $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_4 = 10$, $x_5 = 11$, $x_6 = 13$

$f(x_0) = 48$, $f(x_1) = 100$, $f(x_2 = 294$, $f(x_3) = 900$, $f(x_4) = 1210$, $f(x_5) = 2028$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{100 - 48}{5 - 4} = 52$$
$$f(x_1, x_2) = \frac{294 - 100}{7 - 5} = 97$$
$$f(x_2, x_3) = \frac{900 - 294}{10 - 7} = 202$$
$$f(x_3, x_4) = \frac{1210 - 900}{11 - 10} = 310$$
$$f(x_4, x_5) = \frac{2028 - 1210}{13 - 11} = 409$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{97 - 52}{7 - 4} = 15$$
$$f(x_1, x_2, x_3) = \frac{202 - 97}{10 - 5} = 21$$
$$f(x_2, x_3, x_4) = \frac{310 - 202}{11 - 7} = 27$$
$$f(x_3, x_4, x_5) = \frac{409 - 310}{13 - 10} = 33$$

Third order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3) = \frac{21 - 15}{10 - 4} = 1$$
$$f(x_1, x_2, x_3, x_4) = \frac{27 - 21}{11 - 5} = 1$$
$$f(x_2, x_3, x_4, x_5) = \frac{33 - 27}{13 - 7} = 1$$

4th order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3, x_4) = \frac{1 - 1}{11 - 4} = 0$$
$$f(x_1, x_2, x_3, x_4, x_5) = \frac{1 - 1}{13 - 5} = 0$$

5th order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

Applying Newton's divided difference formula,

$$\begin{aligned} y(x) = f(x) = f\left(x_0\right) + \left(x - x_0\right) f\left(x_0, x_1\right) \\ + \left(x - x_0\right)\left(x - x_1\right) f\left(x_0, x_1, x_2\right) \\ + \left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right) f\left(x_0, x_1, x_2, x_3\right) \\ + \left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right)\left(x - x_3\right) f\left(x_0, x_1, x_2, x_3, x_4\right) \end{aligned}$$

Substituting the $x = 8$ and divided differences in the above equation,

$$\begin{aligned} f(8) =& 48 + (8-4)(52) + (8-4)(8-5)(15) \\ & + (8-4)(8-5)(8-7)(1) \\ & + (8-4)(8-5)(8-7)(8-10)(0) \\ f(8) =& 48 + (4)(52) + (4)(3)(15) + (4)(3)(1)(1) + 0 \\ f(8) =& 448 \end{aligned}$$

4.9 Numerical Solution of algebraic and transcendental equations:

- Algebraic equation is an equation in the form of a polynomial having a finite number of terms.
  Example: $x^3 - 4x - 9 = 0, x^4 + x^3 = 80$

- A transcendental equation is an equation containing a transcendental function of the variable(s) being solved for.
  Example: $xe^x - 2 = 0,\ tanx = 2x,\ xlogx - 1.2 = 0$

- Numerical method of finding approximate roots of the given function is a repetitive type of process known as **iteration process**.

4.10 Newton Raphson Method

Let the given equation be $f(x) = 0$

We first find an interval (a,b) such that f(a) and f(b) are of opposite signs. Then we select a real number in (a,b) as the initial approximation to the required root.

Use the Newton Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

repeatedly, until we obtain the answer with the desired accuracy.

Problem 4.10.1. *Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.*

Solution:: Let $f(x) = x^4 - x - 10$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$f(1) = -10$ (Here value is -ve)

$f(2) = 16 - 2 - 10 = 4$ (Here value is +ve)

$\therefore$ a root of $f(x) = 0$ lies between $a = 1$ and $b = 2.$

Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton's Raphson formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (*)$$

Put $n = 0,$ in (*), the first approximation x_1 is given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{2^4 - 4 - 10}{4 \times 2^3 - 1}$$

$$= 2 - \frac{4}{31} = 1.871$$

Put $n = 1,$ in (*), the second approximation x_2 is given by,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1.871 - \frac{f(1.871)}{f'(1.871)}$$
$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1}$$
$$= 1.871 - \frac{0.3835}{25.199} = 1.856$$

Put $n = 2,$ in (*), the third approximation x_3 is given by,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$
$$= 1.856 - \frac{0.010}{24.574} - 1.856$$

Since $x_2 = x_3 = 1.856$, the required root is $x = 1.856$

Problem 4.10.2. *Using Newton-Raphson method, find the real root of the equation* $x^3 - 2x - 5 = 0$ *correct to 5 decimal places.*

Solution::

$f(x) = x^3 - 2x - 5, \qquad f'(x) = 3x^2 - 2$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$f(1) = -6$ (Here value is -ve)

$f(2) = -1$ (Here value is -ve)

$f(3) = 16$ (Here value is +ve)

Hence root lies between $(a, b) = (2, 3)$.

Let us take initial point $x_0 = 2.5$

Newton's Raphson formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} \qquad (*)$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$x_1 = x_0 - \frac{x_0^3 - 2x_0 - 5}{3x_0^2 - 2}$$
$$= 2.5 - \frac{2.5^3 - 2(2.5) - 5}{3(2.5)^2 - 2}$$
$$= 2.16418$$

Put $n = 1$, in (*), the first approximation x_2 is given by,

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$
$$= 2.16418 - \frac{(2.16418)^3 - 2(2.16418) - 5}{3(2.16418)^2 - 2}$$
$$= 2.09714$$

Similarly, with $n = 2, 3, 4$ in (*) we get

$$x_2 = 2.09714$$
$$x_3 = 2.09455$$
$$x_4 = 2.09455$$

Since $x_3 = x_4 = 2.09455$, the required root is $x = 2.09455$

Problem 4.10.3. *Using Newton-Raphson method, find the real root of the equation* $3x = \cos x + 1$ *correct to four decimal places.*

Solution: Let $f(x) = 3x - \cos x - 1$ (1)

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

Note : Make sure your calculator is in **radian mode**

$f(0) = 3(0) - cos(0) - 1 = -2,$ (Here value is -ve)

$f(1) = 3(1) - cos(1) - 1 = 1.4597,$ (Here value is +ve)

Hence interval (a,b)= (0,1)

$\therefore$ a root of $f(x) = 0$ lies between 0 and $1.$

Let us take $x_0 = 0.6$

Also differentiating (1) we get,

$$f'(x) = 3 + \sin x$$

Newton's Raphson formula gives,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \\ &= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad (*) \end{aligned}$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$\begin{aligned} x_1 &= \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} \\ &= \frac{0.6 \sin 6 + \cos 0.6 + 1}{3 + \sin 0.6} = .6071 \end{aligned}$$

Put $n = 1$ in (*), then second approximation is

$$\begin{aligned} x_2 &= \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} \\ &= \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} \\ &= 0.6071 \end{aligned}$$

Clearly $x_1 = x_2$.

Hence, desired root is 0.6071 correct to 4 decimal places.

Problem 4.10.4. *Using Newton Raphson method, find a real root of* $x \sin x + \cos x$ *near* $x = \pi$ *correct to 3 decimal places.*

Solution: Given $x_0 = \pi = 3.1416$

$$\begin{aligned} f(x) &= x \sin x + \cos x \\ f'(x) &= x \cos x + \sin x - \sin x \\ &= x cos x \end{aligned}$$

The iteration formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n} \qquad (*)$$

Note : Make sure your calculator is in **radian mode**

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$x_1 = x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0}$$

$$= \pi - \frac{\pi \sin \pi + \cos \pi}{\pi \cos \pi} \qquad (\because x_0 = \pi)$$

$$= 2.8233$$

Put $n = 1$, in (*),

$$x_2 = x_1 - \frac{x_1 \sin x_1 + \cos x_1}{x_1 \cos x_1}$$

$$= 2.8233 - \frac{2.8233 \sin(2.8233) + \cos(2.8233)}{2.8233 \cos(2.8233)} \qquad = 2.7986$$

similarly Putting $n = 2$, in (*), and calculating we get

$$x_3 = 2.7984$$

since x_2 and x_3 are same upto three decimal places, we stop the procedure and required root is $x = 2.798$

Problem 4.10.5. *Using Newton Raphson method, find a real root of* $x \log_{10} x - 1.2$ *correct to four decimal places.*

Solution: Here initial appoximation is not given.

$$f(x) = x \log_{10} x - 1.2 \qquad (1)$$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$f(1) = -1.2$ (Here value is -ve)

$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979$ (Here value is -ve)

$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136$ (Here value is +ve)

Hence a root of $f(x) = 0$ lies between a=2 and b=3

Let us take $x_0 = 2$ Also,

$$f'(x) = x\log_{10} x - 1.2$$
$$= \log_{10} x + x\frac{d}{dx}(\log_{10} x)$$
$$= \log_{10} x + x\frac{d}{dx}\left(\frac{log_e x}{log_e 10}\right)$$
$$= \log_{10} x + x\frac{1}{x}\ \log_{10} e$$
$$= \log_{10} x + 0.43429$$

Newton's formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n \log_{10} x_n - 12}{\log_{10} x_n + 0.43429}$$
$$= \frac{0.43429x_n + 1.2}{\log_{10} x_n + 0.43429} \qquad (*)$$

Put $n = 0$,, the first approximation is

$$x_1 = \frac{0.43429x_0 + 1.2}{\log_{10} x_0 + 0.43429}$$
$$= \frac{0.43429(2) + 1.2}{\log_{10} 2 + 0.43429}$$
$$= 2.8132$$

Similarly, Putting $n = 1$ in (*), we get

$$x_2 = \frac{0.43429x_1 + 1.2}{\log_{10} x_1 + 0.43429}$$
$$= \frac{0.43429(2.8132) + 1.2}{\log_{10}(2.8132) + 0.43429}$$
$$= 2.7411$$

Similarly, Putting $n = 2$ in (*), we get

$$x_3 = \frac{0.43429x_2 + 1.2}{\log_{10} x_2 + 0.43429}$$
$$= \frac{0.43429(2.7411) + 1.2}{\log_{10}(2.7411) + 0.43429}$$
$$= 2.7406$$

Similarly, Putting $n = 3$ in (*), we get

$$x_4 = \frac{0.43429x_3 + 1.2}{\log_{10} x_3 + 0.43429}$$
$$= \frac{0.43429(2.7406) + 1.2}{\log_{10}(2.7406) + 0.43429}$$
$$= 2.7406$$

Clearly, $x_3 = x_4$. Hence, the required root is 2.7406 correct to four decimal places.

4.11 Regula Falsi Method

- Let the given equation be $f(x) = 0$
- We first find an interval (a,b) such that f(a) and f(b) are of opposite signs.
- First approximation to the root is given by,
$$\boxed{x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}} \quad (*)$$
- Find $f(x_1)$. If $f(x_1) \neq 0$ then either $f(a)$ and $f(x_1)$ are of opposite signs or $f(x_1)$ and $f(b)$ are of opposite signs.
- If $f(a)$ and $f(x_1)$ are of opposite signs, then next approximation is given by replacing (a, b) by (a, x_1) in equation (*) on the other hand if $f(x_1)$ and $f(b)$ are of opposite signs, then next approximation is given by replacing (a, b) by (x_1, b) in equation (*).
- This process is repeated until the root is obtained with desired accuracy. At each step, the method produces a sequence of shrinking intervals which contains a root.
- Suppose $f(x_1)$ and $f(b)$ are of opposite signs, then
$$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)}$$
Same process is used to find x_3, and so on.

Problem 4.11.1. *Use the Regula-falsi method to find a real root of the equation,* $x^3 - 2x - 5 = 0$ *correct to 2 decimal places.*

Solution:: Let $f(x) = x^3 - 2x - 5$

$$f(0) = -5$$
$$f(1) = -6,$$
$$f(2) = 2^3 - 2(2) - 5 = -1 \quad \text{(negative)}$$
$$f(3) = 3^3 - 2(3) - 5 = 16 \quad \text{(positive)}$$

Since $f(2)$ and $f(3)$ are of opposite signs, a real root lies in (2, 3).

Let us take $a = 2$ and $b = 3$. The first approximation to root is x_1 and is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$
$$= \frac{(2(16) - 3(-1))}{(16 - (-1))}$$
$$= 2.058$$

Now $f(2.058) = (2.0583)^3 - 2(2.058) - 5 = -0.4$ (negative)

The root lies between 2.058 and 3

Taking $a = 2.058$ and $b = 3$. The second approximation to the root is given by

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{(2.058)\, f(3) - 3f(2.058)}{f(3) - f(2.058)}$$
$$= \frac{(2.058)16 - 3(-0.4)}{16 - (-0.4)}$$
$$= 2.081$$

Now $f(2.081) = (2.081)^3 - 2(2.081) - 5 = -0.15$ (negative)

The root lies between 2.081 and 3

Take $a = 2.081$ and $b = 3$ The third approximation to the root is given by

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{(2.081)\, f(3) - 3f(2.081)}{f(3) - f(2.081)}$$
$$= \frac{(2.081)(16) - 3(-0.15)}{16 - (-0.15)} = 2.089$$

Now $f(2.089) = (2.089)^3 - 2(2.089) - 5 = -0.0617 \quad \text{(negative)}$

The root lies between 2.089 and 3

Take $a = 2.089$ and $b = 3$ The 4th approximation to the root is given by

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{(2.089)\, f(3) - 3f(2.089)}{f(3) - f(2.089)}$$
$$= \frac{(2.089)(16) - 3(-0.0617)}{16 - (-0.0617)} = 2.092$$

x_3 and x_4 are almost equal.

Thus the required approximate root correct to 2 decimal places is 2.09

Problem 4.11.2. *Use the Regula-falsi method to find a real root of the equation* $\cos x = xe^x$, *which lies in* $(0, 1)$. *Carryout 3 iterations. Write the answer correct to 5 decimal places.*

Solution:: Let $f(x) = \cos x - xe^x = 0$

Given $(a, b) = (0, 1)$

So that $f(a) = f(0) = 1$, and $f(b) = f(1) = \cos 1 - e = -2.17798$

The first approximation to root (i.e. x_1) is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1}$$
$$= 0.31467$$

Now $f(x_1) = f(0.31467) = 0.51987$

i.e., the root lies between $(a, b) = (0.31467, 1)$.

$f(a) = f(0.31467) = 0.51987$ and $f(b) = f(1) = -2.17798$

The second approximation to the root (i.e. x_2) is given by

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.31467(-2.17798) - 1(0.51987)}{-2.17798 - 0.51987}$$
$$= 0.44673$$

Now $f(x_2) = f(0.44673) = 0.20356$

the root lies between $(a, b) = (0.44673, 1)$ $f(a) = f(0.44673) = 0.20356$ and $f(b) = f(1) = -2.17798$

The third approximation to the root (i.e. x_3) is given by

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.44673(-2.17798) - 1(0.20356)}{-2.17798 - 0.20356}$$
$$= 0.49402$$

After 3 iterations, the root of the equation $\cos x = xe^x$ is $x = 0.49402$ correct to 5 decimal places.

Problem 4.11.3. *Use the Regula-falsi method to find a real root of the equation,* $x\log_{10} x - 1.2 = 0$ *which lies in* $(2, 3)$. *[VTU: Dec 2010, July 2016]*

Solution:: Let $f(x) = x\log_{10} x - 1.2$ Here $(a, b) = (2, 3)$ $f(a) = f(2) = 2\log_{10} 2 - 1.2 = -0.59794 \quad (-ve)$

$f(b) = f(3) = 3\log_{10} 3 - 1.2 = 0.23136 \quad (+ve)$

By method of false position, we have

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$
$$= \frac{2(0.23136) - 3(-0.59794)}{(0.23136) - (-0.59794)} = 2.72102$$

Now, $f(2.72102) = (2.72102)\log_{10} 2.72102 - 1.2 = -0.01709 \ (-ve)$

since, $f(2.72102)$ and $f(3)$ are of opposite sign, so the root lies between $a = 2.72102$ and $b = 3$

$f(a) = f(2.72102) = -0.01709$ and $f(b) = f(3) = 0.23136$

$$\begin{aligned} x_2 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{2.72102f(3) - 3f(2.72102)}{f(3) - f(2.72102)} \\ &= \frac{2.72102(0.23136) - 3(-0.01709)}{0.23136 - (-0.01709)} \\ &= 2.74021 \end{aligned}$$

Now, $f(2.74021) = (2.74021)\log_{10} 2.74021 - 1.2 = -0.00038 \quad (-ve)$

since $f(2.74021)$ and $f(3)$ are of opposite sign, so the root lies between $a = 2.74021$ and $b = 3$

$f(a) = f(2.74021) = -0.00038$ and $f(b) = f(3) = 0.23136$

$$\begin{aligned} x_3 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{2.74021f(3) - 3f(2.74021)}{f(3) - f(2.74021)} \\ &= \frac{2.74021(0.23136) - 3(-0.00038)}{(0.23136) - (-0.00038)} \\ &= 2.74064 \end{aligned}$$

Now, $f(2.74064) = (2.74064)\log_{10} 2.74064 - 1.2 = -0.000005 \quad (-ve)$

since $f(2.74064) \approx 0$, we conclude that $x = 2.7406$ is a root correct to 4 decimal places.

Problem 4.11.4. *Using Regula Falsi Method, compute the real root of the equation $xe^x - 2 = 0$ correct up to three decimals places.* *[VTU Jan 2017]*

Solution:: Here, $f(x) = xe^x - 2$

$f(0) = -2(--\text{ve})$

$f(1) = 1 \times e^1 - 2 = 0.718 \ \ (+\text{ve})$

Try to obtain a smaller interval.

Put $x = 0.5$, then $f(0.5) = 0.5 \times e^5 - 2 = -1.17(-\text{ve})$

As the value of $f(0.5)$ is -ve and $f(1)$ is positive, root lies in $(a, b) = (0.5, 1)$ and $f(a) = f(0.5) = -1.17$ and $f(b) = f(1) = 0.718$ By method of false position, we have

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.5(0.718) - 1(-1.17)}{0.718 - (-1.17)}$$
$$= 0.8098$$

Now, $f\ (x_1) = f(0.8098) = 0.8098e^{0.8098} - 2 = -0.18$ (negative)

Root lies between $a = 0.8098$ and $b = 1$

$f(a) = f(0.8098) = -0.18$ and $f(b) = f(1) = 0.718$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8098(0.718) - 1(-0.18)}{0.718 - (-0.18)}$$
$$= 0.8479$$

Now, $f\ (x_2) = f(0.8479) = 0.8479 - e^{0.8479} - 2 = -0.02$ (negative)

Root lies between $a = 0.8479$ and $b = 1$

$f(a) = f(0.8479) = -0.02$ and $f(b) = f(1) = 0.718$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8479(0.718) - 1(-0.02)}{0.718 - (-0.02)}$$
$$= 0.8519$$

Now, $f\ (x_3) = 0.8519e^{0.8519} - 2 = -0.0031$ (negative)

Root lies between $a = 0.8519$ and $b = 1$

$f(a) = f(0.8519) = -0.0031$ and $f(b) = f(1) = 0.718$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8519(0.718) - 1(-0.0031)}{0.718 - (-0.0031)}$$
$$= 0.8525$$

Now, $f(x_4) = f(0.8525) = 0.8525 - e^{0.8525} - 2 = -0.0005$ (negative)

Root lies between $a = 0.8525$ and $b = 1$

$f(a) = f(0.8479) = -0.0005$ and $f(b) = f(1) = 0.718$

$$x_5 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8525(0.718) - 1(-0.0005)}{0.718 - (-0.0005)}$$
$$= 0.8525$$

Now, $x_4 = x_5 = 0.8525$. Hence approximate root correct to 4 decimal of place is , $x = 0.8525$

4.12 Numerical Integration :

Let $\mathbf{I} = \int_a^b y\,dx$, where y takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots, x_n$

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \frac{b-a}{n}$ so that $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$

$$\therefore \quad \mathbf{I} = \int_{x_0}^{x_0+nh} f(x)dx$$

Numerical Integration is the process of obtaining approximately the value of the definite integral $\mathrm{I} = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{ydx}$ without actually integrating the function but only using the values of y at some points of x equally spaced over $[\mathrm{a}, \mathrm{b}] = [x_0, x_0 + nh]$.

4.13 Three rules to obtain the value of the definite integral:

- **Trapezoidal rule :**
$$\mathrm{I} = \frac{\mathrm{h}}{2}\left[(\mathrm{y}_0 + \mathrm{y}_n) + 2\left(\mathrm{y}_1 + \mathrm{y}_2 + \mathrm{y}_3 + \ldots + \mathrm{y}_{\mathrm{n}-1}\right)\right]$$

- **Simpson's $(1/3)^{\text{th}}$ rule:**
$$\mathrm{I} = \frac{\mathrm{h}}{3}\left[(\mathrm{y}_0 + \mathrm{y}_n) + 4\left(\mathrm{y}_1 + \mathrm{y}_3 + \ldots + \mathrm{y}_{\mathrm{n}-1}\right) + 2\left(\mathrm{y}_2 + \mathrm{y}_4 + \ldots + \mathrm{y}_{\mathrm{n}-2}\right)\right]$$
While using this formula, the given interval of integration must be divided into an even number of sub-intervals(i.e. n=even)

- **Simpson's $(3/8)^{\text{th}}$ rule:**
$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3\left(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}\right) + 2\left(y_3 + y_6 + \ldots + y_{n-3}\right)\right]$$
While using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

4.14 Working procedure for problems:

Step 1: Given the definite integral $\mathrm{I} = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{y}\mathrm{dx}$ for evaluation, first divide the interval $[a, b]$ into n equal parts (strips) of width $h = \frac{(b-a)}{n}$

Step 2: Prepare a table consisting the values of x and the corresponding computed values of y

Step 3: Substitute values from this table into the appropriate rule to obtain the approximate value of the given definite integral.

Note: Number of ordinates = $n + 1$ where n is the number of sub intervals.

Problem 4.14.1. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using*

(i) Simpson's $1/3$ *rule,*

(ii) Simpson's $3/8$ *rule,*

Solution:: Let $n = 6$. Hence $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$. Thus Divide the interval $(0, 6)$ into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.05884	0.0385	0.027
$= y$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Simpson's $1/3$ rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{3}\left[(y_0 + y_6) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)]$$

$$= 1.3662$$

(ii) By Simpson's $3/8$ rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{8}\left[(y_0 + y_6) + 3\left(y_1 + y_2 + y_4 + y_5\right) + 2y_3\right]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)]$$

$$= 1.3571$$

Actual value : Let us evaluate the integral explicitly.

$$\int_0^6 \frac{1}{1+x^2} = \left[tan^{-1}x\right]_0^6$$

$$= tan^{-1}(6) - tan^{-1}(0) = 1.4056$$

(Note : Keep the Calculator in Radian Mode)

Problem 4.14.2. *Use Simpson's $1/3$rd rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates.* *(V.T.U., 2011)*

Solution:: Divide the interval $(0, 0.6)$ into $n = 6$ parts each of width $h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$.

The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
$y = e^{-x^2}$	1 y_0	0.9900 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's $1/3$ rd rule, we have

$$\int_0^{0.6} e^{-x^2} dx = \frac{h}{3}\left[(y_0 + y_6) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$
$$= \frac{0.1}{3}[(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788 + 2(0.9608 + 0.8521)]$$
$$= \frac{0.1}{3}[1.6977 + 10.7308 + 3.6258]$$
$$= \frac{0.1}{3}(16.0543) = 0.5351.$$

Problem 4.14.3. *Compute the value of* $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ *using Simpson's* $\frac{3}{8}$*th rule.*

Solution: Let $y = \sin x - \log_e x + e^x$, $n = 6$. Then $h = \frac{1.4-0.2}{6} = 0.2$

(Note : Keep the Calculator in radian mode)

The values of y are as given below :

x :	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y :	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ th rule, we have

$$\int_{0.2}^{1.4} y dx = \frac{3h}{8}\left[(y_0 + y_6) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$
$$= \frac{3}{8}(0.2)[7.7336 + 2(3.1660) + 3(13.3247)] = 4.053$$

Hence $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx = 4.053$.

Problem 4.14.4. *Calculate the value of the integral*

$$\int_4^{5.2} \log x dx \text{ by}$$

(a) Simpson's $1/3$ *rule (b) Simpson's* $3/8$ *rule Compare the answers with exact solution.*

Solution:: Here $x_0 = 4, x_n = 5.2$

Let $n = 6$. Then $h = \frac{x_n - x_0}{n} = 0.2$.

Taking $h = 0.2$, divide the range of integration $(4, 5.2)$ into six equal parts.The values of $\log x$ for each point of sub-division are given below:

X	$f(x) = \log x = ln(x)$
$x_0 = 4$	$f(0) = 1.3862944$
$x_0 + h = 4.2$	$f(1) = 1.4350845$
$x_0 + 2h = 4.4$	$f(2) = 1.4816045$
$x_0 + 3h = 4.6$	$f(3) = 1.5260563$
$x_0 + 4h = 4.8$	$f(4) = 1.5686159$
$x_0 + 5h = 5.0$	$f(5) = 1.6094379$
$x_0 + 6h = 5.2$	$f(6) = 1.6486586$

(a) By Simpson's $1/3$ rule, we have

$$\int_4^{52} \log x dx = \frac{h}{3}[f(0) + f(6) + 4\{f(1) + f(3) + f(5)\}$$
$$+2\{f(2) + f(4)\}]$$
$$= \frac{0.2}{3}[3.034953 + 4(4.5705787) + 2(3.0502204)]$$
$$= \frac{0.2}{3}[3.034953 + 18.282315 + 6.1004408]$$
$$= \frac{0.2}{3} \times 27.417709 = 1.8278472.$$

(b) By Simpson's $3/8$ rule, we have

$$\int_4^{5.2} \log x dx = \frac{3h}{8}[f(0) + f(6) + 3\{f(1) + f(2) + f(4) + f(5)\} + 2f(3)]$$
$$= \frac{3(0.2)}{8}[3.034953 + 3(6.0947428) + 2(1.5260563)]$$
$$= \frac{0.6}{8}[3.034953 + 18.284228 + 3.0521126]$$
$$= \frac{0.6}{8} \times 24.371294 = 1.827847$$

Exact value of the integral is

$$\int_4^{52} \log x dx = [x(\log x - 1)]_4^{\pi.2}$$
$$= [5.2(\log 5.2 - 1) - 4(\log 4 - 1)] = 3.3730249 - 1.5451774 = 1.8278475$$

Problem 4.14.5. *Evaluate* $\int_0^{\pi/2} \sin x dx$ *by Simpson's rule by taking* $h = \pi/20$. *Compare with exact value.*

Solution: :Here $x_0 = 0$, $x_n = \frac{\pi}{2}$

Given that $h = \frac{\pi}{20}$. $\therefore n = \frac{x_n - x_0}{h} = 10$

x	0	$\pi/20$	$2\pi/20$	$3\pi/20$	$4\pi/20$	$5\pi/20$	$6\pi/20$	$7\pi/20$	$8\pi/20$	$9\pi/20$
x	0	0.1571	0.3142	0.4712	0.6283	0.7854	0.9425	1.0996	1.2566	1.4137
y	0	0.1565	0.3091	0.4540	0.5878	0.7071	0.8090	0.8910	0.9510	0.9877

By simpson's 1/3 rule,

$$\int_{x_0}^{x_{10}} y dx = \frac{h}{3}\left[(y_0 + y_{10}) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right)\right.$$

$$\left.+2\left(y_2 + y_4 + y_6 + y_8\right)\right]$$

$$=\frac{0.1571}{3}[(0+1) + 4(0.1565 + 0.4540 + 0.7071 + 0.8910 + 0.9877)$$

$$+ 2(0.3091 + 0.5878 + 0.8090 + 0.9510)]$$

$$=\frac{0.1571}{3} \times 19.0990 = 1.000150967 = 1.00015$$

Exact Value is

$\int_0^{\pi/2} \sin x dx = (-\cos x)_0^{-12} = 0 - (-1) = 1$

Thus, Simpson's rule gives a better approximation.

4.15 Question Bank : Module 4 - Numerical Methods I

4.15.1 Question Bank :Regula Falsi and Newton Raphson Methods

1. Using Newton-Raphson method, find the real root of the equation $3x = \cos x + 1$ correct to four decimal places. (VTU Model 2022)

2. Find a real root of $x^3 - 9x + 1 = 0$ in $(2, 3)$ by the Regula-Falsi method in four iterations (VTU Model 2022)

3. Use Regula Falsi method to find a real root of the equation $x^3 - 2x - 5 = 0$, correct to 3 decimal places. [VTU - June 2018]

4. Find a real root of $xe^x - cosx = 0$ correct to three decimal places lying in the interval $(0.5, 0.6)$, using Regula-Falsi method. [VTU-Model 2018]

5. Use the Regula-falsi method to find a real root of the equation $\cos x = xe^x$, which lies in $(0, 1)$. Write the answer correct to 4 decimal places. (VTU Model 2022)

6. Use the Regula-falsi method to find a real root of $xlog_{10}x - 1.2 = 0$ [VTU:Jan 2020, June 2019, Model 2018, Dec 2010/July16]

7. Use the Regula-falsi method to find a real root of $cosx = 3x - 1$ [VTU: Dec 2012/July 13/July 15] Ans: 0.607

8. Use the Regula-falsi method to find a real root of $xe^x - 2 = 0$ [VTU- Jan17]

9. Use the Regula-falsi method to find a real root of $xe^x - 3 = 0$, correct to three decimal places. [VTU Jan 2018]

10. Use the regula falsi method to find the forth root of 12 correct to 3 decimal places [VTU: Dec2015/Jan17]

11. Use the Regula Falsi method to find the root of the equation $2x - log_{10}x = 7$ which lies between 3.5 and 4 [VTU-June 12/June17] Ans: 3.7893

12. Find a real root of the equation $x^3 + x^2 + 3x + 4 = 0$ near $x = -1$, correct to four decimal places using Newton- Raphson method. [VTU – Model 2018]

13. Use Newton Raphson method to find a real root of $xsinx + cosx = 0$, near $x = \pi$. Carryout the iterations upto four decimal places of accuracy. [VTU – Model 2022, Jan 2020, June 2019, Model 2018, Dec13/Dec14/Jan15/June17]

14. Find the real root of the equation $xe^x - 2 = 0$ correct to three decimal places using Newton- Raphson method. [VTU- Jan17]

15. Using Newton-Raphson method, find the root that lies near $x = 4.5$ of the equation $tanx = x$ correct to four decimal places. [VTU- Jan 2018, Jan17]

16. Using Newton-Raphson method find the value of cube root of 18 correct to 2 decimals, assuming 2.5 as the initial approximation. [VTU – June17]

17. Find the real root of the equation $xlog_{10}x - 1.2 = 0$ using Newton-Raphson method [VTU – June 2018]

18. Compute one positive root of $2x - \log_{10} x = 7$ by Newton-Raphson method correct to four decimal places.
    Ans : $\mathbf{3.7892}$

19. Use Newton-Raphson method to find a root of the equation $x^3 - 3x - 5 = 0$
    Ans : $\mathbf{2.279}$

20. Find the negative root of the equation $x^3 - 4x + 9 = 0$ from Regula Falsi method. Ans: $\mathbf{-2.7065}$

21. Find the root of the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by the method of false position. Ans : $\mathbf{2.7473}$

22. Find the root of the equation $4\sin x = e^x$, that lies between 0 and 0.5. Correct to 4 places of decimals, using Regula-Falsi method. Ans :$\mathbf{0.3706}$

23. Find the root of the equation $xe^x - 3 = 0$, that lies between 1 and 2, correct to 3 places of decimals, using the method of false position. Ans : $\mathbf{1.049}$

24. Use the Newton-Raphson method to find a root of the equation $xe^x - 2 = 0$ correct to 3 decimal places. (VTU 2005) Ans :$\mathbf{0.853}$

25. Use the Newton-Raphson method to find a root of the equation $cosx = xe^x$ correct to 3 decimal places. Ans : $\mathbf{0.518}$

4.15.2 Question Bank :Newtons Forward and Backward Interpolation formulas

Using Newton's forward interpolation find y at $\text{x} = 5$ from the data

1.

x	4	6	8	10
y	1	3	8	16

(VTU Model 2022)

2. Using Newton's backward interpolation formula find the value of y when $x = 6$ from the given table

x	1	2	3	4	5
y	1	-1	1	-1	1

(VTU Model 2022)

3. Given f(40) = 184, f (50)=204, f (60) = 226, f (70) = 250, f (80) = 276, f (90) = 304, find $f(42)$ using Newton's forward interpolation formula. [VTU Jan 2020]

4. Given f(40) = 184, f (50)=204, f (60) = 226, f (70) = 250, f (80) = 276, f (90) = 304, find $f(42)$ and $f(85)$ using suitable interpolation formulae. [VTU: – Model 2018, June 2018, June12/Jan16]

5. Given $sin45° = 0.7071, sin50° = 0.7660, sin55° = 0.8192, sin60° = 0.8660$ Find $sin57°$ using an appropriate interpolation formula. [VTU: – June 2018]

6. From the following table of half-yearly Premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age	45	50	55	60	65
Premium (in Rupees)	114.84	96.16	83.32	74.48	68.48

[VTU Jan 2018]

7. From the following table find the number of students who have obtained (a) less than 45 marks (b) between 40 and 45 marks.

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

[VTU- June 2019, July15/Jan17]

8. From the following table find the number of students who have obtained less than 70 marks.

Marks	0-19	20-39	40-59	60-79	80-99
No. of students	41	62	65	50	17

[VTU – Jan 17]

9. Estimate the probable number of persons with daily income 20 to 25 Rs from the following table.

Income per day(Rs)	Under 10	10- 20	20- 30	30- 40	40- 50
No. of person	20	45	115	210	115

[VTU: Jan14]

10. Estimate the probable number of getting wages below Rs 35 from the following data :

Wages (in Rs)	0-10	10- 20	20- 30	30- 40
Frequency	9	30	35	42

[VTU Jan 2018]

11. The population of the town is given by the table. Using Newton's forward and backward interpolation formula, calculate the increase in the population from the year 1955 to 1985

Year	1951	1961	1971	1981	1991
Population in 1000	19.96	39.65	58.81	77.21	94.61

[VTU-Jan16/June17]

12. Using Newton's backward interpolation formula find the interpolating polynomial for the function given by the following data. Hence find f(12.5)

x	10	11	12	13
f(x)	22	24	28	34

[VTU - Jan17]

13. Using Newton's forward interpolation formula find y at x=160 for the following data.

x	100	150	200	250	300	350	400
f(x)	10.63	13.03	15.04	16.81	18.42	19.90	21.27

[VTU: June13]

14. Use an appropriate interpolation formula to compute using the following data:

x	1.7	1.8	1.9	2.0	2.1	2.2
f(x)	5.474	6.050	6.686	7.389	8.166	9.025

[VTU Model 2018]

15. Estimate the value of $f(22)$ from the following available data:

x	**20**	**25**	**30**	**35**	**40**	**45**
y	**354**	**332**	**291**	**260**	**231**	**204**

Ans : 352.22304

16. The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x :	100	150	200	250	300	350	400
y :	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Use Newton's forward formula to find y when $x = 218$ ft. Ans : 15.6993 nautical miles.

17. Find the number of men getting wages between Rs. 10 and Rs. 15 from following table

Wages in R s :	$0-10$	$10-20$	$20-30$	$30-40$
Frequency :	9	30	35	42

Ans : 15

18. Estimate the value of $f(42)$ from the following available data:

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

Ans :219

19. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105. Ans : 8666

20. Find a cubic polynomial in x which takes on the values $-3, 3, 11, 27, 57$ and 107, when x $= 0, 1, 2, 3, 4$ and 5 respectively. Ans: $y = x^3 - 2x^2 + 7x - 3$

21. Find the cubic polynomial which takes the following values.

x	0	1	2	3
y	1	2	1	10

Ans: $2x^3 - 7x^2 + 6x + 1$

4.15.3 Question Bank :Newton's Divided Difference interpolation Formula

1. Using Newton's divided difference formula find f(9) from the following data

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

[VTU Model 2022, Jan 2020]

2. Construct the interpolation polynomial for the data given below using Newton's divided difference formula.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

Hence find $f(9)$. [VTU- Model 2018, June 2018, June17] Ans: $f(x) = 2x^3 - 3x^2 + 5x - 4, f(3) = 38$ and $f(9) = 1256$

3. Find the equation of the polynomial which passes through the points $(4, -43)$ $(7, 83)(9, 327)$ and $(12, 1053)$ using Newton's divided difference formula [VTU- Jan17] Ans: $f(x) = x^3 - 4x^2 - 7x - 15$

4. Using Newton's divided difference formula find f(82), f(98) from the following data

x	80	85	90	95	100
f(x)	5026	5674	6362	7088	7854

[VTU: Dec12]

5. Given,

x	-4	-1	0	2	5
y	1245	33	5	9	1335

Determine f(x) as a polynomial in x for the following data using Newton's difference formula. [VTU- Jan17] Ans: $\mathbf{y = 3x^4 - 5x^3 + 6x^2 - 14x + 5}$

6. Using Newton's divided difference formula, find a polynomial for the data :

x	3	7	9	10
F(x)	168	120	72	63

Hence find y at $x = 8$ [VTU- Jan 2018]

7. Using Newton's divided difference formula, fit an interpolating polynomial for the data given below and hence find y at x = 2

x	0	1	4	8	10
f(x)	-5	-14	-125	-21	355

[VTU: June13]

8. Fit an interpolating polynomial for the data $\mathbf{u_{10} = 355, u_0 = -5, u_8 = -21, u_1 = -14\ u_4 = -125}$ by using Newton's general interpolation formula and hence find $\mathbf{u_2}$ Ans: $\mathbf{f(x) = 2x^3 - 17x^2 + 6x - 5, u_2 = -45}$

9. Using Newton's divided difference formula find $f(8), f(15)$ from the following data

X	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Ans: $f(8) = 448, f(15) = 3150$

4.15.4 Question Bank :Lagrange's interpolation formula

1. Using Lagrange's interpolation formula, fit a polynomial which passes through the points $(-1, 0), (1, 2), (2, 9)$ and $(3, 8)$ and hence estimate the value of y when $x = 2.2$ (VTU Model 2022)

2. Use Lagrange's interpolation formula to find y at x = 10 given

x	5	6	9	11
y	12	13	14	16

[VTU- Jan17]

3. Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
f(x)	-12	0	6	12

Hence estimate y at x = 2 [VTU: – June 2018, Jan16]

4. Using Lagrange's formula find the interpolating polynomial that approximates to the function described by the following table.

x	0	1	2	5
f (x)	2	3	12	147

[VTU: Jan 2018, July16]

5. Using Lagrange's formula find y at $x = 4$ by using the following data.

x	0	1	2	5
f (x)	2	3	12	147

[VTU: June 2019],

6. The following table gives the premium payable at ages in years completed. Interpolate the premium payable at age 35 completed, using Lagrange's formula.

Age completed	25	30	40	60
Premium in Rs	50	55	70	95

[VTU- Jan17]

7. Use Lagrange's interpolation formula to find $f(5)$. Given

x	0	1	2	3	4
f(x)	3	6	11	18	27

[VTU- July15]

8. Using Lagrange's interpolation formula to fit a polynomial for the following data:

x	2	10	17
y	1	3	4

[VTU-Model 2018]

9. Use Lagrange's interpolation formula to find $\mathbf{f}(4)$ given

x	0	2	3	6
$f(x)$	-4	2	14	158

Ans: $f(4) = 40$]

10. Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
f(x)	-12	0	6	12

Hence estimate y at $x = 2$ [VTU: Jan16] [Ans: $f(x) = x^3 - 7x^2 + 18x - 12; f(2) = 4$]

11. Using Lagrange's interpolation method, find the value of $f(x)$ at $x = 5$ given the values

x	1	3	4	6
f(x)	3	9	30	132

[Ans: $f(5) = 69.4$]

12. Using Lagrange's interpolation formula to fit a polynomial for the following data:

x	2	10	17
y	1	3	4

[VTU-Model 2018]

4.15.5 Question Bank :Numerical Integration

1. Evaluate $\int_0^5 \frac{1}{4x+5}dx$ by dividing the interval into 10 equal parts. [Ans: 1.61] (Model 2022)

2. Evaluate $\int_0^6 \frac{dx}{(1+x^2)}$ by using Simpson's (3/8)th rule, taking 7 ordinates. [VTU June 2019]

3. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ by using simpson's $(1/3)^{\text{th}}$ rule by dividing $\left[0, \frac{\pi}{2}\right]$ into 6 equal parts. (VTU Model 2022)

4. Use Simpson's (1/3)th rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking 6 sub intervals. [VTU-Jan 2018, Jan17]

5. Evaluate $\int_0^1 \frac{dx}{(1+x)}$ by using Simpson's (3/8)th rule taking seven ordinates and hence deduce the value of $log_e 2$ [VTU: Jan17]

6. Evaluate $\int_0^1 \frac{xdx}{(1+x^2)}$ by using Simpson's one third rule taking seven ordinates and hence find $log_e 2$ [VTU: Model 2018, July 2018, Dec13/Jan16/Jan17]

7. Evaluate $\int_4^{5.2} log_e x dx$ taking 6 equal strips by applying Simpson's (3/8)th rule. [VTU : Model 2022, Jan 2020, June 2019, June13, Jan17]

8. Use Simpson's (3/8)th rule to obtain the approximate value of $\int_{0.2}^{1.4} (sinx - logx + e^x) dx$ by considering 6 parts. [VTU: Jan 2018, Jan17]

9. Using Simpson's (3/8)th rule, Evaluate $\int_0^3 \frac{dx}{(1+x)^2}$ taking 4 equidistant ordinates. [VTU: Model 2018]

10. Evaluate $\int_0^1 \frac{dx}{(1+x^2)}$ by using Simpson's (1/3)th rule, Simpson's (3/8)th rule [VTU: Model 2022, Jan 2020, Dec12/Jan14]

11. Evaluate $\int_0^1 \frac{xdx}{(1+x^2)}$ by using Simpson's (3/8)th rule, dividing the interval into 3 equal parts, and hence find $log_e \sqrt{2}$ [VTU: July15]

12. Using simpson's 1/3 rule with 7 ordinates, evaluate $\int_2^8 \frac{1}{(log_{10} x)} dx$ [VTU: –June 2018]

13. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{sinx} dx$ using Simpson's $\frac{1}{3}$- rule by taking 10 equal parts. [VTU Model 2019]

14. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cosx dx$ using Simpson's (3/8)th rule by dividing the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ into 6 equal parts. [VTU Model 2019]

15. Evaluate $\int_0^1 \frac{\text{dx}}{1+\text{x}^2}$ by using Simpson's $(1/3)^{\text{th}}$ rule taking four equal strips and hence deduct an approximate value of Π. [Ans: 0.7854]

16. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,d\theta$ by using simpson's $(1/3)^{th}$ rule by dividing $\left[0, \frac{\pi}{2}\right]$ into 6 equal parts. Ans: $= 1.1873$

17. Use Simpson's $(1/3)^{th}$ rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking 6 sub intervals. [VTU-Jan17] [Ans: 0.5351]

1. Use Simpson's $(1/3)^{th}$ rule to find $\int_2^{8} \frac{1}{\log_{10} x}dx$ by taking 7 ordinates. [Ans: 9.7203]

2. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using (i) Simpson's $1/3$ rule (ii) Simpson's $(3/8)^{th}$ rule taking seven ordinates and hence deduce the value of $\log_e 2$ [VTU: Jan17] [Ans: $0.6932, 0.693$]

3. Evaluate $\int_0^1 \frac{xdx}{1+x^2}$ by using Weddle's rule taking seven ordinates and hence find $\log_e 2$ [Ans: 0.3466, $\log_e 2 = 0.6932$] [VTU: Dec13/Jan16/Jan17]

4. Evaluate $\int_4^{5.2} \log_e x dx$ taking 6 equal strips by applying using (i) Simpson's $(1/3)^{th}$ rule, (ii) Simpsons $3/8^{th}$ rule and (iii) weddles rule [Ans: $1.8278, 1.8278, 1.8279$] [VTU: June13/Jan17]

5. Use Simpson's $(3/8)^{th}$ rule to obtain the approximate value of $\int_0^{0.3} \left(1 - 8x^3\right)^{\frac{1}{2}}\ dx$ by considering 3 equal intervals. [Ans: 0.2916]

6. Use Simpson's $(3/8)^{th}$ rule to obtain the approximate value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x$ by considering 6 parts. [VTU: Jan17] [Ans: 4.053]

10. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $(1/3)^{th}$ rule. Simpson's $(3/8)^{th}$ rule, Weddle'srule. [VTU: Dec12/Jan14] [Ans: 1.3662, 1.3571, 1.3735]

7. Evaluate $\int_0^1 \frac{xdx}{1+x^2}$ by using Simpson's $(3/8)^{th}$ rule, dividing the interval into 3 equal parts, and hence find $\log_e \sqrt{2}$ [VTU: July15]

8. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Simpson's $(1/3)^{th}$ rule, (ii) Simpsons $3/8^{th}$ rule and (iii) weddles rule . Compare the results with actual value. [VTU 2013] Ans: (i) 1.3662, (ii) 1.3571 (iii) 1.3735

13. Evaluate $\int_0^5 \frac{1}{4x+5}dx$ by dividing the interval into 10 equal parts. [Ans: 1.61]

14. Evaluate $\int_0^\pi \sin x dx$ using 11 ordinates. [Ans: 2.0009]

15. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}dx$ using simpson's $1/3$ rule taking 10 equal parts. [VTU Model 2019] [Ans: 1.3028]

1. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$ by dividing the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ into 6 equal parts. [VTU Model 2019]

Lecture Notes

BMATE201

Mathematics-II for EEE stream

Module 5 - Numerical Methods-II

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Module 5

Numerical Methods II

Syllabus

Introduction to various numerical techniques for handling Mechanical Engineering applications

Numerical Solution of Ordinary Differential Equations (ODEs): Numerical solution of ordinary differential equations of first order and first degree - Taylor's series method, Modified Euler's method, Runge-Kutta method of fourth order and Milne's predictorcorrector formula (No derivations of formulae). Problems.

Self-Study: Adam-Bashforth method.

Applications: Finding approximate solutions to solve mechanical engineering problems.

5.1 Numerical solution of first order differential equations :

Consider the first order differential equation,

$$\frac{dy}{dx} = f(x, y), \text{ with initial condition, } y(x_0) = y_(0),$$

Such problems in which all the initial conditions are given at the initial point only, are called **initial value problems**.

Let $y(x_0)$, $y(x_1)$,..., $y(x_n)$ be the solution values at the equidistant points x_0, x_1, ..., x_n. Computation of the approximate values to these solution values is known as

Numerical solution of the Differential equation.

Analytical methods, when available, generally enable to find the value of y for all values of x. Numerical methods, on the other hand, lead to the values of y corresponding only to some finite set of values of x. Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution involves some error and is an approximate solution.

5.2 Taylors series method :

Taylor's series is a numerical method used to approximate the value of a function f(x) at a specific point x, by using a series of terms that are derived from the function's derivatives evaluated at that point.

Consider the initial value problem $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_(0)$ then the Taylor's series expansion of $y(x)$ at $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0){y_0}' + \frac{(x-x_0)^2}{2!}{y_0}'' + \frac{(x-x_0)^3}{3!}{y_0}''' + \frac{(x-x_0)^4}{4!}{y_0}^{IV} + \cdots$$

Problem 5.2.1. *Find $y(0.1)$ for $y' = x^2y - 1, y(0) = 1$, using Taylor Series method.*

Solution::

Given $y' = x^2y - 1, y(0) = 1, y(0.2) = ?$

Here, $x_0 = 0, y_0 = 1$

Differentiating successively, we get

$$y' = x^2y - 1$$
$$y'' = 2xy + x^2y'$$
$$y''' = 2y + 4xy' + x^2y''$$
$$y^{IV} = 6y' + 6xy'' + x^2y'''$$

Now substituting, we get

$$y_0' = x_0^2 y_0 - 1 = -1$$
$$y_0'' = 2x_0 y_0 + x_0^2 y_0' = 0$$
$$y_0''' = 2y_0 + 4x_0 y_0' + x_0^2 y_0'' = 2$$
$$y_0^{IV} = 6y_0' + 6x_0 y_0'' + x_0^2 y_0''' = -6$$

Putting these values in Taylor's Series, we have

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0^{IV} + \ldots$$

$$Y(0.1) = 1 + 0.1 \cdot (-1) + \frac{(0.1)^2}{2!} \cdot (0) + \frac{(0.1)^3}{3!} \cdot (2) + \frac{(0.1)^4}{4!} \cdot (-6) + \ldots$$
$$= 1 - 0.1 + 0 + 0.00033 + 0 + \ldots$$
$$= 0.90031$$
$$\therefore y(0.1) = 0.90031$$

Problem 5.2.2. *Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x, y(0) = 0$. Compare the numerical solution obtained with the exact solution.*

Solution: We have $y' = 2y + 3e^x$; $x_0 = 0, y_0 = 0$

Hence $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0, y = 0$ we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{kx} = 2y''' + 3e^x, \qquad y^{it}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + \cdots$$
$$= 0 + 3x + \frac{9}{2} x^2 + \frac{21}{6} x^3 + \frac{45}{24} x^4 + \cdots$$
$$= 3x + \frac{9}{2} x^2 + \frac{21}{6} x^3 + \frac{15}{8} x^4 + \cdots$$

Hence $y(0.2) = 3(0.2)+4.5(0.2)^2+3.5(0.2)^3+1.875(0.2)^4+\cdots = 0.8110$

Exact solution : Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x

Its I.F. being e_{-2x}, the solution is

$$ye^{-2x} = \int 3e^x e^{-2x} dx + c = -3e^{-x} + c \text{ or } y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$,

$$\therefore c = 3.$$

Thus the exact solution is $y = 3\left(e^{2x} - e^x\right)$ When $x = 0.2, y = 3\left(e^{0.4} - e^{0.2}\right) = 0.8112$ Comparing (i) and (ii), it is clear that (i) approximates to the exact value up to three decimal places.

Problem 5.2.3. *Solve* $y' = x + y, y(0) = 1$ *by Taylor's series method. Hence find the values of* y *at* $x = 0.1$ *and* $x = 0.2$.

Solution:: Differentiating successively, we get

$$y' = x + y \quad y'(0) = 1 \quad [\because y(0) = 1]$$

$$y'' = 1 + y' \quad y''(0) = 2$$

$$y''' = y'' \quad y'''(0) = 2$$

$$y''' = y''' \quad y'''(0) = 2, \text{etc.}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \cdots$$

Here $x_0 = 0, y_0 = 1$

$$\therefore \quad y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(4)\cdots$$

Thus

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}\cdots$$

$$= 1.1103$$

and

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \cdots$$

$$= 1.2427$$

5.3 Modified Euler's Method :

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \;\; y(x_0) = y_0,$$

To find y_1 i.e. $y(x_1)$ where $x_1 = x_0 + h$, we proceed as follows.

Step 1: We first obtain an initial approximation for $y_1 = y(x_1)$ by applying Euler's formula ${y_1}^{(0)} = y_0 + hf(x_0, y_0)$

Step 2: Since the accuracy is very poor in Euler's formula, the formula is modified and is given by,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$
$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$
$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

In general

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right]$$

and called it as modified Euler's formulae.

Problem 5.3.1. *Given that* $dy/dx = \log_{10}(x + y), y(0) = 1$, *find* $y(0.1)$ *and* $y(0.2)$ *using modified Euler's method.*

Solution: :

Given differential equation is,

$$\frac{dy}{dx} = \log_{10}(x + y) = f(x, y)$$

with initial condition, $x_0 = 0, y_0 = 1$ and $x_1 = 0.1$

Taking, $\quad h = 0.1$, such that

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

and $x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$ **First Stage:**

By Using Euler's formula

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ &= y_0 + h\left[\log_{10}(x_0 + y_0)\right] \\ &= 1 + 0.1\left[\log_{10}(0 + 1)\right] = 1 \quad \text{at } x_1 = x_0 + h = 0.1 \end{aligned}$$

Applying Euler's modified formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(n)}\right)\right]$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right] = 1.0020$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right] = 1.0021 \text{ and}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(2)}\right)\right] = 1.0021$$

clearly, $y_1^{(3)} = y_1^{(2)} = 1.0021 = y_1(\text{ Improved value })$ at $x_1 = 0.1$

Second Stage: Taking $y_1 = 1.0021$ and $x_1 = 0.1$

By Using Euler's formula,

$$y_2 = y_1 + hf\left(x_1, y_1\right)$$

$$\Rightarrow \quad y_2 = y_1 + h\left[\log_{10}\left(x_1 + y_1\right)\right]$$

$$= 1.0021 + 0.1\left[\log_{10}(0.1 + 1.0021)\right]$$

$$= 1.0063, \text{ at } x_2 = 0.2$$

Applying Euler's modified formula,

$$y_2^{(n+1)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(n)}\right)\right]$$

$$y_2{}^{(1)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2\right)\right]$$

$$= 1.0124$$

$$y_2{}^{(2)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2{}^{(1)}\right)\right]$$

$$= 1.0125$$

And

$$y_2{}^{(3)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(2)}\right)\right] = 1.0125$$

Clearly,

$$y_2{}^{(3)} = y_2{}^{(2)} = 1.0125 = y_2$$

at $x_2 = 0.2$

Hence the required value of y at $x = 0.2$ is 1.0125 .

Problem 5.3.2. *Using modified Euler's method find* y *at* $\mathrm{x} = 0.2$ *given* $\frac{dy}{dx} = 3x + \frac{y}{2}$ *with* $\mathrm{y}(0) = 1$ *taking* $\mathrm{h} = 0.1$. *Perform two iteration at each step.*

Solution: : Given differential equation is,

$$\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y)$$

with initial condition, $x_0 = 0, y_0 = 1$. Let $x_1 = 0.1$

Taking, $h = 0.1$, such that

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

and $x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$

Stage 1: By Using Euler's formula

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ &= y_0 + h\left[3x_0 + \frac{y_0}{2}\right] \\ &= 1.05 \end{aligned}$$

Applying Euler's modified formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right]$$

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right] \\ &= y_0 + \frac{h}{2}\left\{\left[3x_0 + \frac{y_0}{2}\right] + \left[3x_1 + \frac{y_1^{(0)}}{2}\right]\right\} \\ &= 1.0020 \\ y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] \\ &= y_0 + \frac{h}{2}\left\{\left[3x_0 + \frac{y_0}{2}\right] + \left[3x_1 + \frac{y_1^{(1)}}{2}\right]\right\} \\ &= 1.0667 \end{aligned}$$

Thus $y_1 = y(0.1) = 1.0667$

Stage 2: By Using Euler's formula

$$\begin{aligned} y_2^{(0)} &= y_1 + hf(x_1, y_1) \\ &= y_1 + h\left[3x_1 + \frac{y_1}{2}\right] \\ &= 1.1 \end{aligned}$$

Applying Euler's modified formula,

$$y_2^{(n+1)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(n)}\right)\right]$$

$$\begin{aligned}
y_2^{(1)} &= y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_1, y_1^{(0)}\right)\right] \\
&= y_1 + \frac{h}{2}\left\{\left[3x_1 + \frac{y_1}{2}\right] + \left[3x_2 + \frac{y_2^{(0)}}{2}\right]\right\} \\
&= 1.1671 \\
y_2^{(2)} &= y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(1)}\right)\right] \\
&= y_1 + \frac{h}{2}\left\{\left[3x_1 + \frac{y_1}{2}\right] + \left[3x_2 + \frac{y_2^{(1)}}{2}\right]\right\} \\
&= 1.1675
\end{aligned}$$

Thus $y_2 = y(0.2) = 1.1675$

Problem 5.3.3. *Using Euler's modified method, obtain a solution of the equation*

$$dy/dx = x + |\sqrt{y}|$$

with initial conditions $y = 1$ *at* $x = 0$*, for the range* $0x0.6$ *in steps of 0.2 .*

Solution:: The various calculations in this method are arranged as follows:

x	$x + \lvert\sqrt{y}\rvert = y'$	Mean slope	Old $y + 0.2$ (mean slope) $=$ new y
0.0	$0 + 1 = 1$	$-$	$1 + 0.2(1) = 1.2$
0.2	$0.2 + \lvert\sqrt{(1.2)}\rvert = 1.2954$	$\frac{1}{2}(1 + 1.2954) = 1.1477$	$1 + 0.2(1.1477) = 1.2295$
0.2	$0.2 + \lvert\sqrt{(1.2295)}\rvert = 1.3088$	$\frac{1}{2}(1 + 1.3088) = 1.1544$	$1 + 0.2(1.1544) = 1.2309$
0.2	$0.2 + \lvert\sqrt{(1.2309)}\rvert = 1.3094$	$\frac{1}{2}(1 + 1.3094) = 1.1547$	$1 + 0.2(1.1547) = 1.2309$
0.2	1.3094	$-$	$1.2309 + 0.2(1.3094) = 1.4927$
0.4	$0.4 + \lvert\sqrt{(1.4927)}\rvert = 1.6218$	$\frac{1}{2}(1.3094 + 1.6218) = 1.4654$	$1.2309 + 0.2(1.4654) = 1.5240$
0.4	$0.4 + \lvert\sqrt{(1.524)}\rvert = 1.6345$	$\frac{1}{2}(1.3094 + 1.6345) = 1.4718$	$1.2309 + 0.2(1.4718) = 1.5253$
0.4	$0.4 + \lvert\sqrt{(1.5253)}\rvert = 1.6350$	$\frac{1}{2}(1.3094 + 1.6350) = 1.4721$	$1.2309 + 0.2(1.4721) = 1.5253$
0.4	1.6350	$-$	$1.5253 + 0.2(1.635) = 1.8523$
0.6	$0.6 + \lvert\sqrt{(1.8523)}\rvert = 1.9610$	$\frac{1}{2}(1.635 + 1.961) = 1.798$	$1.5253 + 0.2(1.798) = 1.8849$
0.6	$0.6 + \lvert\sqrt{(1.8849)}\rvert = 1.9729$	$\frac{1}{2}(1.635 + 1.9729) = 1.8040$	$1.5253 + 0.2(1.804) = 1.8861$
0.6	$0.6 + \lvert\sqrt{(1.8861)}\rvert = 1.9734$	$\frac{1}{2}(1.635 + 1.9734) = 1.8042$	$1.5253 + 0.2(1.8042) = 1.8861$

Hence $y(0.6) = 1.8861$ approximately.

Problem 5.3.4. *Using modified Euler's method find $y(0.1)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2, y(0) = 1$ taking $h = 0.1$.*

Try this Yourself !

Answers: $y_1^{(0)} = 0.9, y_1^{(1)} = 0.9145, y_1^{(2)} = 0.9132, y_1^{(3)} = 0.9133,$
Thus $\mathrm{y}(0.1) = 0.9133$

5.4 R.K. Method of order 4:

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y\left(x_0\right) = y_0$$

We need to find $y\left(x_1\right) = y_1$ With the help of formula

$$y_1 = y_0 + k$$

where

$$\mathrm{k} = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

and

$$\begin{aligned} k_1 &= hf\left(x_0, y_0\right) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\ k_4 &= hf\left(x_0 + h, y_0 + k_3\right) \end{aligned}$$

Problem 5.4.1. *Given* $\frac{dy}{dx} = 3x + \frac{y}{2} \quad y(0) = 1$ *compute* $y(0.2)$ *by taking* $h = 0.2$ *using Runge-kutta method of fourth order.*

Solution:: By data $f(x, y) = 3x + \frac{y}{2}, x_0 = 0, y_0 = 1, h = 0.2$
We shall first compute k_1, k_2, k_3, k_4

$$\begin{aligned} k_1 &= hf\left(x_0, y_0\right) \\ &= (0.2)f(0, 1) \\ &= (0.2)\left[(3)(0) + \frac{1}{2}\right] \\ &= 0.1 \end{aligned}$$

$$
\begin{aligned}
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= (0.2)f\left[0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right] \\
&= (0.2)f(0.1, 1.05) \\
&= (0.2)\left[3(0.1) + \frac{1.05}{2}\right] \\
&= 0.165
\end{aligned}
$$

$$
\begin{aligned}
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= (0.2)f\left[0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right] \\
&= (0.2)f(0.1, 1.0825) \\
&= (0.2)\left[3(0.1) + \frac{1.0825}{2}\right] \\
&= 0.16825
\end{aligned}
$$

$$
\begin{aligned}
k_4 =& hf\left(x_0 + h, y_0 + k_3\right) = (0.2)f\left[0 + \frac{0.2}{2}, 1 + \frac{0.16825}{2}\right] \\
&= (0.2)f(0.2, 1.16825) = (0.2)\left[3(0.2) + \frac{1.16825}{2}\right] \\
&= 0.236825
\end{aligned}
$$

We have,

$$y\left(x_0 + h\right) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

Therefore,

$$y(0.2) = 1 + \frac{1}{6}[0.1 + 2(0.165 + 2(0.16825) + 0.236825)] = 1.1672208$$

Problem 5.4.2. *Using the Runge-Kutta method of fourth order, solve* $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ *with* $y(0) = 1$ *at* $x = 0.2, 0.4$.

Solution:: We have $f(x,y)=\frac{y^2-x^2}{y^2+x^2}$
To find $y(0.2)$ Hence

$$x_0=0, y_0=1, h=0.2$$

$$k_1=hf(x_0,y_0)=0.2f(0,1)=0.2000$$

$$k_2=hf\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1\right)=0.2\times f(0.1,1.1)=0.19672$$

$$k_3=hf\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2\right)=0.2f(0.1,1.09836)=0.1967$$

$$k_4=hf(x_0+h, y_0+k_3)=0.2f(0.2,1.1967)=0.1891$$

$$k=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$=\frac{1}{6}[0.2+2(0.19672)+2(0.1967)+0.1891]=0.19599$$

Hence $y(0.2)=y_0+k=1.196.$

To find $y(0.4)$:

Here $x_1=0.2, y_1=1.196, h=0.2.$

$$k_1=hf(x_1,y_1)=0.1891$$

$$k_2=hf\left(x_1+\frac{1}{2}h, y_1+\frac{1}{2}k_1\right)=0.2f(0.3,1.2906)=0.1795$$

$$k_3=hf\left(x_1+\frac{1}{2}h, y_1+\frac{1}{2}k_2\right)=0.2f(0.3,1.2858)=0.1793$$

$$k_4=hf(x_1+h, y_1+k_3)=0.2f(0.4,1.3753)=0.1688$$

$$k=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$=\frac{1}{6}[0.1891+2(0.1795)+2(0.1793)+0.1688]=0.1792$$

Hence $y(0.4)=y_1+k=1.196+0.1792=1.3752.$

Problem 5.4.3. *Apply the Runge-Kutta method to find the approximate value of* y *for* $x=0.2$*, in steps of 0.1 , if* $dy/dx=x+y^2, y=1$ *where* $x=0$.

Solution:: Given $f(x, y) = x + y^2$. Here we take $h = 0.1$ and carry out the calculations in two steps. Step I. $x0 = 0, y0 = 1, h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1000$$
$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1f(0.05, 1.1) = 0.1152$$
$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1f(0.05, 1.1152) = 0.1168$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1168) = 0.1347$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$\therefore \quad = \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165$
Step II.

$$x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$$
$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1165) = 0.1347$$
$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1f(0.15, 1.1838) = 0.1551$$
$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1f(0.15, 1.194) = 0.1576$$
$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1f(0.2, 1.1576) = 0.1823$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$

Problem 5.4.4. *Using the Runge-Kutta method of order 4 , find y for* $x = 0.1, 0.2$ *given that* $dy/dx = xy + y^2, y(0) = 1$.

Solution:: We have $f(x, y) = xy + y^2$.

To find $y(0.1)$:

Here $x_0 = 0, y_0 = 1, h = 0.1$.

$$\therefore \quad k_1 = hf(x_0, y_0) = (0.1) \times f(0, 1) = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) \times f(0.05, 1.05) = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) \times f(0.05, 1.0577) = 0.1172$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = (0.1) \times f(0.1, 1.1172) = 0.13598$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$

To find $y(0.2)$: Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1$

$$k_1 = hf\left(x_1, y_1\right) = (0.1) \times f(0.1, 1.1169) = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) \times f(0.15, 1.1848) = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) \times f(0.15, 1.1959) = 0.1609$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right) = (0.1) \times f(0.2, 1.2778) = 0.1888$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.1605$$

Thus $\quad y(0.2) = y_2 = y_1 + k = 1.2773.$

5.5 Predictor–Corrector methods

To solve a differential equation over an interval (x_n, x_{n+1}), using previous single-step methods, only the values of y at the beginning of interval is required. However, in the following methods, four prior values are needed for finding the value of y_{n+1} at a given value of x. Also the solution at y_{n+1} is obtained in two stages. This method of refining an initially crude estimate of y_{n+1} by means of a more accurate formula is known as **Predictor–Corrector method**. A Predictor formula is used to predict the value of y_{n+1} and then a Corrector Formula is applied to calculate a still better approximation of y_{n+1}. Now we study one such method namely **Milne's method**.

5.6 Milne's method

Milne's predictor formula :

$$y_4^{(p)} = y_0 \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

Milne's corrector formula :

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)$$

Problem 5.6.1. *Given that* $dy/dx = x - y^2$ *and the data* $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. *Compute y at* $x = 0.8$ *by applying Milne's method*

Solution:: Given, $y' = x - y^2$ in the range $0 \leq x \leq 1$ for the boundary conditions $y = 0$ at $x = 0$.

$$\therefore \quad x_0 = 0.0, \quad y_0 = 0.0000, \quad f_0 = 0.0000$$

$$x_1 = 0.2, \quad y_1 = 0.020, \quad f_1 = 0.1996$$

and

$$x_2 = 0.4, \quad y_2 = 0.0795, \quad f_2 = 0.3937$$

$$x_3 = 0.6, \quad y_3 = 0.1762, \quad f_3 = 0.5689$$

Using the predictor, $y_4^{(p)} = y_0 \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$

$x = 0.8, \quad y_4^{(p)} = 0.3049, \quad f_4 = 0.7070$

and the corrector,

$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)$, yields

$y_4^{(c)} = 0.3046,$

Problem 5.6.2. *Using Milne's method find* $y(4.5)$ *given* $5xy' + y^2 - 2 = 0$ *given* $y(4) = 1$, $y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0143; y(4.4) = 1.0187$.

Solution:: We have $y' = \left(2 - y^2\right)/5x = f(x, y)$

Then the starting values of the Milne's method are

$$
\begin{array}{lll}
x_0 = 0, & y_0 = 1, & f_0 = \frac{2-12}{5\times 4} = 0.05 \\
x_1 = 4.1, & y_1 = 1.0049, & f_1 = 0.0485 \\
x_2 = 4.2, & y_2 = 1.0097, & f_2 = 0.0467 \\
x_3 = 4.3, & y_3 = 1.0143, & f_3 = 0.0452 \\
x_4 = 4.4, & y_4 = 1.0187, & f_4 = 0.0437
\end{array}
$$

Since y_5 is required, we use the predictor

$$y_5^{(p)} = y_1 + \tfrac{4h}{3}\left(2f_2 - f_3 + 2f_4'\right) \qquad \langle h$$

$$x = 4.5, y_5^{(p)} = 1.0049 + \tfrac{4(0.1)}{3}(2 \times 2.0467 - 0.0452 + 2 \times 0.0437) = 1.023$$

$$f_5 = \frac{2 - y_5^2}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424$$

Now using the corrector $y_5^{(c)} = y_3 + \frac{h}{3}\left(f_3 + 4f_4 + f_5\right)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}(0.0452 + 4 \times 0.0437 + 0.0424) = 1.023$$

Hence $y(4.5) = 1.023$ Given that $dy/dx = xy + y^2, y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$. Find y at $x = 0.4$ using Milne's method. **Solution:**: We have $f(x, y) = xy + y^2$.

Now the starting values for the Milne's method are:

x	y	$f = y'$
$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the predictor

$$y_4^{(p)} = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$x_4 = 0.4 \quad y_4^{(p)} = 1.8344 \quad f_4 = 4.0988$$

and the corrector,

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8397 \quad f_4 = 4.1159.$$

Again using the corrector,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159]$$
$$= 1.8391 \quad f_4 = 4.1182$$

Again using the corrector,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$
$$= 1.8392 \text{ which is same as (i)}$$

Hence $y(0.4) = 1.8392$.

5.7 Question Bank : Module 5 - Numerical Methods II

5.7.1 Question Bank :Taylor's series method

1. Use Taylor's series method to find y at $x = 0.1, 0.2, 0.3$ considering terms up to the third degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$ [VTU-Dec 2018, Dec 2012]

2. From Taylor's series method, find $y(0.1)$ considering up to fourth degree term if $y(x)$ satisfies the equation $\frac{dy}{dx} = x - y^2, y(0) = 1$ [VTU-Jan 2015]

3. Employ Taylor's series method to find y at $x = 0.1$ and 0.2 correct to four places of decimal in step size of 0.1 given the linear differential equation $\frac{dy}{dx} - 2y = 3e^x$ whose solution passes through origin. Also find $y(0.1)$ and $y(0.2)$ by analytical method. [VTU-Jan 2018, Jan 2014,July 2013]

4. Using Taylor's series method for $y' = \sqrt{x^2 + y}$, $y(0) = 0.8$, find $y(0.1)$. consider up to third order derivative terms. [VTU-July 2017]

5. Applying Taylor's series method, find y at $x = 0.1$. Given $\frac{dy}{dx} = x + y^2, y(0) = 1$. [VTU-Jan 2014]

6. Using Taylor's series method solve $\frac{dy}{dx} = x^2y + 1, y(0) = 0$. Find the third order solution at $x = 0.4$ [VTU-June 2012]

7. Using Taylor's series method find $y(0.1)$, Given $\frac{dy}{dx} = x^2y - 1, y(0) = 1$ [VTU- Jan 2021, July 2017, June 2012, July 2011]

8. Use Taylor's series method to solve $\frac{dy}{dx} = x^2y - 1, y(0) = 1$ at $x = 0.2$. Consider upto 4th degree terms. [VTU Model 2022, July 2017, july 2016, Jan 2016, Dec 2012]

9. Find $y(0.1)$ correct to 6 decimal places by using Taylor's series method. Given $dy = (xy + 1)dx, y(0) = 1.1$ [VTU-Jun 2010]

10. Use Taylors series method to find $y(4.1)$. Given that $(x^2 + y)y' = 1$ and $y(4) = 4$ [VTU Jan 2018]

11. Use Taylors series method to find $y(0.1)$. Given that $y' + y + 2x = 0$ and $y(0) = -1$. Consider up to third order derivative term. [VTU July 2019]

12. Use Taylors series method to find $y(0.1)$ from $y' = 3x + y^2$ and $y(0) = 1$. Consider up to fourth derivative term. [VTU Model 2022, July 2019]

13. Use Taylors series method to find $y(0.1)$ and $y(0.2)$. Given $y' = x + y, y(0) = 1$ (VTU Sept 2020)

14. Use Taylors series method to find $y(1.1)$ from $y' = e^x - y, y(0) = 2$ (VTU Jan 2021, Sept 2020)

15. Use Taylors series method to find $y(0.2)$ taking h=0.1. Given $y' = e^x - y^2, y(0) = 1$ (VTU Model 2022)

5.7.2 Question Bank :Modified Euler's method

1. Using modified Euler's method find $y(20.2)$ and $y(20.4)$ given that $\frac{dy}{dx} = log_{10}\left(\frac{x}{y}\right)$ With $y(20) = 5$ taking $h = 0.2$. [VTU-July 2017]

2. Apply modified Euler's method to solve the following initial value problems by considering the accuracy up to two approximations in every step $\frac{dy}{dx} = x +$

$|\sqrt{y}|$ in the range $0 \leq x \leq 0.4$ by taking $h = 0.2$ given that $y = 1$ at $x = 0$ initially. [VTU-Jan 2021, Dec 2018, July 2017]

3. Given $\frac{dy}{dx} = \frac{1}{1+x^2} - 2y^2$, $y(0) = 0$. Find $y(0.5)$, by taking $h = 0.25$ using Euler's modified method. [VTU-July 2017, Dec 2011]

4. Solve by using modified Euler's method to obtain $y(1.2)$. Given $y' = \frac{y+x}{y-x}$, $y(1) = 2$. [VTU-July 2015]

5. Solve by using modified Euler's method to obtain y at x=0.2 taking h=0.1. Given $y' = \frac{y-x}{y+x}$, $y(0) = 1$. Carryout 3 iterations. [VTU- Model 2022, Dec 2018, Dec 2012]

6. Solve by using modified Euler's method to find $y(0.4)$ by taking $h = 0.2$. Given
$\frac{dy}{dx} = log(x + y), y(0) = 2$ [VTU-Jan 2014, 2015]

7. Determine the value of y when $x = 0.1$ given that $y(0) = 1$ and $y'' = x^2 + y^2$, using modified Euler's method. Take $h = 0.05$ [VTU-Jan 2014]

8. Given $\frac{dy}{dx} = x + y, y(0) = 1$. compute y(0.2) by using modified Euler's method. [VTU-Jan 2013]

9. Use modified Euler's method to find y at x=0.1.Given $\frac{dy}{dx} = 3x+\frac{y}{2}, y(0) = 1$, $h = 0.1$ [VTU Sept 2020, Jan 2018]

10. $\frac{dy}{dx} = 1+\frac{y}{x}$ with $y(1) = 2$. Find the value of y at x=1.2 using Eulers Modified method. (VTU - Dec 2018)

11. Find $y(0.2)$ by using modified Euler's method, given that given that $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Take $h = 0.1$ and carry out two modifications at each step. [VTU-July 2019]

12. Use modified Euler's method to find y at $x = 0.4$. Given $\frac{dy}{dx} = x + \sin y, y(0) = 1, h = 0.2$ (VTU Jan 2020)

13. Use modified Euler's method to find y at $x = 1.2$. Given $\frac{dy}{dx} = 1 + \frac{y}{2}, y(1) = 2, h = 0.2$ (VTU Sept. 2020)

14. Use modified Euler's method to find y at $x = 0.1$. Given $\frac{dy}{dx} = x^2 + y, y(0) = 1, h = 0.05$ (VTU Model 2022, Jan 2021)

5.7.3 Question Bank :Runge-kutta method

1. Apply Runge-kutta method of order 4 to find an approximate value of y for $x = 0.2$ in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$ [VTU-June, July 2014]

2. Use fourth order Runge-kutta method to solve $(x + y)\frac{dy}{dx} = 1, y(0.4) = 1$ at $x = 0.5$ correct to four decimal places. [VTU-July 2013]

3. Use fourth order Runge-kutta method to find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$ taking $h = 0.2$. [VTU-Dec 2018, Jan 2018, Jan 2014, Dec 2012, Dec 2011]

4. Solve $(y^2 - x^2)dx = (y^2 + x^2)dy$ for $x = 0.1$. Given that $y = 1$ at $x = 0$ initially, by applying Runge-kutta method of order 4. [VTU- Model 2022, July 2017, Dec 2012, June 2012, July 2011]

5. Use fourth order Runge-kutta method to solve $\frac{dy}{dx} = x + y$ at $x = 0.2$ given that $y(0) = 1$ [VTU-Jan 2013]

6. Use fourth order Runge-kutta method to solve $\frac{dy}{dx} = -xy^2, y(0) = 2$.Compute $y(0.2)$by taking $h = 0.1$ [VTU-June 2012]

7. Use fourth order Runge-kutta method to solve the following initial value problem
   $\frac{dy}{dx} = 3x + \frac{y}{2}, y(0) = 1$ .compute $y(0.2)$ with $h = 0.1$ [VTU-Model 2022, Jan 2021, Dec 2015, Jan 2016, July 2016]

8. Given $\frac{dy}{dx} = x + y, y(0) = 1.$ compute y(0.2) by using fourth order Runge-kutta method. (take h=0.2) [VTU-Jan 2013]

9. Use fourth order Runge-kutta method to solve the following initial value problem
$10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ .compute $y(0.2)$ with $h = 0.1$ [VTU Dec 2010]

10. Use Runge-kutta method to find $y(0.1)$ from $\frac{dy}{dx} = x^2 + y$ given that $y(0) = -1$ [VTU- July 2019]

11. Use fourth order Runge-kutta method to solve the following initial value problem
$\frac{dy}{dx} + y + xy^2 = 0, y(0) = 1$ Find y at $x = 0.1$ (Jan 2020)

12. Use fourth order Runge-kutta method to solve the following initial value problem
$\frac{dy}{dx} = 3e^x + 2y, y(0) = 1$ Find y at $x = 0.1, h = 0.1$ (VTU : Model 2022, Sept 2020)

5.7.4 Question Bank :Milne's Method

1. Applying Milne's Predictor - Corrector method, to find $y(1.4)$, from $\frac{dy}{dx} = x^2 + \frac{y}{2}$, given that $y(1) = 2, y(1.1) = 2.2156, y(1.2) = 2.4549, y(1.3) = 2.7514$ (VTU - Model 2022)

2. Applying Milne's Predictor-Corrector method, find $y(0.8)$, from $\frac{dy}{dx} = x^3 + y$, given that $y(0) = 2, y(0.2) = 2.073, y(0.4) = 2.452, y(0.6) = 3.023$ (VTU - Model 2022)

3. Using Milne's Predictor-Corrector method, find $y(4.5)$, given $\frac{dy}{dx} = \frac{2-y^2}{5x}$ and $y(4.1) = 1.0049, \quad y(4.2) = 1.0097$ $y(4.3) = 1.0143, \quad y(4.4) = 1.0187$ (VTU - Model 2022)

4. Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762$. Compute y at $x = 0.8$ by applying Milne's method. [VTU-Jan 2021, Sept 2020, July 2015, Jan 2016, June 2010]

5. The following table gives the solution of $5xy' + y^2 - 2 = 0$. find the value of y at $x = 4.5$ using Milne's predictor and corrector formulae. Use the corrector formula twice.

X	4	4.1	4.2	4.3	4.4
Y	1	1.0049	1.0097	1.0143	1.0187

[VTU-July 2017]

6. Given $\frac{dy}{dx} = xy + y^2$; y(0)=1 , y(0.1)=1.1169 , y(0.2) =1.2773, y(0.3)=1.5049 find y(0.4) correct to three decimal places using Milne's predictor corrector method . Apply the corrector formula twice. [VTU Jan 2014, Dec 2012, June 2012]

7. If $\frac{dy}{dx} = 2e^x - y, y(0) = 2, y(0.1) = 2.010, y(0.20 = 2.040, y(0.3) = 2.090$ find $y(0.4)$ correct to 4 decimal places by Milnes predictor corrector method. Apply the corrector formula twice. [VTU Jan 2018, July 2013]

8. Given $2\frac{dy}{dx} = (1 + x^2)y^2$ and $y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12, y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's method. [VTU- Model 2022, Dec 2011]

9. Given $\frac{dy}{dx} = \frac{x+y}{2}$, Given that $y(0) = 2, y(0.5) = 2.636, y(1) = 3.595$ and $y(1.5) = 4.968$ Find the value of y at $x = 2$ using Milnes Predictor and Corrector Formulae.

10. Given $\frac{dy}{dx} = \frac{1}{x+y}$, Given that $y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755$ and $y(0.6) = 2.2493$. Find the value of y at $x = 0.8$ using Milnes Predictor and Corrector Formulae. [VTU-July 2019]

11. Apply Milne's predictor-corrector formulae to compute y(0.3) given, $\frac{dy}{dx} = x + y^2$ with

x	0.0	0.1	0.2	0.3
y	1.0000	1.1000	1.2310	1.4020

(VTU Model 2019)