Lecture Notes

BMATM201

Mathematics-II for Mechanical Stream

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Module 1

Integral Calculus

Syllabus :

Introduction to Integral Calculus in Mechanical Engineering applications. Multiple Integrals:
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Beta and Gamma functions: Definitions, properties, relation between Beta and Gamma functions. Problems.
Self-Study: Volume by triple integration, Center of gravity.
Applications: Applications to mathematical quantities (Area, Surface area, Volume), Analysis of probabilistic models. (RBT Levels: L1, L2 and L3)

1.1 Evaluation of double integrals

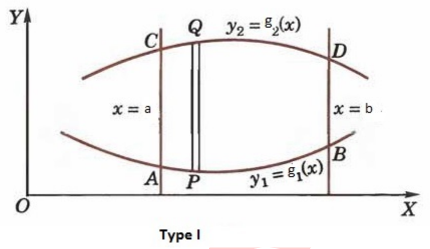
Let us consider a function f (x, y) defined in Cartesian region. If this region is denoted by R, then double integral is defined as

$$\int\int_R f(x,y)dxdy$$

where R is called the region of integration and is a region in the (x, y) plane.

There are two types of regions of double integrals. These are as follows:

Type I :

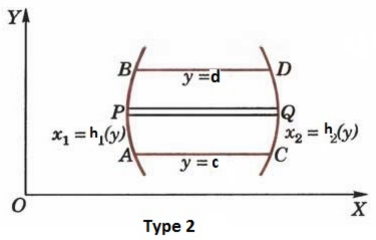


The region is defined as $R = (x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)$ and the integral is defined as,

$$\int\int_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

We call $g_1(x)$ and $g_2(x)$ as the **inner limits** of integration and we call a and b as the **outer limits** of integration. Here, inner limits are functions of x and outer limits are constants. To evaluate this double integral, we first integrate $f(x, y)$ w.r.to y treating x as a constant, between the limits $y = g_1(x)$ to $y = g_2(x)$ and then the resulting expression is integrated w.r.to x between the outer limits.

Type II :



The region is defined as $R = (x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)$ and the integral is defined as,

$$\int\int_R f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$$

We call $h_1(y)$ and $h_2(y)$ as the **inner limits** of integration and we call c and d as the **outer limits** of integration. Here, inner limits are functions of y and outer limits are constants. To evaluate this double integral, we first integrate w.r.to x treating y as a constant, between the limits $y = h_1(y)$ to $y = h_2(y)$ and then the resulting expression is integrated w.r.to y between the outer limits.

Remember:

- The outer limits of integration are always constants.
- Both inner and outer limits of integration are constant if and only if the region of integration R is a rectangle.
- When the double integral is evaluated, the result is a real number, not a variable function.

Problem 1.1.1. *Evaluate* $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y dx dy$ *(VTU Jan 2017, Dec 2011)*

Solution:

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y dx dy &= \int_0^1 y \int_0^{\sqrt{1-y^2}} x^3 dx dy \\
&= \frac{1}{4} \int_0^1 y [x^4]_0^{\sqrt{1-y^2}} dy \\
&= \frac{1}{4} \int_0^1 y[(1-y^2)^2] dy \\
&= \frac{1}{4} \int_0^1 [y + y^5 - 2y^3] dy \\
&= \frac{1}{4} [\frac{y^2}{2} + \frac{y^6}{6} - \frac{y^4}{2}]_0^1 \\
&= \frac{1}{24}
\end{aligned}$$

Problem 1.1.2. *Evaluate:* $\int_0^1 \int_x^{\sqrt{x}} (x^2 y + x y^2)\, dy dx$

Solution: Let

$$
\begin{aligned}
I &= \int_0^1 \left[\int_x^{\sqrt{x}} \left(x^2y + xy^2\right) dydx\right] \\
&= \int_0^1 \left[\int_x^{\sqrt{x}} \left(x^2y + xy^2\right) dydx\right] \\
&= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_x^{\sqrt{x}} dx \\
&= \int_0^1 \left[\left(\frac{x^3}{2} + \frac{x \cdot xy^{3/2}}{3}\right) - \left(\frac{x^4}{2} + \frac{x^4}{3}\right)\right] dx \\
&= \left[\frac{x^4}{8} + \frac{x^{7/2}}{(7/2)(3)} - \frac{5}{6}\left(\frac{x5}{5}\right)\right]_0^1 \\
&= \left(\frac{1}{8} + \frac{2}{21} + \frac{1}{6}\right) - (0) \\
&= \left(\frac{21+16-28}{168}\right) \\
&= \frac{9}{168} = \frac{3}{56}
\end{aligned}
$$

1.2 Triple Integrals

Let $f(x, y, z)$ be a function of three variable defined in a three dimensional region R. The triple integral of $f(x, y, z)$ over R is denoted by

$$\int\int_R\int f(x, y, z)dxdydz$$

To evaluate a triple integral in the form $\int_a^b \int_{f(x)}^{g(x)} \int_{f(x,y)}^{g(x,y)} f(x, y, z)dxdydz$, we first integrate the function w.r.t 'z' between the limits $f(x, y)$ to $g(x, y)$ keeping x and y fixed. The resulting expression is integrated w.r.t 'y' between the limits $f(x)$ to $g(x)$, keeping x constant. The result just obtained is finally integrated w.r.t x from a to b.

Thus

$$\int_a^b \int_{f(x)}^{g(x)} \int_{f(x,y)}^{g(x,y)} f(x, y, z)dxdydz = \int_a^b \left\{\int_{f(x)}^{g(x)} \left[\int_{f(x,y)}^{g(x,y)} f(x, y, z)dz\right] dy\right\} dz$$

Problem 1.2.1. *Evaluate* $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dxdydz$ *(VTU Jan 2019, Jan 2017, Jan 2016, Jan 2015, July 2011, July 2009)*

Solution: Integrating first w.r.t. y keeping x and z constant, we have

$$\begin{aligned}
I &= \int_{-1}^{1}\int_{0}^{z}\left|xy+\frac{y^2}{2}+yz\right|_{x-z}^{x+z}dxdz\\
&= \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(2z)+\frac{1}{2}4xz\right]dxdz\\
&= 2\int_{-1}^{1}\left|\frac{x^2z}{2}+z^2x+\frac{x^2}{2}z\right|_{0}^{2}dz\\
&= 2\int_{-1}^{1}\left(\frac{z^3}{2}+z^3+\frac{z^3}{2}\right)dz\\
&= 4\left|\frac{z^4}{4}\right|_{-1}^{1}\\
&= 0.
\end{aligned}$$

Problem 1.2.2. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dzdydx$ *(VTU June 2019, Model 2018, July 2017, Jan 2016, July 2014, July 2013)*

Solution: Let

$$\begin{aligned}
I &= \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dzdydx\\
&= \int_{-c}^{c}\left[\int_{-b}^{b}(ax^2+ay^2+\frac{z^3}{3})\right]_{-a}^{a}dydx\\
&= \int_{-c}^{c}\int_{-b}^{b}(2ax^2+2ay^2+\frac{2a^3}{3})dydx\\
&= \int_{-c}^{c}\left[(2ayx^2+\frac{2ay^3}{3}+\frac{2ya^3}{3})\right]_{-b}^{b}dx\\
&= \int_{-c}^{c}(4abx^2+\frac{4ab^3}{3}+\frac{4ba^3}{3})dx\\
&= 4ab\left[\frac{x^3}{3}+\frac{xb^2}{3}+\frac{xa^2}{3}\right]_{-c}^{c}
\end{aligned}$$

$$\therefore\ I = \frac{8abc}{3}(a^2+b^2+c^2)$$

Problem 1.2.3. *Evaluate the integral* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$ *(VTU Jan 2020, June 2018, Jan 2013, Dec 2010, Jan 2008)*

Solution: The given triple integral is

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2}xy\left(1-x^2-y^2\right) dydx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \frac{1}{2}x \left(y - x^2y - y^3\right) dy\right] dx$$

$$= \frac{1}{2}\int_0^1 x \left[\frac{y^2}{2} - x^2\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x \left[\frac{(1-x^2)}{2} - x^2\frac{(1-x^2)}{2} - \frac{(1-x^2)^2}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 x \left[\frac{2(1-x^2) - 2x^2(1-x^2) - (1-x^2)^2}{4}\right] dx$$

$$= \frac{1}{8}\int_0^1 x\left(1-x^2\right)^2 dx \qquad \text{(on simplification)}$$

$$= \frac{1}{8}\int_0^1 x\left(1-2x^2+x^4\right) dx$$

$$= \frac{1}{8}\int_0^1 \left(x-2x^3+x^5\right) dx$$

$$= \frac{1}{8}\left(\frac{x^2}{2} - 2\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$= \frac{1}{48} \qquad \text{(After simplification)}$$

Problem 1.2.4. *Evaluate* $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dzdydx$ *(VTU July 2016, Dec 2010, Jan 2010, July 2007)*

Solution: We have

$$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx.$$

$$= \int_0^a \int_0^x e^{x+y+z}\Big|_0^{x+y} dy dx$$

$$= \int_0^a \int_0^x \left[e^{2(x+y)} - e^{x+y}\right] dy dx$$

$$= \int_0^a \left[\int_0^x e^{2(x+y)} dy - \int_0^x e^{x+y} dy\right] dx$$

$$= \int_0^a \left\{\left[\frac{e^{2(x+y)}}{2}\right]_0^x - \left[e^{x+y}\right]_0^x\right\} dx$$

$$= \int_0^a \left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x\right] dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{4} - \frac{e^{2x}}{2} + e^x\right]_0^a$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a.$$

1.3 Evaluation of Double integral when Region is Given

Let $f(x, y)$ be a function defined over a domain D (or Region R) in xy plane and we need to find the double integral of $f(x, y)$. Draw the figure from the given data to identify the specific region R.

If we divide, the required region into vertical stripes and carefully find the end points for x and y i.e. the limits of the region, then we can use the formula Then express I in any one of the following form. $I = \int_a^b \int_{f(x)}^{g(x)} f(x, y) dy dx$ If we divide, the required region into vertical stripes and carefully find the end points for x and y. $I = \int_c^d \int_{f(y)}^{g(y)} f(x, y) dx dy$

Problem 1.3.1. *Evaluate* $\int\int xy(x + y) dx dy$ *over the region between* $y = x$ *and* $y = x^2$ *(VTU Jan 2015, Dec 2010, Jan 2010)* .

Solution: Let I=$\int\int xy(x+y)dxdy$ Since $y=x^2$ and $y=x$

$x^2=x$ which gives us point of intersection $(0,0)$and $(1,1)$

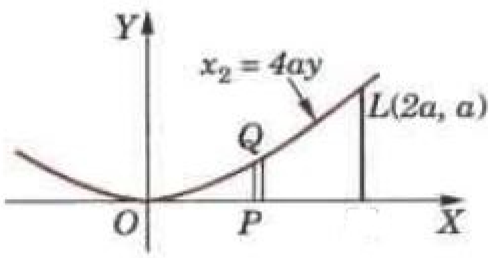
Clearly x varies from 0 to 1

and y varies from x^2 to x.

$$\begin{aligned}
\therefore I &= \int_0^1 \int_{x^2}^{x} (x^2y+xy^2)dydx \\
&= \int_0^1 [(\frac{x^2y^2}{2}+\frac{xy^3}{3})]_{x^2}^{x} dx \\
&= \int_0^1 [(\frac{x^4}{2}-\frac{x^6}{2}+\frac{x^4}{3}-\frac{x^7}{3})]dx \\
&= \left[(\frac{x^5}{10}-\frac{x^7}{14}+\frac{x^5}{15}-\frac{x^8}{24})\right]_0^1 \\
&= \frac{3}{56}
\end{aligned}$$

Problem 1.3.2. *Evaluate* $\int\int_R xydxdy$, *where R is the region bounded by* x*- axis, the ordinate* $x=2a$ *and* $x^2=4ay$ *(VTU Jan 2016)*

Solution: The line $x=2a$ and the parabola $x^2=4ay$ intersect at L.



When $\mathrm{x}=2\mathrm{a}$ and $\mathrm{x}^2=4$ ay $\therefore$ $4\mathrm{a}^2=4\mathrm{ay} \Rightarrow \mathrm{y}=\mathrm{a}$ $\therefore$ The point of intersection of $x=2a$ and $x^2=4ay$ is $L(2a,a)$

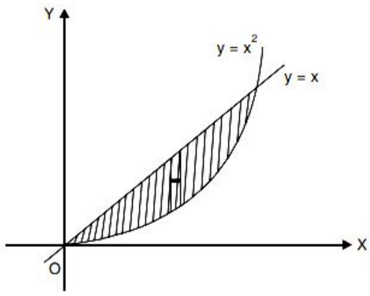
Integrating first over a vertical strip PQ, i.e., w.r.t. y from $P(y=0)$ to Q $(y=x^2/4a)$

on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xydxdy = \int_{x=0}^{2a}\int_{y=0}^{\frac{x^2}{4a}} xydydx$$
$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} dx$$
$$= \int_0^{2a} \frac{x^5}{32a^2}dx$$
$$= \left[\frac{x^6}{32a^2 \times 6}\right]_0^{2a}$$
$$= \frac{a^4}{3}$$

Problem 1.3.3. *Evaluate* $\int\int xy(x+y)dxdy$ *over the region between* $y = x$ *and* $y = x^2$ *(VTU June 2018, Jan 2015, Dec 2010, Jan 2010)*

Solution: The bounded curves are $y = x^2$ and $y = x$.



The common points are given by solving the two equations. So, we have

$$x^2 = x = x(x-1) = 0$$
$$\Rightarrow x = 0 \text{ or } 1$$

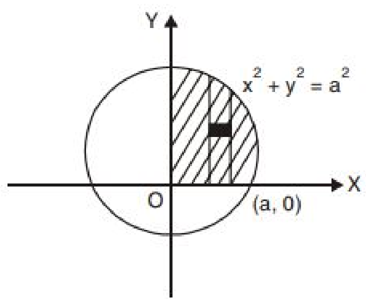
when $x = 0$, we have $y = 0$ and when $x = 1, y = 1$(from $y = x$) Considering a vertical strip y varies from $y = x^2$ to $y = x$ and then we slide the strip from

$x = 0$ to $x = 1$

$$\therefore \iint_R xy(x+y)dxdy = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)dydx$$
$$= \int_0^1 x\left[\frac{xy^2}{2}+\frac{y^3}{3}\right]_{x^2}^{x} dx$$
$$= \int_0^1 x\left\{x\left(\frac{x^2}{2}-\frac{x^4}{2}\right)+\left(\frac{x^3}{3}-\frac{x^6}{3}\right)\right\} dx$$
$$= \int_0^1 \left(\frac{5}{6}x^4-\frac{x^6}{2}-\frac{x^7}{3}\right) dx$$
$$= \left[\frac{5}{6}\frac{x^5}{5}-\frac{x^7}{14}-\frac{x^8}{24}\right]_0^1$$
$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24}=\frac{3}{56}.$$

Problem 1.3.4. *Evaluate* $\iint xy\mathrm{dx}\,dy$ *over the area in the first quadrant bounded by the circle* $x^2 + y^2 = a^2$

Solution:



$$\iint xydxdy = \int_{x=0}^{a}\left[\int_{y=0}^{\sqrt{a^2-x^2}} xydy\right]dx$$

$$\{\because x^2+y^2=a^2$$

$$\Rightarrow y^2 = a^2 - x^2$$

$$y = \sqrt{a^2-x^2}$$

$$= \int_0^a x\cdot\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x\left(\frac{a^2-x^2}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a \left(a^2x - x^3\right)dx$$

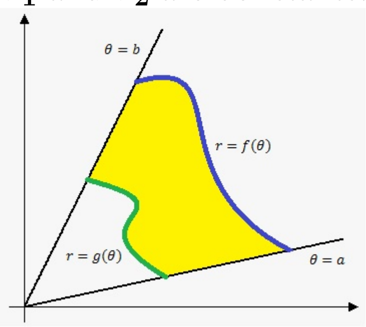
$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= 1/2\left[a^4/2 - a^4/4\right]$$

$$= a^4/8$$

1.4 Double Integrals in Polar Coordinates

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)drd\theta$, we first integrate w.r.to. r between the limits $r = r_1$ and $r = r_2$, treating θ as a constant and then the resulting expression is integrated w.r.to. θ from θ_1 to θ_2. In this integral r_1 and r_2 are functions of θ and θ_1 and θ_2 are constants.



1.5 Evaluation of Double integrals by changing in to polar form

Let (r, θ) be the polar co-ordinates of the point (x,y).Where $x = rcos\theta$,$y = rsin\theta$ and $r^2 = x^2 + y^2$

Then $\int\int_R dxdy = \int\int_R rdrd\theta$

$\because dxdy = Jdrd\theta$

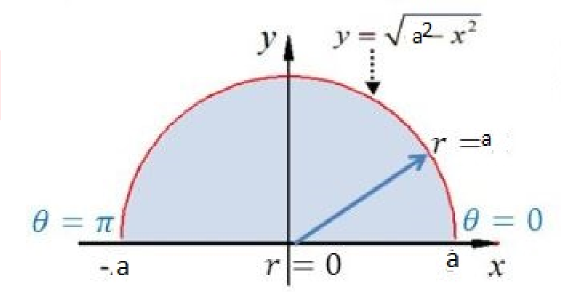
Where $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$

Problem 1.5.1. *Evaluate* $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}dxdy$ *by changing in to polar form.*

Solution: In the given Region,

y varies from 0 to $\sqrt{a^2 - x^2}$.

x varies from $-a$ to a



After changing to polar form :

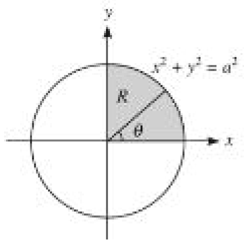
r varies from 0 to a

$\boldsymbol{\theta}$ varies from 0 to $\boldsymbol{\pi}$. Where $\boldsymbol{x = rcos\theta}$, $\boldsymbol{y = rsin\theta}$ and
$\boldsymbol{r^2 = x^2 + y^2, dxdy = rdrd\theta}$

$$\therefore \boldsymbol{I} = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dxdy$$
$$= \int_{0}^{a}\int_{0}^{\pi} \sqrt{r^2}\,rdrd\theta$$
$$= \int_{0}^{a}\int_{0}^{\pi} r^2 drd\theta$$
$$= \int_{0}^{a} \pi r^2 dr$$
$$= \frac{\pi a^3}{3}$$

Problem 1.5.2. *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dydx$.

Solution: Let $\boldsymbol{I} = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dydx$
Here $\boldsymbol{y}$ varies from $\boldsymbol{y = 0}$ to $\boldsymbol{y = \sqrt{a^2 - x^2}}$ i.e. $\boldsymbol{x^2 + y^2 = a^2}$
and $\boldsymbol{x}$ varies from $\boldsymbol{x = 0}$ to $\boldsymbol{x = a}$ (see the Figure)



Putting $\boldsymbol{x = r\cos\theta, y = \pi\sin\theta}$, we get $\boldsymbol{r = a}$.
Therefore in the given region $\boldsymbol{R}$ (which is first quadrant) $\boldsymbol{r}$ varies from $\boldsymbol{r = 0}$ to a

and θ varies from $\theta = 0$ to $\frac{\pi}{2}$. Hence

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{a} r^2 \sin^2 \theta r r dr d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi} r^4 \sin^2 \theta dr d\theta \\ &= \frac{a^5}{5} \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2} \right) \theta \\ &= \frac{a^5 \pi}{20} \end{aligned}$$

Problem 1.5.3. *Evaluate* $\int_0^{\infty} \int_0^{\infty} e^{-\left(x^2+y^2\right)} dx dy$ *using polar cordinates*

Solution:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad dx dy = r\, dr d\theta$$

are the polar coordinates for the above integral.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} \mathrm{e}^{-\mathrm{r}^2} r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} \mathrm{e}^{-\mathrm{r}^2} \frac{1}{2} d\left(r^2\right) d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} \left[-\mathrm{e}^{-\mathrm{r}^2} \right]_0^{\infty} d\theta \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{2}} \left[e^{-r^2} \right]_0^{\infty} d\theta \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{2}} [0 - 1] d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{1}{2} [\theta]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right] \\ &= \frac{\pi}{4} \end{aligned}$$

1.6 Application of multiple integrals to find area and volume

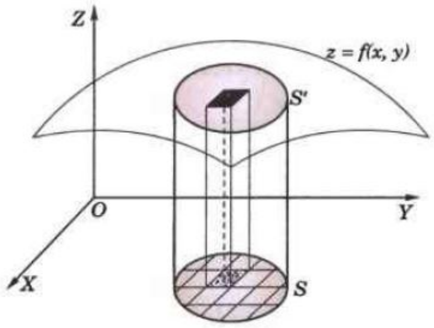
Area enclosed by a plane curve in Cartesian form:

$$\text{Area} = \iint_R dxdy$$

Area enclosed by a plane curve in polar form:

$$\text{Area} = \iint_R rdrd\theta$$

Consider a surface $z = f(x, y)$. Let the orthogonal projection of this surface on XY-plane be the area S

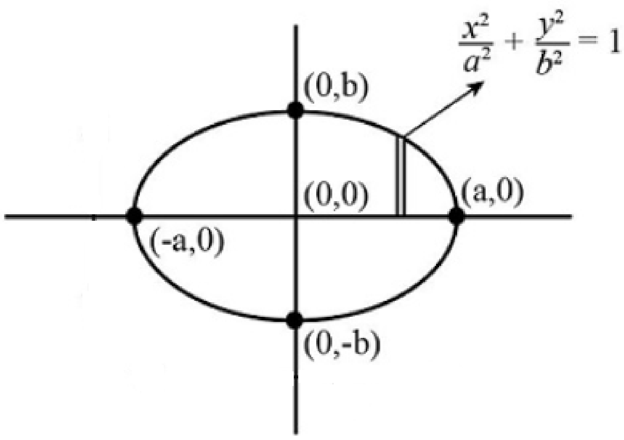


Then the volume of the solid bounded by $z = f(x, y)$ and its projection R on the XY plane is given by

$$\boxed{\text{Volume} = \iint z\ dxdy} \text{ or } \iint f(x, y)\ dxdy$$

Problem 1.6.1. *Find the area of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

solution This ellipse is symmetric w.r.t x- axis and y- axis.

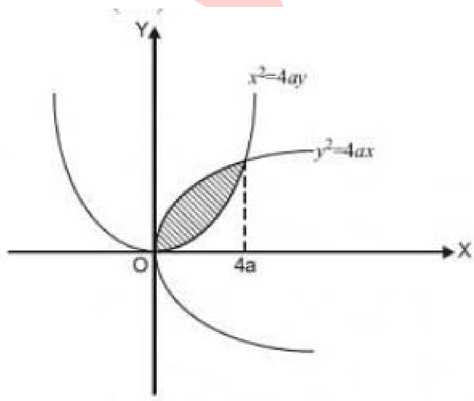


$\therefore$ Area of the ellipse = 4 times Area of the region in the first quadrant.
In this region x varies from 0 to a and y varies from 0 to $\frac{b\sqrt{a^2-x^2}}{a}$

$$\therefore \text{Area of ellipse} = 4[\int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{a}} dydx]$$
$$= 4\left[\int_0^a \frac{b\sqrt{a^2-x^2}}{a} dx\right]$$
$$putx = asin\theta \therefore dx = acos\theta d\theta$$
$$whenx = 0, \theta = 0andwhenx = a, \theta = \frac{\pi}{2}$$
$$\therefore Areaofellipse = 4[\int_0^{\frac{\pi}{2}} [\frac{b\sqrt{a^2 - a^2sin^2\theta}acos\theta}{a}]d\theta]$$
$$A = 4ab\int_0^{\frac{\pi}{2}} cos^2\theta d\theta$$
$$A = 4ab\frac{\pi}{4} = \pi ab sq.units.$$

Problem 1.6.2. *Show that the area between the parabolas* $y^2 = 4ax$ *and* $x^2 = 4ay$ *is* $\frac{16}{3}a^2$.

Solution:



Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at $O(0,0)$ and $A(4a, 4a)$. As such for the shaded area between these parabolas, considering vertical strip PQ, y varies from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$

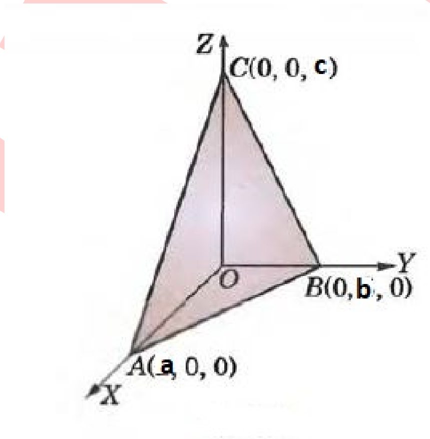
and x varies from 0 to $4a$.

Hence the required area is

$$\begin{aligned} Area &= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dydx \\ &= \int_0^{4a}\left(2\sqrt{(ax)} - x^2/4a\right)dx \\ &= \left|2\sqrt{a}\cdot\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a} \\ &= \frac{32}{3}a^2 - \frac{16}{3}a^2 \\ &= \frac{16}{3}a^2. \end{aligned}$$

Problem 1.6.3. *Find the volume of the tetrahedron bounded by the coordinate surfaces $x = 0, y = 0$ and $z = 0$ and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.*

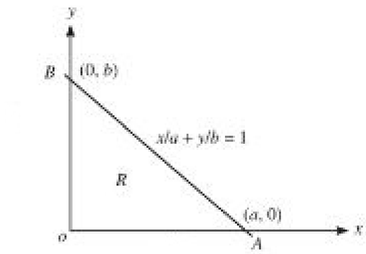
Solution:



The volume V of the terahedron is given by

$$V = \iint_R zdxdy$$

where R is the triangular region in the xy-plane (Figure) whose sides are $x = 0, y = 0$ and $x/a + y/b = 1$.

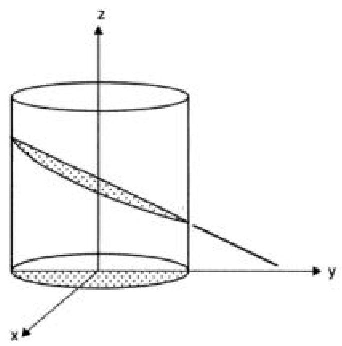


Therefore,

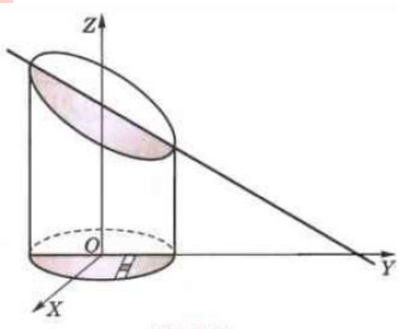
$$\begin{aligned}
V &= \iint_R z\,dx\,dy \\
&= \int_0^a \int_0^{b(1-\frac{x}{a})} \left(1 - \frac{y}{b} - \frac{x}{a}\right) dy\,dx \\
&= \int_0^a \left[y - \frac{y^2}{2b} - \frac{xy}{a} \right]_0^{b-\frac{bx}{a}} dx \\
&= \int_0^a \left(\frac{bx^2}{2a^2} - \frac{bx}{a} + \frac{b}{2} \right) dx \\
&= \frac{abc}{6} \quad \text{(after simplification)}
\end{aligned}$$

Problem 1.6.4. *Find the volume bounded by the cylinder* $x^2 + y^2 = 4$ *and the planes* $y + z = 4$ *and* $z = 0$.

Solution:



or



The projection R is half of the the circle $x^2 + y^2 = 4$ in the xy-plane.

To find the required volume, $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy-plane.

In the region R in the xy-plane, x varies from 0 to $\sqrt{4 - y^2}$ and y varies from -2

to $\mathbf{2}$.

Thus,

$$
\begin{aligned}
V &= 2\int_{-2}^{2}\left[\int_{0}^{\sqrt{4-y^2}} z\,dx\right]dy \\
&= 2\int_{-2}^{2}\left[\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\right]dy \\
&= 2\int_{-2}^{2}(4-y)[x]_0^{\sqrt{4-y^2}}dy \\
&= 2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy \\
&= 2\left[4\int_{-2}^{2}\sqrt{4-y^2}\,dy-\int_{-2}^{2}y\sqrt{4-y^2}\,dy\right] \\
&= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy \quad \text{[The second term vanishes as the integrand is an odd function.]} \\
&= 16\int_{0}^{2}\sqrt{4-y^2}\,dy \quad \text{(as the integrand is an even function)} \\
&= 16\left[\frac{y\sqrt{4-y^2}}{2}+\frac{4}{2}\sin^{-1}\frac{y}{2}\right]_0^2 \\
&= 16\left[2\sin^{-1}1\right]=\frac{32\pi}{2} \\
&= 16\pi.
\end{aligned}
$$

1.7 Evaluation of Double integral by changing the order of integration

In a double integral with variable limits, the change of order of integration changes the limits of integration. i.e. The double integral $\int_a^b\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$ takes the equivalent form $\int_c^d\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$ or vice versa. This process of converting a given double integral into its equivalent double integral by changing the order of integration is often called change of order of integration or reversing the order of integration. Some of the problems connected with double integrals, which are complicated, can be made easy to evaluate by a change in the order of integration. one such double integral is, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy$. Here it is difficult to integrate $\frac{e^{-y}}{y}$ first w.r.to y.

The following steps are used in change of order of integration.

- Use the limits of given integral and draw a a rough sketch of the region of integration.
- Then use the sketch to figure out how to write the limits of integration in the reverse order.
- In order to do that, you will have to write down on your sketch the equations of the various sides of the sketch, written in a suitable form depending on whether you are viewing the region as Type I or Type II.
- Note that this process does not change the function, which you are integrating. It changes only the limits of integration.
- Sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits.

Problem 1.7.1. *Evaluate* $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dxdy$ *by changing the order of integration.*

Solution: Given region:

x varies from 0 to 1

y varies from 0 to $\sqrt{1-x^2}$.

After changing the order :

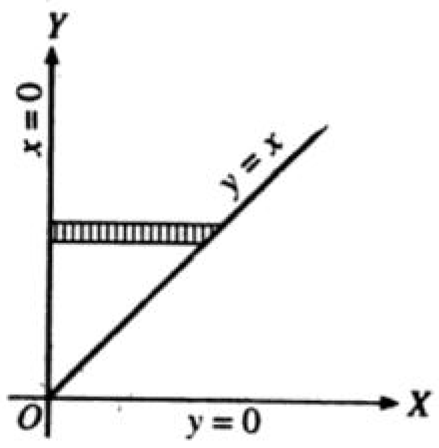
x varies from 0 to $\sqrt{1-y^2}$

y varies from 0 to 1.

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 dx dy$$
$$= \int_0^1 [y^2 x]_0^{\sqrt{1-y^2}} dy$$
$$= \int_0^1 [y^2\sqrt{1-y^2}] dy$$
$$y = \sin\theta \implies dy = \cos\theta d\theta$$
θ varies from 0 to $\frac{\pi}{2}$
$$\therefore I = \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta$$
$$I = \frac{\pi}{16}$$

Problem 1.7.2. *Change the order of integration and evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy$

Solution: In the given integral the limits of integration area given by the lines $y = x$, $y = \infty, x = 0$ and $x = \infty$ and the first integration is w.r. to y.
Therefore the region of integration is bounded $y = x$, $y = \infty, x = 0$ and $x = \infty$



If we want to reverse the order of integration, we have to first integrate w,r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by considering strip parallel to the x-axis.
Starting from the line $x = 0$ and terminating on the line $x = y$(new inner limits)

we slide the strip from $\mathbf{y} = 0$ are to $y = \infty$.(new outer limits).

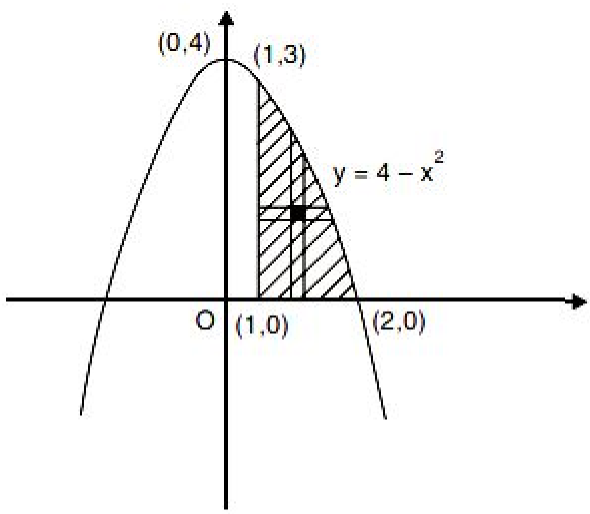
$$\begin{aligned}\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx dy &= \int_0^\infty \int_0^y \frac{e^{-y}}{y} dy dx \\ &= \int_0^\infty \frac{e^{-y}}{y} [x]_0^y dy \\ &= \int_0^\infty \frac{e^{-y}}{y} \cdot y dy \\ &= \int_0^\infty e^{-y} dy \\ &= \left[\frac{e^{-y}}{-1}\right]_0^\infty \\ &= 1\end{aligned}$$

Problem 1.7.3. *Change the order of integration and hence evaluate*

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy.$$

Solution: Here, the limits for x are $x = 1$ and $x = \sqrt{4-y}$ and limits for y are 0 and 3.

$x = \sqrt{(4-y)} \Rightarrow x^2 = 4 - y \Rightarrow y = 4 - x^2$, which is a parabola.



When $x = 1$, on $y = 4 - x^2, y = 3$

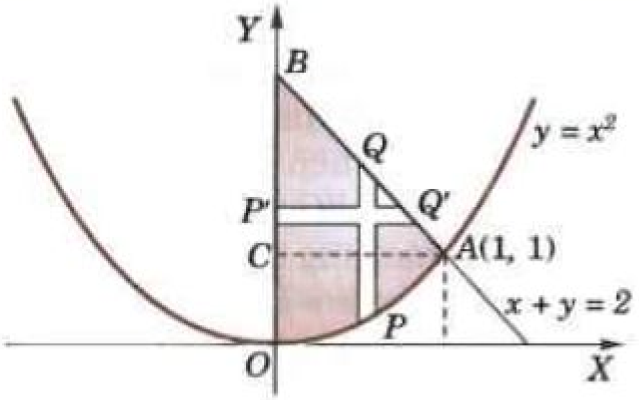
On changing the order, considering a horizontal strip, new inner limits are $y = 0$ to

$y = 4 - x^2$ and new outer limits are $y = 0$ to $y = 3$

Now,
$$\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\,dxdy = \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\,dydx$$
$$= \int_{1}^{2} \left[xy + y^2/2\right]_0^{4-x^2} dx$$
$$= \int_{1}^{2} \left(4x - x^3 + 8 - 4x^2 + \frac{x^4}{2}\right) dx$$
$$= \left[2x^2 - \frac{x^4}{4} + 8x - \frac{4}{3}x^3 + \frac{x^5}{10}\right]_1^2$$
$$= 6 - \frac{15}{4} + 8 - \frac{28}{3} + \frac{31}{10}$$
$$= \frac{241}{60}$$

Problem 1.7.4. *Change the order of integration in* $I = \int_0^1 \int_x^{2-x} xy\,dxdy$ *and hence evaluate the same.*

Solution: Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as in the figure.



On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line $AC(y = 1)$,

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region

OAC is

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xydx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xydx$$

Hence, on reversing the order of integration,

$$\begin{aligned}
I &= \int_0^1 dy \int_0^{\sqrt{y}} xydx + \int_1^2 dy \int_0^{2-y} xydx \\
&= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y} \\
&= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy \\
&= \frac{1}{6} + \frac{5}{24} \\
&= \frac{3}{8}.
\end{aligned}$$

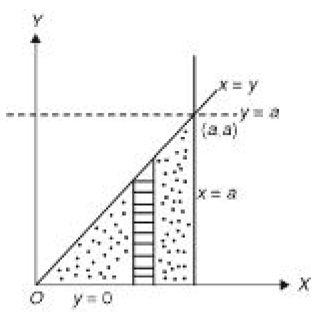
Problem 1.7.5. *Evaluate the integral* $\int_0^{\pi} \int_a \frac{x}{x^2+y^2} dxdy$ *by changing the order of integration.*

Solution: The given integral is

$$I = \int_0^a \int_a^y \frac{x}{x^2+y^2} dxdy.$$

Here inner limits are $x = a$ to $x = y$ and outer limits are $y = 0$ to $y = a$, and the first integration is w.r.to y

The region of integration is bounded by the lines $x = y, x = a, y = 0$ and $y = a$. Thus the region of integration is shown in the figure below:



On changing the order of integration, we first integrate with respect to y along the strip parallel to y axis. The strip extends from $y = 0$ to $y = x$(new inner limits). To cover the whole region, we then integrate with respect to x from $x = 0$ to $x = a$(new outer limits).

Hence

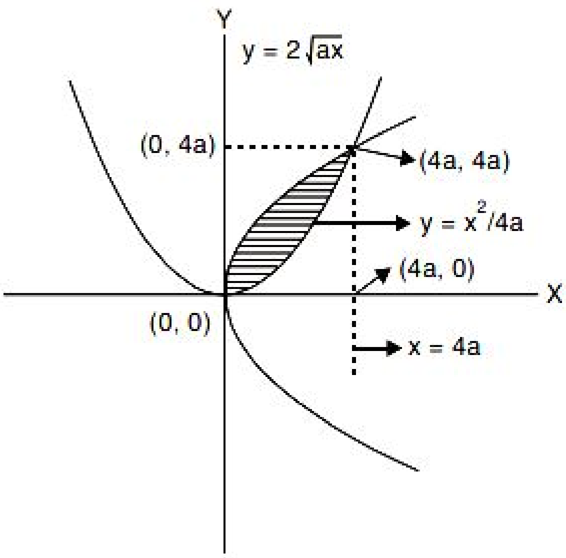
$$I = \int_0^a \int_0^x \frac{x}{x^2 + y^2} dxdy = \int_a^a x \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$
$$= \int_0^a \frac{\pi}{4} dx = \frac{\pi}{4}a.$$

Problem 1.7.6. *Change the order of integration and hence evaluate*

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xydydx$$

(VTU Jan 2017, Model 2015)

Solution:



Let us first find the point of intersection :

We have $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$

or $\frac{x^2}{4a} = 2\sqrt{ax}$

i.e. $x^4 = 64a^3x$

i.e., $x\left(x^3 - 64x^3\right) = 0$

$\Rightarrow x = 0$ and $x = 4a$

From $y = x^2/4a$, we get $y = 0$ and $y = 4a$

Thus the points of intersection of the parabola $y = x^2/4a$ and $y = 2\sqrt{ax}$ are $(0,0)$ and $(4a, 4a)$

on changing the order of integration we take a horizontal strip and first integrate w.r.to x

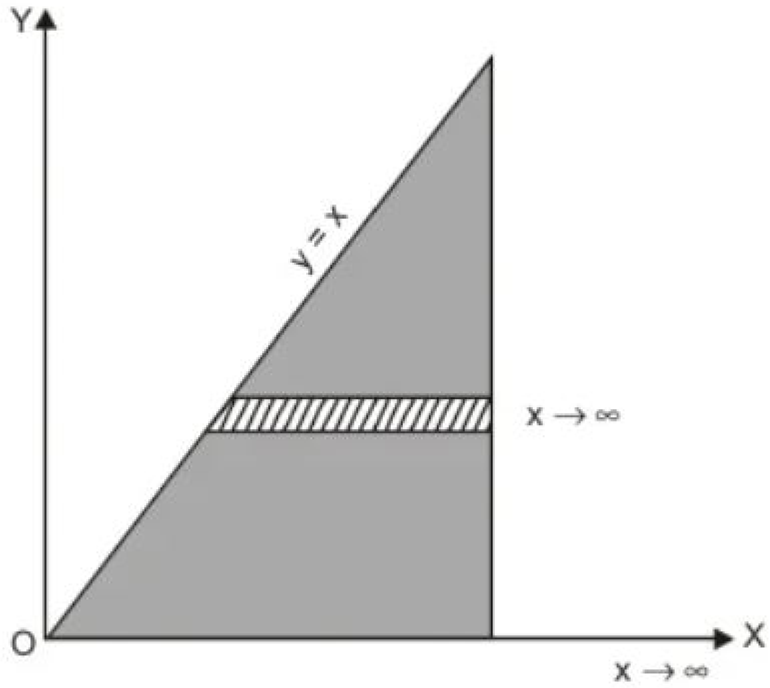
we have x varying from $y^2/4a$ to $2\sqrt{ay}$(new inner limits) and y varying from 0 to $4a$ (new outer limits)

Thus

$$
\begin{aligned}
I &= \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} xy dx dy \\
&= \int_{y=0}^{4a} y \cdot \left[\frac{x^2}{2}\right]_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy \\
&= \frac{1}{2} \int_{y=0}^{4a} y \left[4ay - \frac{y^4}{16a^2}\right] dy \\
&= \frac{1}{2} \int_{y=0}^{4a} \left(4ay^2 - \frac{y^5}{16a^2}\right) dy \\
&= \frac{1}{2} \left[\frac{4ay^3}{3} - \frac{1}{16a^2} \cdot \frac{y^6}{6}\right]_{y=0}^{4a} \\
&= \frac{1}{2} \left[4a\left(\frac{64a^3}{3}\right) - \frac{1}{96a^2}\left(4096a^6\right)\right] \\
&= \frac{1}{2} \left[\frac{256a^3}{3} - \frac{128a^4}{3}\right] \\
&= \frac{64a^4}{3}
\end{aligned}
$$

Problem 1.7.7. *Evaluate the integral* $\int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx dy$ *by changing the order of integration.* *(VTU July 2016, July 2009, Jan 2009, July 2008, July 2007)*

Solution: Here $y = 0$ and $y = x$, $x = 0$ and $x = \infty$



On changing the order, we consider a horizontal strip.

Here x start from $x = y$ and goes to $x \to \infty$ and y varies from $y = 0$ to $y \to \infty$

$$\therefore \int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dxdy$$

$$= \int_{y=0}^\infty \int_{x=y}^\infty xe^{-\frac{x^2}{y}} \cdot dydx$$

Put $e^{-\frac{x^2}{y}} = t \Rightarrow -\frac{2x}{y}e^{-\frac{x^2}{y}} dx = dt$

$$\therefore \int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dxdy = \int_0^\infty \left[-\frac{y}{2}e^{-\frac{x^2}{y}}\right]_y^\infty dy$$

$$= \int_0^\infty \left[0 + \frac{y}{2}e^{-\frac{y^2}{y}}\right] dy = \int_0^\infty \frac{y}{2}e^{-y}dy$$

$$= \left[\frac{y}{2}\left(-e^{-y}\right) - \frac{1}{2}\left(e^{-y}\right)\right]_0^\infty$$

$$= (0 - 0) + \left(0 + \frac{1}{2}\right)$$

$$= \frac{1}{2}.$$

1.8 Beta and Gamma functions

Beta function : An expression of the form

$$B(m,n) = \int_{x=0}^{1} x^{(m-1)}(1-x)^{(n-1)}dx$$

where $m, n > 0$is called an Beta function.

Alternative definition of Beta function:

$$B(m,n) = 2\int_{0}^{\frac{\pi}{2}} \sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$

Gamma Function : An expression of the form

$$\Gamma(n) = \int_{x=0}^{\infty} e^{-x}\, x^{n-1}dx$$

where $n > 0$ is called an Gamma function.

Alternative definition of Gamma function:

$$\Gamma(n) = 2\int_{0}^{\infty} e^{-t^2}\, t^{2n-1}dt$$

1.9 Properties:

$$B(m,n) = B(n,m)$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n+1) = n!, \quad \text{when n is an integer}$$

Also

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Note: When n is a -ve fraction :

We have $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

Problem 1.9.1. *Evaluate* $\Gamma(4)$

Solution: Gamma integral $\Gamma(n+1) = n\,!$ where n is a positive integer.

$$\therefore \Gamma(4) = (4-1)! = 3! = 3 \times 2 = 6$$

Problem 1.9.2. *Evaluate* $\Gamma\left(\frac{9}{2}\right)$

Solution: We know $\Gamma(n+1) = n\Gamma(n)$

$$\begin{aligned}
\therefore \Gamma\left(\frac{9}{2}\right) &= \Gamma\left(\frac{7}{2}+1\right) \\
&= \frac{7}{2}\Gamma\left(\frac{7}{2}\right) \\
&= \frac{7}{2}\Gamma\left(\frac{5}{2}+1\right) \\
&= \frac{7}{2} \times \frac{5}{2} \times \Gamma\left(\frac{5}{2}\right) \\
&= \frac{7}{2} \times \frac{5}{2} \times \Gamma\left(\frac{3}{2}+1\right) \\
&= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{3}{2}\right) \\
&= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{1}{2}+1\right) \\
&= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \\
&= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \quad \left(\because \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right) \\
&= \frac{105}{16}\sqrt{\pi}
\end{aligned}$$

Problem 1.9.3. *Compute* $\Gamma(-\frac{1}{2})$

Solution: We have $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

$$\text{Put } n = \left(\frac{-1}{2}\right)$$

$$\Gamma(-1/2) = \frac{\Gamma(1/2)}{\frac{-1}{2}} = -2\sqrt{\pi}$$

Problem 1.9.4. *Show that* $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\mathrm{dx}$

Solution: We have

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$
$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \cdots\cdots\cdots (1)$$

Now consider

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Put $x = \frac{1}{y}$ so that $dx = -\frac{1}{y^2}dy$,

When $x = 1 \Rightarrow y = 1$ and $x \to \infty \Rightarrow y = 0$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\text{dx} = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right) \text{dy}$$
$$= \int_0^1 \frac{\frac{1}{y^{m-1}}}{(1+y)^{m+n}} \cdot \frac{1}{y^2}\text{dx}$$
$$= \int_0^1 \frac{1}{y^{m-1}} \frac{y^{m+n}}{(1+y)^{m+n}} \frac{1}{y^2}\text{dy}$$
$$= \int_0^1 \frac{y^{m+n-m+1-2}}{(1+y)^{m+n}}dy$$
$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy$$
$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Hence Equation (1) becomes

$$B(\text{m},\text{n}) = \int_0^1 \frac{x^{\text{m}-1}}{(1+x)^{\text{m}+n}}\text{dx} + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\text{dx}$$
$$\therefore B(\text{m},\text{n}) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{\text{m}+n}}\text{dx}$$

Relation Between Beta and Gamma function :

Problem 1.9.5. *Prove that* $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ *(VTU Jan 2017, July 2016, July 2013, July 2011, July 2008)*

Proof: We have

$$B(m,n) = 2\int_0^{\frac{\pi}{2}}[\sin^{2m-1}\theta cos^{2n-1}\theta d(\theta)]$$

$$\Gamma(n) = 2\int_0^{\infty}[e^{-x^2}x^{2n-1}dx]$$

$$\Gamma(m) = 2\int_0^{\infty}[e^{-y^2}y^{2m-1}dy]$$

$$\Gamma(m+n) = 2\int_0^{\infty}[e^{-r^2}r^{2(m+n)-1}dr]$$

$$Now\Gamma(m)\Gamma(n) = 2\int_0^{\infty}[e^{-x^2}x^{2n-1}dx \times 2\int_0^{\infty}[e^{-y^2}y^{2m-1}dy]]$$

$$= 4\int_0^{\infty}\int_0^{\infty}[e^{-(x^2+y^2)}x^{2n-1}y^{2m-1}dxdy]$$

Let us evaluate this double integral by changing in to polar co-ordinate systems.

put $x = rcos(\theta)$,$y = rsin(\theta)$, and $r^2 = x^2 + y^2$,$dxdy = rdrd(\theta)$

In the region of integration, r varies from 0 to ∞

θ varies from 0 to $\frac{\pi}{2}$.

$$\Gamma m\Gamma n = 4\int_0^{\infty}\int_0^{\frac{\pi}{2}}\left[e^{-r^2}r^{2n-1}r^{2m-1}\sin^{2m-1}\theta cos^{2n-1}\theta rdrd(\theta)\right]$$

$$= 2\int_0^{\infty}\left[e^{-r^2}r^{2(m+n)-1}dr\right]\ 2\int_0^{\frac{\pi}{2}}\left[\sin^{2m-1}\theta cos^{2n-1}\theta d(\theta)\right]$$

$$= \Gamma(m+n)B(m,n)$$

$$i.e.B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Hence the proof.

Problem 1.9.6. *S.T* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Solution: We know that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $\mathrm{m} > 0, \mathrm{n} > 0$

Taking $\mathrm{m} = \mathrm{n} = \frac{1}{2}$, we have

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$= \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \qquad \ldots\ldots(1) \qquad [\because \Gamma(1) = 1]$$

But using

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta cos^{2n-1}\theta d(\theta)$$

we can write

$$\begin{aligned}
B\left(\frac{1}{2},\frac{1}{2}\right) &= 2\int_0^{\frac{\pi}{2}} \sin^{2\times\frac{1}{2}-1}\theta cos^{2\times\frac{1}{2}-1}\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^0\theta cos^0\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} d\theta \\
&= 2[\theta]_0^{\frac{\pi}{2}} \\
&= 2\left[\frac{\pi}{2} - 0\right] \\
&= \pi \qquad \ldots\ldots(2)
\end{aligned}$$

From Equation (1) and (2),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

1.10 Evaluation of Integrals Using Beta and Gamma functions :

Some Useful Formulae :

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^p\theta cos^q\theta d(\theta) = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}}$$

$$\boxed{\Gamma(n)\Gamma(1-n) = \frac{\pi}{sinn\pi}}$$

In particular,

$$\boxed{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi}$$

Problem 1.10.1. *Show that* $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

Solution: We know that,

$$\int_0^{\pi/2} \sin^p \theta \cos^p \theta d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$
$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$
$$\int_0^{\frac{\pi}{2}} \sqrt{\cot \theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2} \theta}{\sin^{1/2} \theta} d\theta$$
$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2} \theta \cos^{1/2} \theta d\theta$$
$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$
$$= \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)}$$
$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)$$

Problem 1.10.2. *Evaluate* $\int_0^2 (4-x^2)^{\frac{3}{2}} dx$ *using Beta and Gamma function.*

Solution: *Let* $I = \int_0^2 (4-x^2)^{\frac{3}{2}} dx$

Put $x = 2sin\theta \; dx = 2cos\theta d\theta$

when $x = 0, \theta = 0$

and when $x = 2, \theta = \frac{\pi}{2}$

$\therefore I = \int_0^{\frac{\pi}{2}} [4 - 4\sin^2 \theta]^{\frac{3}{2}} 2cos\theta d\theta$

$\therefore I = 16 \int_0^{\frac{\pi}{2}} cos^4\theta d\theta$

$I = 16 \int_0^{\frac{\pi}{2}} sin^0\theta cos^4\theta d\theta$

$= 16 \frac{1}{2} B(\frac{1}{2}, \frac{5}{2})$

$= 8[\frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})+(\frac{5}{2})}]$

$= 8 \frac{3\pi}{\Gamma 3}$

$= \frac{24\pi}{8} = 3\pi$

Problem 1.10.3. *S.T.* $\Gamma(n) = \int_0^1 (\log 1/x)^{n-1} dx, n > 0$

Solution: We have $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$

Putting $x = \log\frac{1}{y} = -\log y$

(or) $y = e^{-x}$ sothat $dy = -e^{-x}dx$

$$dx = \frac{-1}{y}dy$$

Equation (1) becomes

$$\Gamma(n) = -\int_1^0 (\log\frac{1}{y})^{n-1}\cdot y\cdot\frac{1}{y}dy$$
$$= \int_0^1 (\log\frac{1}{y})^{n-1}dy$$
$$\therefore \Gamma(n) = \int_0^1 (\log\frac{1}{x})^{n-1}dx$$

Problem 1.10.4. *Prove that*

$$B(m,n) = B(m+1,n) + B(m,n+1)$$

Solution:

$$\begin{aligned}\text{R.H.S.} &= B(m+1,n) + B(m,n+1)\\ &= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}\\ &= \frac{m\Gamma(m)\cdot\Gamma(n)+\Gamma(m)n\Gamma(n)}{\Gamma(m+n+1)}\\ &= \frac{\Gamma(m)\,\Gamma(n)\quad(m+n)}{(m+n)\Gamma(m+n)}\\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\\ &= B(m,n)\\ &= \text{L.H.S.}\end{aligned}$$

Problem 1.10.5. *Prove that* $\Gamma(n)\Gamma\left(n+1/2\right) = 2^{1-2n}\sqrt{\pi}\Gamma(2n)$.

Solution:

By the definition of beta function, we have

$$B(n,n) = 2\int_0^{\pi/2} (\sin\theta)^{2n-1}(\cos\theta)^{2n-1}d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\ \cos\theta)^{2n-1}d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2n-1} d\theta$$

$$= \frac{2}{2^{2n-1}}\int_0^{\pi/2} (\sin 2\theta)^{2n-1}d\theta$$

put $2\theta = \phi$

$$= 2^{1-2n}\int_0^{\pi} (\sin\phi)^{2n-1}d\phi$$

$$B(n,n) = 2^{1-2n}\cdot 2\int_0^{\pi/2} (\sin\phi)^{2n-1}d\phi$$

$$= 2^{1-2n}\cdot B(n,1/2)$$

Converting both sides to Gamma function

$$\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = 2^{1-2n}\cdot B(n,1/2)$$

$$= 2^{1-2n}\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)}$$

rearranging the terms

$$\Gamma(n)\Gamma\left(n+1/2\right) = 2^{1-2n}\sqrt{\pi}\Gamma(2n)$$

Problem 1.10.6. *Express the following in terms of gamma functions* $\int_0^{\pi/2}\sqrt{(\tan\theta)}d\theta$

Solution:

$$\int_0^{\pi/2} \sin^p\theta \cos^p\theta d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$
$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$
$$\therefore \int_0^{\pi/2} \sqrt{(\tan\theta)}d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta d\theta$$
$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$
$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}$$
$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Problem 1.10.7. *Prove that* $\int_0^1 \frac{x^2dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

Solution:

$$I_1 = \int_0^1 \frac{x^2dx}{\sqrt{(1-x^4)}}$$
$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{2\sqrt{(\sin\theta)}}d\theta$$

(Putting $x^2 = \sin\theta, dx = \frac{\cos\theta d\theta}{2\sqrt{(\sin\theta)}}$)

$$\therefore I_1 = \frac{1}{2}\int_0^{\pi/2} \sqrt{(\sin\theta)}d\theta$$
$$= \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$
$$= \frac{1}{4}\frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)}$$
$$= \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(1/4)}$$

$$
\begin{aligned}
I_2 &= \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} \\
&= \int_0^{\pi/4} \frac{\sec^2\theta d\theta}{2\sqrt{(\tan\theta)}\sec\theta} \quad \left(\text{ Putting } x^2 = \tan\theta, dx = \frac{\sec^2\theta d\theta}{2\sqrt{(\tan\theta)}}\right) \\
&= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin 2\theta)}} \\
&= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\phi d\phi \quad \left(\text{ Putting } 2\theta = \phi, d\theta = \frac{1}{2}d\phi\right) \\
&= \frac{1}{4\sqrt{2}} \beta\left(\frac{1}{4}, \frac{1}{2}\right) \\
&= \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} \\
\therefore \int_0^1 \frac{x^2dx}{\sqrt{(1-x^4)}} &\times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{\pi}{4\sqrt{2}}
\end{aligned}
$$

1.11 Question Bank : Module 1-Integral Calculus

1.11.1 Question Bank :Evaluation of double integrals

1) $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y dx dy$ (VTU Jan 2017)

2) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) dx dy$ (VTU July 2011)

3) $\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r dr d\theta dz$ (VTU July 2009)

1.11.2 Question Bank :Triple Integrals

1) $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dy dx dz$ **Ans:** 0
(VTU Jan 2019, Jan 2017, Jan 2016, Jan 2015, July 2011, July 2009)

2) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$ **Ans:** $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$
(VTU July 2016, Dec 2010, Jan 2010, July 2007)

3) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz dz dy dx$ (VTU July 2016)

4) $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dxdydz$ (VTU June 2019, Model 2018, July 2017, Jan 2016, July 2014, July 2013)

5) $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dydxdz$ **Ans:**8π (VTU Jan 2017)

6) $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dzdydx$ (VTU July 2016, July 2008)

7) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyzdzdydx$ (VTU Jan 2020, June 2018, Jan 2013, Dec 2010, Jan 2008)

8) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}dzdydx$ (VTU Dec 2011, July 2008)

9) $\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyzdzdydx$ (VTU Jan 2018)

10) $\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{1}{a^2-x^2-y^2-z^2}dzdydx$ (VTU July 2017)

11) Evaluate $\int_1^3\int_{y=\frac{1}{x}}^1\int_{z=0}^{\sqrt{x}} xyzdzdydx$ (VTU July 2017)

12) $\int_1^e\int_1^{log_e y}\int_1^{e^x} logzdzdxdy$ (VTU July 2017)

1.11.3 Question Bank :Evaluation of Double integral when Region is Given

1) Evaluate $\int\int_R xydxdy$, where R is the region bounded by x- axis, the ordinate $x=2a$ and $x^2=4ay$ (VTU Jan 2016)

2) $\int\int xy(x+y)dxdy$ over the region between $y=x$ and $y=x^2$ (VTU June 2018, Jan 2015, Dec 2010, Jan 2010)

3) Evaluate $\int\int ydxdy$ over the region bounded by the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
(VTU Jan 2020, Jan 2013)

1.11.4 Question Bank :Change of Variables

Evaluate by changing to polar coordinates.

1. $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\,dxdy$ (VTU Jan 2018)
2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$ (VTU Jan 2019, Model 2018,July 2017, Jan 2017, July 2016, Jan 2016, July 2013, Dec 2011)
3. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dxdy$ (VTU July 2015)

1.11.5 Question Bank :Changing the order of integration

Change the order of integration of the following Integrals & hence evaluate them :

1. $\int_0^1 \int_x^{\sqrt{x}} xy\,dydx$ (VTU Jan 2019, July 2016, June 2010, Jan 2008)
2. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dxdy$ (VTU July 2011)
3. $\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\,dxdy$ (VTU July 2016)
4. $\int_0^1 \int_{x^2}^{2-x} xy\,dydx$ (VTU July 2017, Jan 2016, Dec 2011)
5. $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dydx$ (VTU Jan 2017, Model 2015)
6. $\int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{ax}} dydx$ (VTU July 2014)
7. $\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}}\,dxdy$ (VTU July 2015)
8. $\int_0^\infty \int_0^x xe^{\frac{-x^2}{y}}\,dydx$ (VTU July 2016, July 2009, Jan 2009, July 2008, July 2007)
9. $\int_0^1 \int_{x^2}^{2-x} xy\,dxdy$ (VTU Jan 2012, July 2008)
10. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$ (VTU June 2018, Dec 2010)
11. $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dxdy$ (VTU Model 2018, Jan 2018, July 2017, Jan 2008)
12. $\int_0^a \int_0^{2\sqrt{ax}} x^2\,dydx$ (VTU June 2019, Jan 2013)
13. $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\,dydx$ (VTU Jan 2014)

14. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dxdy$ (VTU July 2012)
15. $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} (2-x)dydx$ (VTU Jan 2020)

1.11.6 Question Bank :Area and Volume

1. By using double integral, find the area bounded by the coordinate axes and the line $x+y=1$
(VTU July 2017)
2. Show that area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$ (VTU June 2018,Jan 2016)
3. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant (VTU July 2016)
4. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration. (VTU June 2019, Jan 2018, July 2016)
5. Find the area lying inside the circle $r = asin\theta$ and outside the cardioid $r = a(1 - cos\theta)$
(VTU Jan 2017)
6. Find the area enclosed by the curve $r = a(1 + Cos\theta)$ above the initial line(or between $\theta = 0$ and $\theta = \pi$) (VTU Jan 2020, July 2016)
7. Using Double integration, find the area of a loop of the lemniscate $r^2 = a^2cos2\theta$ (VTU Model 2018)
8. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ using triple integral.(or find the volume of a sphere of radius a using triple integral). (VTU July 2017, Jan 2017, Jan 2009)
9. Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
(VTU July 2016)

10. Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, x + y + z = 1$ and $z = 0$
    (VTU July 2015)

11. Find the area bounded by the circle $x^2 + y^2$ and the line $x + y = a$
    (VTU June 2019)

1.11.7 Question Bank :Beta and Gamma functions

1) Prove that $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
(VTU Jan 2020, June 2019, Jan 2019, June 2018, Model 2018, Jan 2018, July 2017, Jan 2017, July 2016, July 2013, July 2011, July 2008)

2) Prove that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{sin\theta}d\theta = \pi$
(VTU Jan 2020, June 2019, Jan 2019, Model 2018, Jan 2018, July 2017, July 2016, Jan 2015, Dec 2010, Jan 2008)

3) Evaluate $\int_0^a x^4\sqrt{a^2 - x^2}dx$ using Beta and Gamma function. (VTU July 2017, July 2016, July 2011)

4) Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^4}}dx$ using bea and Gamma functions. (VTU July 2015)

5) Define Beta and Gamma function. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (VTU June 2018, June 2017, Jan 2016)

6) For m and n positive, prove that $\beta(m, n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ (VTU Jan 2017, Dec 2011, Jan 2008)

7) Evaluate $\int_0^a y^4\sqrt{a^2 - y^2}dy$ using Beta and Gamma function. (VTU Jan 2017)

8) Prove that $\beta(m, \frac{1}{2}) = 2^{2m-1}\,\beta(m, m)$ (VTU Jan 2016)

9) Prove that $\int_0^\infty xe^{-x^8}dx \int_0^\infty x^2e^{-x^4}dx = \frac{\pi}{16\sqrt{2}}$ (VTU Jan 2013)

10) $\int_{-1}^1 (1 + x)^{m-1}(1 - x)^{n-1}dx = 2^{m+n-1}B(m, n)$ (VTU July 2011)

11) Prove that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$ (VTU Dec 2011)

12) Prove that $\sqrt{\pi}\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)$ (VTU June 2010)

13) Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{cot\theta}d\theta$ expressing in terms of Gamma Function. (VTU Jan 2010, July 2009)

14) Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx. \int_0^1 \frac{1}{\sqrt{1-x^4}}dx = \frac{\pi}{4}$ (VTU July 2016)

15) Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx. \int_0^1 \frac{1}{\sqrt{1+x^4}}dx = \frac{\pi}{4\sqrt{2}}$ (VTU July 2015, July 2014)

Module 2

Vector Calculus:

Syllabus:

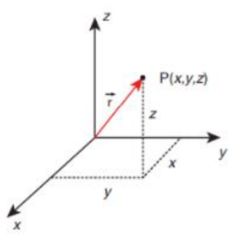
Introduction to Vector Calculus in Mechanical Engineering applications
Vector Differentiation: Scalar and vector fields. Gradient, directional derivative, curl and divergence – physical interpretation, solenoidal and irrotational vector fields. Problems. Vector Integration: Line integrals, Surface integrals. Applications to work done by a force and flux. Statement of Green's theorem and Stoke's theorem. Problems.
Self-Study: Volume integral and Gauss divergence theorem.
Applications: Heat and mass transfer, oil refinery problems, environmental engineering, velocity and acceleration of moving particles, analysis of streamlines. (RBT Levels: L1, L2 and L3)

2.1 Basic Concepts

- **Vector :** A vector is a physical quantity having both magnitude and direction. Example : Force, Acceleration, Velocity etc.

- A vector is represented by a directed line segment with an arrow over it. i.e. $\vec{PQ}$ is a vector with initial point P and terminal point Q whose direction is from P to Q and magnitude is the length PQ.

- **Position vector of a point P :**Position vector is a vector which represents the position of a point in a space with respect to the origin. The position vector of a point P(x,y,z) in space is given by $\boxed{\vec{r} = xi + yj + zk}$



- **Note :** If x, y, z are functions of a single variable 't' then the vector $\vec{r}$ is said to be vector point function of 't'(i.e. vector equation of a curve at any point 't'). i.e. $\vec{r} = r(\vec{t}) = x(t)i + y(t)j + z(t)k$.

- **Magnitude** of a vector $\vec{r}$ denoted by $|\vec{r}|$ or r is defined as $\boxed{r = \sqrt{x^2 + y^2 + z^2}}$

- **Dot product :** If $\vec{A} = a_1i + a_2j + a_3k$ and $\vec{B} = b_1i + b_2j + b_3k$ are two vectors, then the dot product is given by the formula,
$$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$$
Sometimes the dot product is called the scalar product.

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

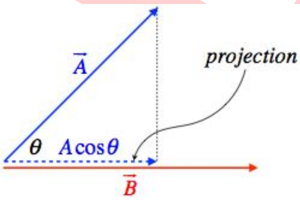
- $i \cdot i = j \cdot j = k \cdot k = 1$

- $i \cdot j = j \cdot k = k \cdot i = 0$

- If $\vec{A} = a_1i + a_2j + a_3k$ and $\vec{B} = b_1i + b_2j + b_3k$ then $\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$
- $i \times i = j \times j = k \times k = 0$
- $i \times j = k, j \times k = i, k \times i = j$
- $j \times i = -k, k \times j = -i, i \times k = -j$

- If θ is the angle between two vectors $\vec{A}$ and $\vec{B}$, then $cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$
- If $\vec{A} \cdot \vec{B} = 0$, then $\vec{A}$ and $\vec{B}$ are perpendicular.
- If $\vec{A} \times \vec{B} = 0$, then $\vec{A}$ and $\vec{B}$ are parallel.
- **Component of a vector along a given direction:** A **component** of a vector $\vec{A}$ along a given direction $\vec{B}$ is the resolved part of $\vec{A}$ and is given by $\boxed{\vec{A} \cdot \hat{n}}$ where $\boxed{\hat{n} = \dfrac{\vec{B}}{|\vec{B}|}}$



2.2 Vector differential operator

The vector differential operator is denoted by ∇ (Read it as **del** or **nabla**) and is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

2.3 The gradient of the scalar point function:

The gradient of the scalar point function Φ denoted by $grad\Phi$ or $\nabla\Phi$ is a vector function defined by

$$\begin{aligned} grad\,\Phi = \nabla\Phi &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\Phi \\ &= \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k \\ &= \sum\frac{\partial\Phi}{\partial x}i \end{aligned}$$

Note : Gradient of a scalar function is a vector quantity

2.4 Divergence of a vector point function :

The divergence of a vector point function $\vec{F} = f_1 i + f_2 j + f_3 k$ denoted by $divF$ is defined by,

$$\begin{aligned} div\,\vec{F} &= \nabla\cdot\vec{F} \\ &= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot(f_1 i + f_2 j + f_3 k) \\ &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ &= \sum\frac{\partial f_1}{\partial x} \end{aligned}$$

Note : Divergence of a vector point function is a scalar quantity

2.5 Curl of a vector point function

The curl of a continuously differentiable vector point function $\vec{F} = f_1i + f_2j + f_3k$ denoted by $curl\vec{F}$ is defined by

$$\begin{aligned} curl\ \vec{F} &= \nabla \times \vec{F} \\ &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (f_1i + f_2j + f_3k) \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \\ &= \sum \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i \end{aligned}$$

Note : Curl of a vector point function is a vector quantity.

Problem 2.5.1. *Find grad Φ when $\Phi = 3x^2y - y^3z^2$ at the point (1,-2,-1) (VTU July 2017)*

Solution:

$$\begin{aligned} \nabla\phi &= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left(3x^2y - y^3z^2\right) \\ &= \frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right)i + \frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right)j \\ &\quad + \frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)k \\ &= (6xy)i + \left(3x^2 - 3y^2z^2\right)j + \left(-2y^3z\right)k \qquad \cdots (1) \end{aligned}$$

Thus $\nabla\phi$ at the point $(1, -2, -1)$ is obtained by putting these values of x, y, z in $R.H.S.$ of (1) Thus

$\nabla\phi\ |_{(1,-2,-1)}$

$$\begin{aligned} &= 6(1)(-2)i + \left[3 - 3(-2)^2(-1)^2\right]j + \left[-2(-2)^3(-1)\right]k \\ &= -12i + j(3 - 12) + k(-16) \\ &= -12i - 9j - 16k \end{aligned}$$

Problem 2.5.2. *If $\Phi = log(x^2 + y^2 + z^2)$, then find $\nabla\Phi$ & $|\nabla\Phi|$ at (1 , 2 , 1)*

Solution: Given

$$\phi(x, y, z) = \log\left(x^2 + y^2 + z^2\right)$$

$$\text{grad}\,\phi = \nabla\phi$$

$$= \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get,

$$\frac{\partial\phi}{\partial x} = \frac{1}{x^2 + y^2 + z^2} \cdot 2x$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{x^2 + y^2 + z^2} \cdot 2y$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{x^2 + y^2 + z^2} \cdot 2z$$

At the point $(1, 2, 1)$

$$\frac{\partial\phi}{\partial x} = \frac{2 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{\partial\phi}{\partial y} = \frac{2 \cdot 2}{1^2 + 2^2 + 1^2} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{\partial\phi}{\partial z} = \frac{2 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore\ , \text{grad}\,\phi\,|_{(1,2,1)} = \frac{1}{3} i + \frac{2}{3} j + \frac{1}{3} k$$

$$= \frac{1}{3}[i + 2j + k]$$

$$|\ grad\phi\ | = \sqrt{\frac{1^2 + 2^2 + 1^2}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

Problem 2.5.3. *Find* $divF$ *and* $curlF$ *where* $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ **(VTU Jul 2015. Jul 2013, Jan 2008, Aug 2002)**

Solution: Let $\phi = x^3 + y^3 + z^3 - 3xyz$

The given expression can be written as

$$\begin{aligned}
\vec{F} &= grad\phi = \nabla\phi \\
&= \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k \\
&= \frac{\partial(x^3 + y^3 + z^3 - 3xyz)}{\partial x}i + \frac{\partial(x^3 + y^3 + z^3 - 3xyz)}{\partial y}j \\
&\quad + \frac{\partial(x^3 + y^3 + z^3 - 3xyz)}{\partial z}k \\
\vec{F} &= (3x^2 - 3yz)i + (3y^2 - 3xz) + (3z^2 - 3xy)k
\end{aligned}$$

$$\begin{aligned}
&div(\vec{F}) = \nabla \cdot \vec{F} \\
&= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \cdot ((3x^2 - 3yz)i + (3y^2 - 3xz) + (3z^2 - 3xy)k) \\
&= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy) \\
&= 6x + 6y + 6z \\
&= 6(x + y + z)
\end{aligned}$$

$$\begin{aligned}
curl(\vec{F}) &= \nabla \times \vec{F} \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix} \\
&= i\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right] \\
&\quad - j\left[\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right] \\
&\quad + k\left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right] \\
&= i(-3x + 3x) - j(-3y + 3y) + k(-3z + 3z) \\
&= 0i - 0j + 0k = \vec{0}
\end{aligned}$$

Problem 2.5.4. *If* $\Phi = 2x^3y^2z^4$*, find* $div(grad\Phi)$ *at (1,1,1)* *(VTU Jan 2018)*

Solution:

$$\begin{aligned}
\text{grad}\,\varphi &= \nabla\varphi \\
&= \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z} \\
&= \hat{i}\frac{\partial}{\partial x}\left(2x^3y^2z^4\right) + \hat{j}\frac{\partial}{\partial y}\left(2x^3y^2z^4\right) + \hat{k}\frac{\partial}{\partial z}\left(2x^3y^2z^4\right) \\
\text{grad}\,\varphi &= \hat{i}\left(6x^2y^2z^4\right) + \hat{j}\left(4x^3yz^4\right) + \hat{k}\left(8x^3y^2z^3\right)
\end{aligned}$$

and

$$\begin{aligned}
\text{div}(\text{grad}\,\varphi) &= \nabla\cdot\nabla\varphi \\
&= \frac{\partial}{\partial x}\left(6x^2y^2z^4\right) \\
&\quad + \frac{\partial}{\partial y}\left(4x^3yz^4\right) \\
&\quad + \frac{\partial}{\partial z}\left(8x^3y^2z^3\right) \\
&= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 \\
\text{div}(\text{grad}\,\varphi)\,|_{(1,1,1)} &= 12 + 4 + 24 = 40
\end{aligned}$$

Problem 2.5.5. *Find* $div\vec{F}$ *and* $curl F$ *at the point (1,2,3) if* $\vec{F} = grad(x^3y + y^3z + z^3x - x^2y^2z^2)$ *(VTU 2007)*

Solution: Let $\Phi = x^3y + y^3z + z^3x - x^2y^2z^2$

$$\begin{aligned}
\text{Given}\ \ \vec{F} &= \text{grad}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right) \\
&= \nabla(\Phi) \\
&= \frac{\partial}{\partial x}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right)\hat{i} \\
&\quad + \frac{\partial}{\partial y}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right)\hat{j} \\
&\quad + \frac{\partial}{\partial z}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right)\hat{k}
\end{aligned}$$

$$\begin{aligned}
\vec{F} &= \left(3x^2y + z^3 - 2xy^2z^2\right)\hat{i} + \left(x^3 + 3y^2z - 2x^2yz^2\right)j \\
&\quad + \left(y^3 + 3z^2x - 2x^2y^2z\right)\hat{k} \\
\therefore \text{div}\,\vec{F} &= \nabla \cdot \vec{F} \\
&= \frac{\partial}{\partial x}\left(3x^2y + z^3 - 2xy^2z^2\right) \\
&\quad + \frac{\partial}{\partial y}\left(x^3 + 3y^2z - 2x^2yz^2\right) \\
&\quad + \frac{\partial}{\partial z}\left(y^3 + 3z^2x - 2x^2y^2z\right) \\
&= 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6zx - 2x^2y^2 \\
\text{div}\,\vec{F}\,|_{(1,2,3)} &= 12 - 36 + 54 - 72 + 18 - 8 = -32
\end{aligned}$$

$$Curl(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \begin{matrix}(3x^2y + z^2 \\ -2xy^2z^2)\end{matrix} & \begin{matrix}(x^3 + 3y^2z \\ -2xy^2z^2)\end{matrix} & \begin{matrix}(y^3 + 3z^2x \\ -2x^2y^2z)\end{matrix} \end{vmatrix}$$

$$\begin{aligned}
&= i\left[3y^2 - 4x^2yz - 3y^2 + 4x^2yz\right] \\
&\quad - j\left[3z^2 - 4xy^2z - 3z^2 + 4xy^2z\right) \\
&\quad + k\left[3x^2 - 4xyz^2 - 3x^2 + 4xyz^2\right] \\
&= 0\hat{i} + 0\hat{j} + 0\hat{k} \\
&= \vec{0}
\end{aligned}$$

Problem 2.5.6. *Prove that* $\nabla \cdot \vec{r} = 3$ *and* $\nabla \times \vec{r} = 0$

Solution: (i)

$$\begin{aligned}
\nabla \cdot \vec{r} &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (xi + yj + zk) \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\
&= 1 + 1 + 1 = 3
\end{aligned}$$

$$\nabla \times \vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (xi + yj + zk)$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$
$$= i\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - j\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + k\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$
$$= 0i + 0j + 0k = \vec{0}$$

Problem 2.5.7. *If* $\vec{V} = \vec{w} \times \vec{r}$*, P.T.* $w = \frac{1}{2}(\nabla \times \vec{V})$ *or* $curl\vec{V} = 2\vec{w}$ *where* $\vec{w}$ *is a constant vector.* **(VTU Jan 2015)**

Solution: Given $\vec{v} = \vec{w} \times \vec{r}$

$\vec{r} = xi + yj + zk \quad [\because \vec{r}$ is the position vector of $(x, y, z)]$

and let $\vec{w} = w_1 i + w_2 j + w_3 k, \quad w_1, w_2, w_3$ are constants.

$$\vec{w} \times \vec{r} = \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$
$$= i\left[w_2 z - w_3 y\right] - j\left[w_1 z - w_3 x\right] + k\left[w_1 y - w_2 x\right]$$

$$curl\vec{v} = \nabla \times (\vec{w} \times \vec{r})$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$
$$= i\left[\frac{\partial}{\partial y}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_3 x - w_1 z)\right]$$
$$- j\left[\frac{\partial}{\partial x}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_2 z - w_3 y)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(w_3 x - w_1 z) - \frac{\partial}{\partial y}(w_2 z - w_3 y)\right]$$
$$= i[w_1 + w_1] - j[-w_2 - w_2] + k[w_3 + w_3]$$
$$= 2[w_1 i + w_2 j + w_3 k]$$
$$= 2\vec{w} \Rightarrow \vec{w} = \frac{1}{2}\,\text{curl}\,\vec{v}$$

Problem 2.5.8. *Find divergence and curl of the vector* $\vec{V} = xyzi + 3x^2yj + (xz^2 - y^2z)k$ *at (2,-1,1).* *(VTU Jan 2017)*

Solution:

$$\vec{v} = (xyz)i + \left(3x^2y\right)j + \left(xz^2 - y^2z\right)k$$
$$\text{Div. } \vec{v} = \nabla \cdot \vec{v}$$
$$= \left[\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right] \cdot \vec{v}$$
$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}\left(3x^2y\right) + \frac{\partial}{\partial z}\left(xz^2 - y^2z\right)$$
$$= yz + 3x^2 + 2xz - y^2$$
$$\text{Div. } \vec{v}\,|_{(2,-1,1)} = -1 + 12 + 4 - 1 = 14$$

$$\begin{aligned}
\text{Curl } \vec{v} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} \\
&= -2yzi - \left(z^2 - xy\right) j + (6xy - xz)k \\
&= -2yzi + \left(xy - z^2\right) j + (6xy - xz)k \\
\text{Curl at } (2,-1,1) &= -2(-1)(1)i + \{(2)(-1) - 1\}j \\
&\quad + 6(2)(-1) - 2(1)k \\
&= 2\hat{i} - 3\hat{j} - 14\hat{k}
\end{aligned}$$

Problem 2.5.9. *Find curl (curl $\vec{A}$) where $\vec{A} = xy\hat{i} + y^2z\hat{j} + z^2y\hat{k}$*

$$\begin{aligned}
\vec{A} &= xy\hat{i} + y^2z\hat{j} + z^2y\hat{k} \\
\text{curl } \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2y \end{vmatrix} \\
&= \hat{i}\left(z^2 - y^2\right) - \hat{j}(0 - 0) + \hat{k}(0 - x) \\
\text{curl } \vec{A} &= \left(z^2 - y^2\right)\hat{i} - 0j - x\hat{k} \\
\therefore \text{curl}(\text{curl } \vec{A}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & 0 & -x \end{vmatrix} \\
&= \hat{i}(0 - 0) - \hat{j}(-1 - 2z) + \hat{k}(0 + 2y) \\
\text{curl } (\text{curl } \vec{A}) &= 0i + (2z + 1)j + 2yk
\end{aligned}$$

Problem 2.5.10. *If $F = (x+y+1)i + j - (x+y)k$, then show that $\vec{F} \cdot curl\vec{F} = 0$* **(VTU 2018, Jan 2014, July 2013)**

Solution:

$\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ and

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= \hat{i}[-1-0] - \hat{j}[-1-0)] + \hat{k}[0-1]$$

$$\text{curl } \vec{F} = -\hat{i} + \hat{j} - \hat{k}$$

$$\therefore \vec{F} \cdot \text{curl } \vec{F} = ((x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}) \cdot (-\hat{i} + \hat{j} - \hat{k})$$

$$= -(x+y+1) + 1 + (x+y)$$

$$= 0$$

Problem 2.5.11. *If* $\vec{r} = xi + yj + zk$ *and* $r = |\vec{r}|$ *then prove that* $\nabla r^n = nr^{n-2}\vec{r}$ **(VTU Jul 2015, Jul 2013)**

Solution: We have $\vec{r} = xi + yj + zk$ and $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\nabla r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(r^n)$$

$$= i\frac{\partial}{\partial x}(r^n) + j\frac{\partial}{\partial y}(r^n) + k\frac{\partial}{\partial z}(r^n)$$

$$= i\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + j\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + k\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \nabla r^n = nr^{n-1}\left[\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right]$$

$$= \frac{nr^{n-1}}{r}[xi + yj + zk]$$

$$= nr^{n-2}\vec{r}$$

Problem 2.5.12. *Prove that* $\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^3}\vec{r} = \frac{-1}{r^2}\hat{r}$

Solution: We have $\vec{r} = xi + yj + zk$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\nabla\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= i\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + j\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + k\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\left[\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right]$$

$$= -\frac{1}{r^3}(xi + yj + zk)$$

$$= -\frac{\vec{r}}{r^3} = -\frac{1}{r^2}\frac{\vec{r}}{r} = \frac{-1}{r^2}\hat{r}$$

Problem 2.5.13. *Compute curl of* $V(x, y, z) = xyz^2\hat{\imath} + xy^2z\hat{\jmath} + x^2yz\hat{k}$

$$\text{Curl}\, V = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & xy^2z & x^2yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(x^2yz\right) - \frac{\partial}{\partial z}\left(xy^2z\right)\right)\hat{\imath} + \left(\frac{\partial}{\partial z}\left(xyz^2\right) - \frac{\partial}{\partial x}\left(x^2yz\right)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}\left(xy^2z\right) - \frac{\partial}{\partial y}\left(xyz^2\right)\right)\hat{k}$$

$$= \left(x^2z - xy^2\right)\hat{\imath} + (2xyz - 2xyz)\hat{\jmath} + \left(y^2z - xz^2\right)\hat{k}$$

$$= \left(x^2z - xy^2\right)\hat{\imath} + \left(y^2z - xz^2\right)\hat{k}$$

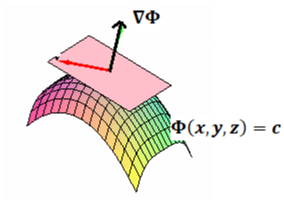
Normal vector :

If $\Phi(x, y, z)$ represents equation of a surface, then $\vec{\nabla}\Phi$ is a vector normal to the surface.

Thus

$$\boxed{\text{normal vector} = \vec{\nabla}\Phi}$$

$$\boxed{\text{unit normal vector, } \hat{n} = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|}}$$



Problem 2.5.14. *Find a unit normal to the surface* $xy^3z^2 = 4$ *at the point* $(-1,-1,2)$.

Solution: Given surface is

$$xy^3z^2 = 4$$

$$\Rightarrow \phi(x,y,z) = xy^3z^2$$

Hence the normal vector is given by

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = y^3z^2,$$

At the point $(-1,-1,2)$,

$$\frac{\partial\phi}{\partial x} = (-1)^3 \cdot 2^2 = -4$$

$$\frac{\partial\phi}{\partial y} = 3(-1)(-1)^2 \cdot 2^2 = -12$$

$$\frac{\partial\phi}{\partial z} = 2(-1)(-1)^3 \cdot 2 = 4$$

$\therefore$ at the point $(-1,-1,2)$,

$\nabla\phi = -4i - 12j + 4k = -4(i + 3j - k)$

$\therefore$ unit normal to the given surface at the point $(-1,-1,2)$ is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{-4(i+3j-k)}{4\sqrt{1+9+1}} = -\frac{(i+3j-k)}{\sqrt{11}}$$

2.6 Angle between two surfaces :

If $\Phi_1(x, y, z) = c_1$ and $\Phi_2(x, y, z) = c_2$ are two surfaces, then the angle between these two surfaces is equal to the angle between their normals $\nabla\Phi_1$ and $\nabla\Phi_2$.

Hence the angle is given by

$$\boxed{cos\theta = \frac{\nabla\Phi_1 \cdot \nabla\Phi_2}{|\nabla\Phi_1|\,|\nabla\Phi_2|}}$$

Problem 2.6.1. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2, -1, 2)$.*

Solution: The given surfaces are

$$x^2 + y^2 + z^2 = 9 \cdots (1) \text{ and } x^2 + y^2 - z = 3 \cdots (2)$$

Let

$$\Phi_1 = x^2 + y^2 + z^2 \text{ and } \Phi_2 = x^2 + y^2 - z$$

Now, vector normal to the first surface is given by

$$\nabla\Phi_1 = i\frac{\partial\Phi_1}{\partial x} + j\frac{\partial\Phi_1}{\partial y} + k\frac{\partial\Phi_1}{\partial z}$$
$$= 2xi + 2yj + 2zk$$
$$\nabla\Phi_1 \mid_{(2,-1,2)} = 4i - 2j + 4k$$

Now, vector normal to the second surface is given by

$$\nabla\Phi_2 = i\frac{\partial\Phi_2}{\partial x} + j\frac{\partial\Phi_2}{\partial y} + k\frac{\partial\Phi_2}{\partial z}$$
$$= 2xi + 2yj - k$$
$$\nabla\Phi_2 \mid_{(2,-1,2)} = 4i - 2j - k$$

If θ is the angle between the surfaces (1) and (2) at $(2,-1,2)$, then

$$\begin{aligned}\cos\theta &= \frac{\nabla\Phi_1\cdot\nabla\Phi_2}{|\nabla\Phi_1||\nabla\Phi_2|}\\ &= \frac{(4i-2j+4k)\cdot(4i-2j-k)}{\sqrt{16+4+16}\ \sqrt{16+4+1}}\\ &= \frac{16+4-4}{\sqrt{36}\sqrt{21}}\\ &= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}\\ \Rightarrow \theta &= \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)\end{aligned}$$

Problem 2.6.2. *Find the angle between the surfaces* $x\log z = y^2-1$ *and* $x^2y = 2-z$ *at the point* $(1,1,1)$

Recall that angle between the surfaces at a point of their intersection is defined to be the angle between the normals to the surfaces at the point of intersection. Thus we need to find the normals to both the surfaces at $(1,1,1)$.

Let $\varphi_1(x,y,z) = x\log z - y^2 + 1$ and $\phi_2(x,y,z) = x^2y - 2 + z$

Normal to the first surface is given by

$$\begin{aligned}grad\varphi_1 &= \nabla\varphi_1\\ &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\\ &= \frac{\partial}{\partial x}\left(x\log z - y^2+1\right)\hat{i}\\ &\quad + \frac{\partial}{\partial y}\left(x\log z - y^2+1\right)\hat{j}\\ &\quad + \frac{\partial}{\partial z}\left(x\log z - y^2+1\right)\hat{k}\\ &= (\log z)\hat{i} + (-2y)\hat{j} + \left(\frac{x}{z}\right)\hat{k}\\ \therefore grad\varphi_1\,|_{(1,1,1)} &= -2\hat{j} + \hat{k}\end{aligned}$$

Normal to the second surface is given by

$$
\begin{aligned}
grad\phi_2 &= \nabla\phi_2 \\
&= \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} + \frac{\partial\phi_2}{\partial z}\hat{k} \\
&= \frac{\partial}{\partial x}\left(x^2y - 2 + z\right)\hat{i} + \frac{\partial}{\partial y}\left(x^2y - 2 + z\right)\hat{j} \\
&\quad + \frac{\partial}{\partial z}\left(x^2y - 2 + z\right)\hat{k} \\
&= 2xy\hat{i} + \left(x^2\right)\hat{j} + (1)\hat{k} \\
\nabla\phi_2 \,|_{(1,1,1)} &= 2\hat{i} + \hat{j} + \hat{k}
\end{aligned}
$$

Angle between the surfaces is given by

$$
\begin{aligned}
\cos\theta &= \frac{\nabla\Phi_1 \cdot \nabla\Phi_2}{|\nabla\Phi_1||\nabla\Phi_2|} \\
&= \frac{\left[0i - 2\hat{j} + \hat{k}\right]\cdot\left[2\hat{i} + \hat{j} + \hat{k}\right]}{\sqrt{0^2 + (-2)^2 + 1^2}\,\sqrt{2^2 + 1^2 + 1^2}} \\
&= \frac{0 - 2 + 1}{\sqrt{5}\,\sqrt{6}} = -\frac{1}{\sqrt{30}} \\
\Rightarrow \theta &= \cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)
\end{aligned}
$$

Problem 2.6.3. *Find the angle between the normal to the surface $xy = z^2$ at (1,4,2) and (-3,-3,3)*

Solution: The given surface is $xy - z^2 = 0$ $\therefore$

$$\phi = xy - z^2$$

We know $\nabla\phi$ is normal to the surface at the point (x, y, z)

Let $\vec{n}_1, \vec{n}_2$, be the normals to the surface at the points $(1, 4, 2)$ and $(-3, -3, 3)$ respectively.

Now

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \qquad \cdots (1)$$

$$\frac{\partial\phi}{\partial x} = y, \quad \frac{\partial\phi}{\partial y} = x, \quad \text{and} \quad \frac{\partial\phi}{\partial z} = -2z$$

Substituting in (1) we get

$$\nabla\phi = yi + xj - 2zk$$

At the point $(1, 4, 2)$, $\nabla\phi = 4i + j - 4k$

and at the point $(-3, -3, 3)$, $\nabla\phi = -3i - 3j - 6k$

$\therefore\ \vec{n}_1 = 4i + j - 4k,\quad \vec{n}_2 = -3i - 3j - 6k$

If θ is the angle between the normals, then

$$\begin{aligned}
\cos\theta &= \frac{\vec{n}_1}{|\vec{n}_1|} \cdot \frac{\vec{n}_2}{|\vec{n}_2|} \\
&= \frac{(i + j - 4k)}{\sqrt{16 + 1 + 16}} \cdot \frac{(-3i - 3j - 6k)}{\sqrt{9 + 9 + 36}} \\
&= \frac{-12 - 3 + 24}{\sqrt{33}\sqrt{54}} \\
&= \frac{9}{\sqrt{3}\ \sqrt{11}\ 3\sqrt{3}\ \sqrt{2}} \\
&= \frac{1}{\sqrt{22}}
\end{aligned}$$

Hence $\theta = cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$

Problem 2.6.4. *Find the constants* a *&* b *so that the surface* $ax^2 - byz = (a+2)x$ *is orthogonal to the surface* $4x^2y + z^3 = 4$ *at the point* $(1, -1, 2)$. ***Ans :***

Solution: The given surfaces are

$$ax^2 - byz - (a + 2)x = 0$$

$$4x^2y + z^3 - 4 = 0$$

$$\text{Let}\quad \phi_1 = ax^2 - byz - (a + 2)x \qquad \cdots (1)$$

and

$$\phi_2 = 4x^2y + z^3 - 4 \qquad \cdots (2)$$

Given that surfaces (1) and (2) cut orthogonally at the point $(1, -1, 2)$.

$$\therefore\ \nabla\phi_1 \cdot \nabla\phi_2 = 0$$

we have

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$\text{where } \frac{\partial \phi_1}{\partial x} = 2ax - a - 2,$$
$$\frac{\partial \phi_1}{\partial y} = -bz \text{ and}$$
$$\frac{\partial \phi_1}{\partial z} = -by$$
$$\therefore \nabla \phi_1 = (2ax - a - 2)i - bzj - byk$$

Similarly,

$$\nabla \phi_2 = i\frac{\partial \phi_2}{\partial x} + j\frac{\partial \phi_2}{\partial y} + k\frac{\partial \phi_2}{\partial z}$$
$$\text{where } \frac{\partial \phi_2}{\partial x} = 8xy,$$
$$\frac{\partial \phi_2}{\partial y} = 4x^2$$
$$\text{and } \frac{\partial \phi_2}{\partial z} = 3z^2$$
$$\therefore \nabla \phi_2 = 8xyi + 4x^2j + 3z^2k$$

At the point (1,-1,2),

$$\nabla \phi_1 = (2a - a - 2)i - b(2)j - b(-1)k$$
$$= (a - 2)i - 2bj + bk \text{ and}$$
$$\nabla \phi_2 = -8i + 4j + 12k$$
$$\nabla \phi_1 \cdot \nabla \phi_2 = 0$$
$$\Rightarrow ((a - 2)i - 2bj + bk) \cdot (-8i + 4j + 12k) = 0$$
$$\Rightarrow -8(a - 2) - 8b + 12b = 0$$
$$\Rightarrow -8a + 4b + 16 = 0$$
$$\Rightarrow \quad 2a - b = 4 \qquad \cdots (3)$$

Since the point of intersection $(1, -1, 2)$ is a common point on the surfaces (1) and (2), it should satisfy (1). Hence we get

$$
\begin{aligned}
a + 2b - (a + 2) &= 0 \\
\Rightarrow \quad 2b &= 2 \\
\Rightarrow \quad b &= 1 \\
\therefore \Rightarrow \quad 2a &= 4 + b = 4 + 1 = 5 \ \text{(from (3))} \\
\Rightarrow \quad a &= \frac{5}{2} \\
\therefore \quad a &= \frac{5}{2}, \quad b = 1
\end{aligned}
$$

2.7 Directional Derivative :

The directional derivative of a scalar point function ϕ in a given direction $\vec{a}$ is the rate of change of ϕ in that direction. It is given by the component of $\nabla\phi$ in the direction of $\vec{a}$

$$\therefore \text{ the directional derivative } = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

We can write

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{|\nabla\phi|\,|\vec{a}|}{|\vec{a}|}\ \cos\theta = |\nabla\phi|\cos\theta$$

, where θ is the angle between $\nabla\phi$ and $\vec{a}$.
So, the directional derivative at a given point is maximum if $\cos\theta$ is maximum. i.e., $\cos\theta = 1 \Rightarrow \theta = 0$

$\therefore$ **The maximum directional derivative at a point is in the direction of $\nabla\phi$ and the maximum directional derivative is $|\nabla\phi|$.**

Problem 2.7.1. *Find the directional derivative of* $\Phi = x^2yz + 4xz^2$ *at* $(1, -2, -1)$ *in the direction of the vector* $2i - j - 2k$ **(VTU Aug 2005, Aug 2004, Mar 2000)**

Solution: Given

$$\phi(x, y, z) = x^2yz + 4xz^2$$

We know

$$\text{grad}\,\phi = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\phi}{\partial y} = x^2z, \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz$$

At the point $(1, -2, -1)$,

$$\frac{\partial\phi}{\partial x} = 2\cdot 1(-2)(-1) + 4(-1)^2 = 8$$

$$\frac{\partial\phi}{\partial y} = 1^2\cdot(-1) = -1$$

$$\frac{\partial\phi}{\partial z} = 1^2(-2) + 8\cdot 1(-1) = -2 - 8 = -10$$

$\therefore$ at the point $(1, -2, -1)$, $\nabla\phi = 8i - j - 10k$

Given direction is $\vec{a} = 2i - j - 2k$

$\therefore$ the directional derivative of ϕ at the point $(1, -2, -1)$ in the direction of $\vec{a}$ is given by

$$\nabla\phi\cdot\frac{\vec{a}}{|\vec{a}|} = (8i - j - 10k)\cdot\frac{(2i - j - 2k)}{\sqrt{4+1+4}} = \frac{16+1+20}{\sqrt{9}} = \frac{37}{3}$$

Problem 2.7.2. *Find the maximum value of the directional derivative of* $\phi = x^3yz$ *at the point* $(1, 4, 1)$

Solution: Given

$$\phi = x^3yz$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

We know, The directional derivative is maximum in the direction of $\nabla\phi$ and the maximum value $= |\nabla\phi|$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 3x^2yz, \quad \frac{\partial\phi}{\partial y} = x^3z, \quad \frac{\partial\phi}{\partial z} = x^3y$$

At the point $(1, 4, 1)$,
$\frac{\partial\phi}{\partial x} = 3(1)(4)(1) = 12,$
$\frac{\partial\phi}{\partial y} = 1^3(1) = 1,$
$\frac{\partial\phi}{\partial z} = 1^3(4) = 4$
$\therefore$ at the point $(1, 4, 1)$, $\quad \nabla\phi = 12i + j + 4k$
Maximum value of the directional derivative is

$$\begin{aligned} |\nabla\phi| &= |12i + j + 4k| \\ &= \sqrt{144 + 1 + 16} \\ &= \sqrt{161} \end{aligned}$$

Problem 2.7.3. *Find the directional derivative of* $f = xy^2 + yz^3$ *at the point* $(2, -1, 1)$ *in the direction of the normal to the surface* $xlogz - y^2 = -4$ *at the point* $(-1, 2, 1)$. **(VTU Feb 2004)**

Solution:: Given

$$\begin{aligned} f &= xy^2 + yz^3 \\ \therefore \ \nabla f &= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \\ &= y^2 i + \left(2xy + z^3\right) j + 3yz^2 k \end{aligned}$$

At the point $(2, -1, 1)$,

$$\begin{aligned} \nabla f &= (-1)^2 i + (-4 + 1)j + 3(-1)1^2 k \\ \Rightarrow \nabla f &= i - 3j - 3k \end{aligned}$$

The directional derivative of f in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point $(-1, 2, 1)$ is required.
Let $g = x \log z - y^2 + 4$.
Normal to this surface is,

$$\begin{aligned} \nabla g &= i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z} \\ &= \log z i - 2yj + \frac{x}{z}k \end{aligned}$$

At the point $(-1,2,1)$,

$$\begin{aligned}\vec{a} &= \nabla g\\ &= \log 1 i - 4j + \left(\frac{-1}{1}\right)k\\ &= 0i - 4j - k = -4j - k\\ \therefore \vec{a} &= -4j - k\end{aligned}$$

Required directional derivative is

$$\begin{aligned}\nabla f \cdot \frac{\vec{a}}{|\vec{a}|} &= (i - 3j - 3k)\cdot\frac{(-4j-k)}{\sqrt{16+1}}\\ &= \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}\end{aligned}$$

Problem 2.7.4. *Find the directional derivative of $\varphi(x,y,z) = 2xy + 3y^2 - z^3$ at the point $P(1,-1,2)$ in the direction of the vector $\vec{v} = \hat{i} - \hat{j} + \hat{k}$.*

Solution: First we find that

$$\begin{aligned}\text{grad}\,\varphi &= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\\ &= 2y\hat{i} + (2x+6y)\hat{j} + (-3z^2)\hat{k}\\ \nabla\varphi|_{(1,-1,2)} &= -2\hat{i} - 4\hat{j} - 12\hat{k}\end{aligned}$$

The directional derivative of ϕ in the direction of v is given by

$$\begin{aligned}\nabla\varphi\cdot\frac{\vec{v}}{|\vec{v}|} &= (-2\hat{i} - 4\hat{j} - 12\hat{k})\cdot\left(\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}\right)\\ &= \frac{-2+4-12}{\sqrt{3}}\\ &= -\frac{10}{\sqrt{3}}\end{aligned}$$

12. Find the directional derivative of the function $\phi = xy + yz + zx$ in the direction of the vector $2i + 3j + 6k$ at the point $(3,1,2)$. Ans : $45/7$

13. Find the directional derivative of the function $2yz + z^2$ in the direction of the vector $i + 2j + 2k$ at the point $(1,-1,3)$ Ans : $\frac{20}{3}$

14. Find the directional derivative of $x^3 + y^3 + z^3$ at the point $(1,-1,2)$ in the direction of $i + 2j + k$ Ans : $\frac{7\sqrt{6}}{2}$

2.8 Solenoidal and irrotational vector fields

A vector $\vec{F}$ said to be **solenoidal** if $div\vec{F} = 0$. i. e.

$$\boxed{\nabla \cdot \vec{F} = 0}$$

A vector $\vec{F}$ said to be **irrotational** if $curl\vec{F} = \vec{0}$. i. e.

$$\boxed{\nabla \times \vec{F} = \vec{0}}$$

Note : (1) An irrotational vector field is also called as conservative field or potential field.

(2) If $\vec{F}$ is irrotational, then there always exists a scalar function Φ such that $\nabla\phi = \vec{F}$

Problem 2.8.1. *Show that* $V(x, y, z) = (4xy - z^3)\,\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$ *is irrotational*

Solution:

$$Curl\ \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right)\hat{i} + \left(\frac{\partial}{\partial z}(4xy - z^3) - \frac{\partial}{\partial x}(-3xz^2)\right)\hat{j} + \left(\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right)\hat{k}$$

$$= (0 - 0)\hat{i} + \left(-3z^2 - (-3z^2)\right)\hat{j} + (4x - 4x)\hat{k} = 0$$

Hence $\vec{V}$ is irrotational.

Problem 2.8.2. *Show that* $V(x, y, z) = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}$ *is solenoidal.*

Solution:

$$\text{div } \vec{V} = \nabla \cdot \vec{V}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}3(x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

Hence, $\boldsymbol{V}$ is solenoidal.

Problem 2.8.3. *Find the value of* $\boldsymbol{a}$ *so that* $\boldsymbol{V}(x,y,z) = (x+3y)\hat{\imath} + (y-2z)\hat{\jmath} + (x-az)\hat{k}$ *is solenoidal.*

Solution: Given that $\vec{V}$ is solenoidal.

Hence div $\boldsymbol{V} = 0$

$$\Rightarrow \nabla \cdot \boldsymbol{V} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot ((x+3y)\hat{\imath} + (y-2z)\hat{\jmath} + (x-az)\hat{k}) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x-az) = 0$$

$$\Rightarrow 1 + 1 - a = 0$$

$$\Rightarrow a = 2$$

Problem 2.8.4. *Prove that* $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ *is both solenoidal and irrotational.* *(VTU Jan 2016)*

Solution: For solenoidal, we have to prove $\nabla \cdot \vec{F} = 0$.

Now,

$\nabla \cdot \vec{F}$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left[\begin{array}{c} (y^2 - z^2 + 3yz - 2x)\,\hat{i} \\ +(3xz + 2xy)\hat{j} \\ +(3xy - 2xz + 2z)\hat{k} \end{array}\right]$$

$$= \frac{\partial}{\partial x}\left(y^2 - z^2 + 3yz - 2x\right) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= (-2) + (2x) + (-2x + 2)$$

$$= 0$$

Thus, $\vec{F}$ is solenoidal.

For irrotational, we have to prove Curl $\vec{F} = 0$.

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= (3x - 3x)\hat{i} - (-2z + 3y - 3y + 2z)\hat{j} + (3z + 2y - 2y - 3z)k$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

Hence $\vec{F}$ is irrotational.

Problem 2.8.5. *Find a, b, c if* $(x+y+az)i+(bx+2y-z)j+(-x+cy+2z)k$ *is irrotational.*

Solution: Let $\vec{F} = (x + y + az)i + (bx + 2y - z)j + (-x + cy + 2z)k$

Given $\vec{F}$ is irrotational.

$$\therefore \quad \nabla \times \vec{F} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & -x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow i\left[\frac{\partial}{\partial y}(-x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z)\right] - j\left[\frac{\partial}{\partial x}(-x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az)\right] + k\left[\frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az)\right] = 0$$

$$\Rightarrow i(c + 1) - j(-1 - a) + k(b - 1) = \vec{0}$$

$$\Rightarrow (c + 1)i + (1 + a)j + (b - 1)k = \vec{0}$$

$$\Rightarrow c + 1 = 0, 1 + a = 0, b - 1 = 0$$

$$\Rightarrow a = -1, b = 1 \text{ and } c = -1$$

Problem 2.8.6. *Prove that* $\frac{xi+yj}{x^2+y^2}$ *is both solenoidal & irrotational.* *(VTU Model 2014)*

Solution::

$$\begin{aligned}\vec{F} &= \frac{x\hat{i}+y\hat{j}}{x^2+y^2}\\ &= \frac{x}{x^2+y^2}\hat{i}+\frac{y}{x^2+y^2}\hat{j}\end{aligned}$$

$$\begin{aligned}\text{div }\vec{F} &= \nabla\cdot\vec{F}\\ &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\\ &= \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}+\frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}\\ &= \frac{(y^2-x^2)}{(x^2+y^2)^2}+\frac{(x^2-y^2)}{(x^2+y^2)^2}\\ &= 0\\ &\Rightarrow \vec{F}\text{ is solenoidal.}\end{aligned}$$

$$\begin{aligned}\text{curl }\vec{F} = \nabla\times\vec{F} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0\end{vmatrix}\\ &= \hat{i}[0-0]-\hat{j}[0-0]+\hat{k}\left(\frac{-2xy}{(x^2+y^2)^2}+\frac{2xy}{(x^2+y^2)^2}\right)\\ &= (0)\hat{i}+(0)\hat{j}+(0)\hat{k}=\vec{0}\end{aligned}$$

curl $\vec{F}=\vec{0}\Rightarrow\vec{F}$ is irrotational.

Problem 2.8.7. *A vector field is given by* $\vec{A}=(x^2+xy^2)i+(y^2+x^2y)j$. *Show that the field is irrotational and find the scalar potential.* *(VTU Jan 2017)*

Solution:

$$\begin{aligned}Curl\vec{A} &= \nabla\times\vec{A}\\ &= \begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+xy^2 & y^2+x^2y & 0\end{vmatrix}\\ &= \hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(2xy-2xy)=0\end{aligned}$$

Hence, $\vec{A}$ is irrotational. If ϕ is the scalar potential, then

$$\vec{A} = \text{grad}\,\phi$$

$$i.e.\ \vec{A} = \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k$$

$$(x^2 + xy^2)i + (y^2 + x^2y)j = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

Equating the components on both sides

$$\frac{\partial \phi}{\partial x} = (x^2 + xy^2) \qquad \cdots (1)$$

$$\frac{\partial \phi}{\partial y} = (y^2 + x^2y) \qquad \cdots (2)$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \cdots (3)$$

Integrating (1), only w.r.to x, treating y and z as constants,

$$\phi = \frac{x^3}{3} + \frac{x^2}{2}y^2 + f(y,z) \qquad \cdots (4)$$

Integrating (1), only w.r.to y, treating x and z as constants,

$$\phi = \frac{y^3}{3} + \frac{y^2}{2}x^2 + g(x,z) \qquad \cdots (5)$$

Integrating (1), only w.r.to z, treating x and y as constants,

$$\phi = 0 + h(x,y) \qquad \cdots (6)$$

Equating all the expressions for ϕ from equations (4), (5) and (6), we get

$$\frac{x^3}{3} + \frac{x^2}{2}y^2 + f(y,z) = \frac{y^3}{3} + \frac{y^2}{2}x^2 + g(x,z) = 0 + h(x,y)$$

Comparing we get

$$f(y,z) = \frac{y^3}{3} + 0$$

$$g(x,z) = \frac{x^3}{3} + 0$$

$$h(x,y) = \frac{y^3}{3} + \frac{y^2}{2}x^2 + \frac{x^3}{3}$$

Substituting these values in (4), (5) and (6), we get a unique expression for ϕ as

$$\phi = \frac{y^3}{3} + \frac{y^2}{2}x^2 + \frac{x^3}{3}$$

Problem 2.8.8. *Show that* $\vec{F} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ *is irrotational and hence find its scalar potential.* **(VTU July 2015, Jan 2013)**

Solution: let

$$\vec{f} = \left(x^2 - yz\right) i + \left(y^2 - zx\right) j + \left(z^2 - xy\right) k$$

$$\text{Then curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \sum i(-x + x) = \vec{0}$$

$\therefore \vec{f}$ is Irrotational. Then there exists ϕ such that $\vec{f} = \nabla\phi$

$$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = \left(x^2 - yz\right) i + \left(y^2 - zx\right) j + \left(z^2 - xy\right) k$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int \left(x^2 - yz\right) dx = \frac{x^3}{3} - xyz + f_1(y, z) \qquad \cdots (1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z, x) \qquad \cdots (2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x, y) \qquad \cdots (3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = \frac{x^3 + y^3 + z^3}{3} - xyz + \text{Constant}$$

Which is the required scalar potential.

Problem 2.8.9. *Find the constants* a *and* b *such that* $F = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ *is irrotational. Find* Φ *such that* $F = \nabla\Phi$ **(VTU July 2014, Jan 2014, Jun 2012, Feb 2005)**

Solution:: Given that F is irrotational. Hence

$$\nabla \times \vec{F} = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = 0$$

$$\Rightarrow i(-1+1) - \hat{j}\left(bz^2 - 3z^2\right) + \hat{k}(6x - ax) = 0$$

$$\Rightarrow 0i - z^2(b-3)\hat{j} + x(6-a)\hat{k} \qquad = 0$$

$$\Rightarrow b = 3 \text{ and } a = 6$$

$$\therefore \vec{F} = \left(6xy + z^3\right)\hat{i} + \left(3x^2 - z\right)\hat{j} + \left(3xz^2 - y\right)\hat{k}$$

Given, $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \left(6xy + z^3\right)\hat{i} + \left(3x^2 - z\right)\hat{j} + \left(3xz^2 - y\right)\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy + z^3 \Rightarrow \phi = 3x^2y + xz^3 + f_1(y,z) \qquad \cdots(1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \Rightarrow \phi = 3x^2y - yz + f_2(x,z) \qquad \cdots(2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \Rightarrow \phi = xz^3 - yz + f_3(x,y) \qquad \cdots(3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = 3x^2y + xz^3 - yz + c$$

Problem 2.8.10. *Find the constants* a, b, c *such that the vector field* $\vec{f} = (x + y + az)i + (x + cy + 2z)k + (bx + 2y - z)j$ *is irrorational. Also find* Φ *such that* $\vec{F} = \nabla\Phi$ **(VTU Jul 2015)**

Solution: Given that the vector field is irrotational. Therefore $\nabla \times \vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & x + cy + 2z \end{vmatrix} = 0$$

$\hat{i}(c+1) - \hat{j}(1-a) + \hat{k}(b-1) = 0$

$c + 1 = 0, \quad 1 - a = 0, \quad b - 1 = 0$ Hence $\vec{f} = (x+y+z)i + (x+2y-$

$c = -1, \quad a = 1, \quad b = 1$

$z)j + (x - y + 2z)k$

Then there exists ϕ such that $\vec{f} = \nabla\phi$

$$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$
$$= (x + y + z)i + (x + 2y - z)j + (x - y + 2z)k$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x + y + z \Rightarrow \phi = \frac{x^2}{2} + xy + xz + f_1(y, z) \qquad \cdots (1)$$
$$\frac{\partial\phi}{\partial y} = x + 2y - z \Rightarrow \phi = xy + 2\frac{y^2}{2} - zy + f_2(x, z) \qquad \cdots (2)$$
$$\frac{\partial\phi}{\partial z} = x - y + 2z \Rightarrow \phi = xz - yz + 2\frac{z^2}{2} + f_3(x, y) \qquad \cdots (3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2 + \text{constant}$$

Problem 2.8.11. *Find constants a, b and c if the vector $\vec{f} = (2x + 3y + az)i + (bx + 2y + 3z)j + (2x + cy + 3z)k$ is Irrotational.*

Solution: Given

$$\vec{f} = (2x + 3y + az)i + (bx + 2y + 3z)j + (2x + cy + 3z)k$$

$$\text{Curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 3y + az & bx + 2y + 3z & 2x + cy + 3z \end{vmatrix}$$
$$= (c - 3)i - (2 - a)j + (b - 3)k$$

If the vector is irrotational then curl $\vec{f} = \vec{0}$

$$\therefore 2 - a = 0 \Rightarrow a = 2, b - 3 = 0 \Rightarrow b = 3, c - 3 = 0 \Rightarrow c = 3$$

Problem 2.8.12. *Find the constants a, b, c such that the vector field $\vec{f} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrorational. Hence find its scalar potential.* *(VTU July 2014)*

Solution:: Given vector is $\vec{A} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is Irrotational.

$$\Rightarrow \text{curl } \vec{A} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow (c + 1)i + (a - 4)j + (b - 2)k = \vec{0}$$

$$\Rightarrow (c + 1)i + (a - 4)j + (b - 2)k = 0i + 0j + 0k$$

Comparing both sides,

$$c + 1 = 0, a - 4 = 0, b - 2 = 0$$

$$c = -1, a = 4, b = 2$$

Now $\vec{A} = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k,$

on substituting the values of a, b, c

we have $\vec{A} = \nabla\phi$.

$$\Rightarrow \vec{A} = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k$$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + bk\frac{\partial\phi}{\partial z}$$

Comparing both sides, we have

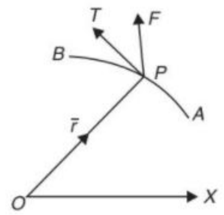
$$\frac{\partial\phi}{\partial x} = x + 2y + 4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4zx + f_1(y, z)$$

$$\frac{\partial\phi}{\partial y} = 2x - 3y - z \Rightarrow \phi = 2xy - 3y^2/2 - yz + f_2(x, z)$$

$$\frac{\partial\phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2 + f_3(y, x)$$

Hence $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

2.9 Line Integrals-Vector line Integral:



Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.
Line integral of a vector function $\vec{F}$ along the curve $C = AB$ is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

$$\therefore \boxed{\text{Line integral } = \int_c \vec{F} \cdot d\vec{R} = \int_c \vec{F} \cdot d\vec{r}}$$

Note : When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$

If $F(R)$or$\vec{F} = f_1(x, y, z)i + f_2(x, y, z)j + f_3(x, y, z)k$

and $\vec{r} = xi + yj + zk$

Then $d\vec{R} = dxi + dyj + dzk$

and $\displaystyle\int_c \vec{F} \cdot d\vec{r} = \int_c (f_1dx + f_2dy + f_3dz)$

This is a scalar.

Note:

- **Circulation :** If $\vec{F}$ represents the velocity of a fluid particle then the line integral $\int_C \vec{F} \cdot d\vec{R}$ is called the circulation of $\vec{F}$ around the curve. When the circulation of $\vec{F}$ around every closed curve in a region E vanishes, $\vec{F}$ is said to be irrotational in E.

- **Work :** If $\vec{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement $\delta R = \vec{F} \cdot \delta R$. Therefore, the total work done by $\vec{F}$ during the displacement from A to B is given by the integral $\int_A^B \vec{F} \cdot d\vec{R}$

Problem 2.9.1. *If* F $= \left(5xy - 6x^2\right) i + (2y - 4x)j$, *evaluate* $\int_C F \cdot dr$ *along the curve* C *the* xy *-plane,* $y = x^3$ *from the point* $(1, 1)$ *to* $(2, 8)$.

Solution:: Given $\vec{F} = \left(5xy - 6x^2\right) \hat{i} + (2y - 4x)\hat{j}$ Let

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ then}$$
$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$
$$\vec{F} \cdot d\vec{r} = \left(5xy - 6x^2\right) dx + (2y - 4x)dy$$
$$\int_C \vec{F} \cdot dr = \int_C \left(5xy - 6x^2\right) dx + (2y - 4x)dy$$

Along the curve $y = x^3$ $\left(\therefore dy = 3x^2dx\right)$, x varies from $x = 1$ to $x = 2$ and we get

$$\int_C F \cdot dr = \int_1^2 \left(5x\left(x^3\right) - 6x^2\right) dx + \left(2x^3 - 4x\right) 3x^2dx$$
$$= \int_1^2 \left(5x^4 - 6x^2 + 6x^5 - 12x^3\right) dx$$
$$= \left[5\frac{x^5}{5} - 6\frac{x^3}{3} + 6\frac{x^6}{6} - 12\frac{x^4}{4}\right]_1^2$$
$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$
$$= (32) - (3) = 35$$

Problem 2.9.2. *Find the total work done in moving a particle in the force field* $F = 3xyi - 5zj + 10xk$ *along the curve* $x = t^2 + 1, y = 2t^2, z = t^3$ *from* $t = 1$ *to* $t = 2$ *(VTU June 2019)*

Solution:: We have $\vec{r} = xi + yj + zk$ will give $d\vec{r} = dxi + dyj + dzk$

$$\vec{F} \cdot \vec{dr}$$
$$= (3xyi - 5zj + 10xk) \cdot (dxi + dyi + dzk)$$
$$= 3xydx - 5zdy + 10xdz$$
$$= 3(t^2 + 1)(2t^2)(2tdt) - 5(t^3)(4tdt) + 10(t^2 + 1)(3t^2dt)$$
$$= (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)dt$$

$$\vec{F}\cdot\vec{dr} = (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$$

$$\int_C F\cdot dr = \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$$

$$= \left[12\,\frac{t^6}{6} + 10\,\frac{t^5}{5} + 12\,\frac{t^4}{4} + 30\,\frac{t^3}{3}\right]_1^2$$

$$= 12\left[\frac{2^6-1}{6}\right] + 10\left[\frac{2^5-1}{5}\right] + 12\left[\frac{2^4-1}{4}\right] + 30\left[\frac{2^3-1}{3}\right]$$

$$= 2(63) + 2(31) + 3(15) + 10(7)$$

$$= 303$$

Problem 2.9.3. *If* $\vec{F} = xyi + yzj + zxk$, *Evaluate* $\int_c \vec{F}\cdot\vec{dr}$. *Where C is the curve represented by* $x = t, y = t^2, z = t^3, \quad -1 \le t \le 1$ *(VTU Jan 2020)*

Solution::we have $\vec{F} = xyi + yzj + zxk$ and $\vec{r} = xi + yj + zk$ will give $d\vec{r} = dxi + dyj + dzk$
$\vec{F}\cdot d\vec{r} = xydx + yzdy + zxdz$ since $x = t, y = t^2, z = t^3$,
$dx = dt, dy = 2tdt, dz = 3t^2dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\frac{dr}{dt}dt$$

$$= \int_C (xyi + yzj + zxk)\cdot\left(1i + 2tj + 3t^2k\right)dt$$

$$= \int_C \left(xy + 2yzt + 3zxt^2\right)dt$$

$$\begin{aligned}\int_C \vec{F}\cdot dr &= \int_C \left(t(t^2)+2(t^2)(t^3)t+3(t^3)(t)t^2\right)dt\\ &= \int_{-1}^{1}\left(t^3+2t^6+3t^6\right)dt\\ &= \int_{-1}^{1}\left(t^3+5t^6\right)dt\\ &= \left[\frac{t^4}{4}+\frac{5t^7}{7}\right]_{-1}^{1}\\ &= \left[\left(\frac{1}{4}+\frac{5}{7}\right)-\left(\frac{1}{4}-\frac{5}{7}\right)\right]\\ &= \frac{27}{28}+\frac{13}{28}=\frac{40}{28}=\frac{10}{7}\end{aligned}$$

Problem 2.9.4. *If* $F = \left(3x^2+6y\right)i - 14yzj + 20xz^2k$. *Evaluate* $\int F\cdot dr$ *from* $(0,0,0)$ *to* $(1,1,1)$ *along the path* $x = t, y = t^2, z = t^3$.

Solution:: Given path is $x = t, y - t^2, z = t^3$ Therefore, $dx = dt, dy = 2tdt$ and $dz = 3t^2dt$

Also, (x,y,z) varies from $(0,0,0)$ to $(1,1,1) \Rightarrow t = 0$ to $t = 1$. Thus,

$$\begin{aligned}\int_C F\cdot dr &= \int_C \left(3x^2+6y\right)dx - 14yzdy + 20xz^2dz\\ &= \int_{i=0}^{1}\left(3t^2+6t^2\right)(dt) - 14(t^2)(t^3)(2tdt) + 20(t)(t^6)(3t^2dt)\\ &= \int_{t=0}^{1}\left(9t^2-28t^6+60t^9\right)dt\\ &= \left(3t^3-4t^7+6t^{10}\right)_0^1 = [3-4+6] = 5\end{aligned}$$

Problem 2.9.5. *Find the work done by the force* $\vec{F} = (2y+3)i + xzj + (yz-x)k$ *to shift the particle from* $(0,0,0)$ *to* $(2,1,1)$ *along the curve* $x = 2t^2, y = t, z = t^3$.

Solution:: $\vec{r} = xi + yj + zk$ and $d\vec{r} = dxi + dyj + dzk$

Work done is given by

$$W = \int_C F\cdot dr = \int_C (2y+3)dx + xzdy + (yz-x)dz \cdots (1)$$

Equation to the given curve is $x = 2t^2, y = t, z = t^3$; so that $dx = 4tdt, dy = dt$ and $dz = 3t^2dt$

further $(x, y, z) = (0, 0, 0) \to (2, 1, 1) \Rightarrow t = 0 \to t = 1$. Substituting in (1), we get

$$\begin{aligned} W &= \int_0^1 (2t+3)4tdt + 2t^2t^3dt + \left(t(t^3) - 2t^2\right)3t^2dt \\ &= \int_0^1 \left(8t^2 + 12t + 2t^5 + 3t^6 - 6t^4\right)dt \\ &= \left(\frac{8t^3}{3} + 6t^2 + \frac{t^3}{3} + \frac{3t^7}{7} - \frac{6t^5}{5}\right)_0^1 \\ &= \left(\frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right) = \frac{288}{35} \end{aligned}$$

$\therefore$ work done $= \frac{288}{35}$.

Problem 2.9.6. *Find the total work done by the force represented by* $\vec{F} = 3xyi - yj + 2zxk$ *in moving a particle round the circle* $x^2 + y^2 = 4$ *(VTU-2010)*

Solution:: Total work done is

$$W = \int_C \vec{F} \cdot \vec{dr}$$

$x^2 + y^2 = 4$ can be represented in the parameter from $x = rcos\theta, y = r\sin\theta$

$x = 2\cos\theta \quad y = 2\sin\theta$ and $z = 0, 0 \le \theta \le 2\pi$

$dx = -2\sin\theta d\theta \quad dy = 2\cos\theta d\theta$

work done $= \int_C \vec{F}\cdot d\vec{r} = \int_C 3xydx - ydy + 2zxdz$

$$= \int_{\theta=0}^{2\pi} 3(4\cos\theta\sin\theta)(-2\sin\theta d\theta) - 2\sin\theta(2\cos\theta d\theta) + 0$$

$$= -24\int_{\theta=0}^{2\pi}\sin^2\theta\cos\theta d\theta - 4\int_{\theta=0}^{2\pi}\sin\theta\cos\theta d\theta$$

$$= -24\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} - 2\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi}$$

In the first term, We substitute $\sin\theta = t,$

$\Rightarrow \cos\theta d\theta = dt$

$\Rightarrow \int t^2 dt = \frac{t^3}{3}$

$$= \frac{-24}{3}\left[\sin^3 2\pi - \sin^3 0\right] + \left[\cos 4\pi - \cos 0\right]$$

$$= 0 + (1-1) = 0$$

Problem 2.9.7. *Find the work done in moving a particle in the force field* $F = 3x^2 i + (2xz - y)j + zk$*, along a) The straight line from (0,0,0) to (2,1,3) b) The curve defined by* $x^2 = 4y, 3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. *(VTU-2017)*

Solution:

$$\text{work done} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C \left(3x^2 i + (2xz - y)j + zk\right)\cdot(dxi + dyj + dzk)$$

$$= \int_C \left(3x^2\right)dx + (2xz - y)dy + zdz \qquad \cdots (1)$$

a) The equations of the straight line from (0,0,0) to (2,1,3) are $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t($ say)

Then $x = 2t, y = t, z = 3t$ are its parametric equations.

and $dx = 2dt$, $dy = dt$ and $dz = 3dt$

The points $(0,0,0)$ to $(2,1,3)$ corresponds to $t=0$ and $t=1$ respectively.

$$\begin{aligned}
\text{workdone } &= \int_C \vec{F}\cdot d\vec{r}\\
&= \int_C \left(3x^2\right)dx + (2xz-y)dy + zdz\\
&= \int_0^1 \left[(3(2t)^2\,(2dt) + [2(2t)(3t)-t]\,dt\right.\\
&\qquad \left.+(3t)(3dt)\right]\\
&= \int_0^1 \left(36t^2+8t\right)dt = 16
\end{aligned}$$

b) Let $x=t$ in $x^2=4y$ and $3x^3=8z$.

Then the parametric equations of C are

$$x=t,\quad y=\frac{t^2}{4},\quad z=\frac{3t^3}{8} \text{ and}$$

$$dx=dt,\quad dy=\frac{2tdt}{4}=\frac{t}{2}dt,\quad dz=\frac{9t^2}{8}dt$$

and t varies from 0 to 2

$$\begin{aligned}
\text{Work done } &= \int_C \vec{F}\cdot d\vec{r}\\
&= \int_C \left(3x^2\right)dx + (2xz-y)dy + zdz\\
&= \int_0^2 3t^2dt + \left(2(t)\,\frac{3t^3}{8}-\frac{t^2}{4}\right)\frac{tdt}{2} + \left(\frac{3t^3}{8}\right)\frac{9t^2}{8}dt\\
&= \int_0^2 \left[3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right]dt\\
&= t^3 - \frac{t^4}{32} + \frac{17}{128}t^6\Big|_0^2 = 16
\end{aligned}$$

Problem 2.9.8. *A vector field is given by* $\vec{F} = sinyi + x(1+cosy)j$*. Evaluate the line integral over a circular path given by* $x^2+y^2=a^2, z=0$ *[VTU Jan 2018]*

Solution:: We have,

$$\text{Work done} = \int_C \vec{F}\cdot\vec{dr}$$
$$= \int_C [(\sin y)\hat{i} + x(1+\cos y)\hat{j}]\cdot\left[dx\hat{i} + dy\hat{j} + 0k\right]$$
$$(\because z = 0,\ \text{hence } dz = 0)$$
$$= \int_C \sin y dx + x(1+\cos y)dy$$
$$\text{Work done} = \int_C (\sin y dx + x\cos y dy + x dy)$$

The parametric equations of given path $x^2 + y^2 = a^2$ are $x = a\cos\theta, y = a\sin\theta$,

$dx = -a\sin\theta d\theta$ and $dy = a\cos\theta d\theta$

and θ varies from 0 to 2π

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (\sin y dx + x\cos y dy + x dy)$$
$$= \int_C d(x\sin y) + \int_C x dy$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} d[a\cos\theta\sin(a\sin\theta)]d\theta$$
$$+ \int_0^{2\pi} a\cos\theta\cdot a\cos\theta d\theta$$
$$= \int_0^{2\pi} d[a\cos\theta\sin(a\sin\theta)] + \int_0^{2\pi} a^2\cos^2\theta\, d\theta$$
$$= [a\cos\theta\sin(a\sin\theta)]_0^{2\pi} + \int_0^{2\pi} a^2\cos^2\theta d\theta$$
$$= 0 + a^2\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$
$$= \frac{a^2}{2}\cdot 2\pi = \pi a^2$$

Problem 2.9.9. *If $F = 3xyi - y^2j$. Evaluate $\int F.dR$, where C is the curve in the xy-plane $y = 2x^2$ from (0,0) to $(1,2)$* *[VTU 2010]*

Solution: Since the particle moves in the xy -plane $(z = 0)$
We take $\vec{r} = x_i + y_j$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (3xyi - y^2 j) \cdot (dxi + dyj)$$

where C is the parabola $y = 2x^2$. Then $dy = 4xdx$ and x varies from 0 to 1.

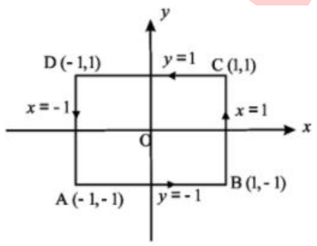
$$\int_C \vec{F} \cdot d\vec{r} = \int_C (3xydx - y^2 dy) \qquad \cdots (1)$$

Substituting $y = 2x^2$, where x goes from 0 to 1
Therefore (1) becomes

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_{x=0}^{1} \left[3x\left(2x^2\right)dx - \left(2x^2\right)^2 (4xdx)\right] \\ &= \int_0^1 \left(6x^3 - 16x^5\right) dx \\ &= \frac{-7}{6}\end{aligned}$$

Problem 2.9.10. *Evaluate the line integral* $\int_C \left[\left(x^2 + xy\right) dx + \left(x^2 + y^2\right) dy\right]$ *where* $\mathbf{c}$ *is the square formed by the lines* $x = \pm 1$ *and* $y = \pm 1$.

Solution::



$$\text{Here } \int_C \vec{F} \cdot d\vec{r} = \int_C \left[\left(x^2 + xy\right) dx + \left(x^2 + y^2\right) dy\right]$$

In the counter clockwise direction

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r} \quad \cdots (1)$$

Along AB:
Here $y = -1$. $\therefore dy = 0$

$$\begin{aligned}\therefore \int_{\mathrm{AB}} \vec{F} \cdot d\vec{r} &= \int_{-1}^{1} \left(x^2 - x\right) dx \\ &= \int_{-1}^{1} x^2 dx - \int_{-1}^{1} x dx \\ &= 2\int_{0}^{1} x^2 dx - 0 = 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3}\end{aligned}$$

Along BC:

Here $x = 1$. $\therefore dx = 0$

$$\begin{aligned}\therefore \int_{\mathrm{BC}} \vec{F} \cdot d\vec{r} &= \int_{-1}^{1} \left(1 + y^2\right) dy \\ &= 2\int_{0}^{1} \left(1 + y^2\right) dy \\ &= 2\left(y + \frac{y^3}{3}\right)_0^1 \\ &= 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}\end{aligned}$$

Along CD :

Here $y = 1$. $\therefore dy = 0.$

$$\begin{aligned}\therefore \int_{CD} \vec{F} \cdot d\vec{r} &= \int_{1}^{-1} \left(x^2 + x\right) dx \\ &= (-1)\int_{-1}^{1} \left(x^2 + x\right) dx \\ &= (-1)\left[2\int_{0}^{1} x^2 dx + 0\right] \\ &= -\frac{2}{3}\end{aligned}$$

Along DA:

Here $x = -1$. $\therefore dx = 0.$

$$\therefore \int_{DA} \vec{F}\cdot d\vec{r} = \int_{1}^{-1} \left(1+y^2\right) dy$$
$$= (-1)\int_{-1}^{1} \left(1+y^2\right) dy$$
$$= (-2)\int_{0}^{1} \left(1+y^2\right) dy$$
$$= (-2)\left(y+\frac{y^3}{3}\right)_0^1$$
$$= (-2)\left(1+\frac{1}{3}\right) = -\frac{8}{3}$$

Hence the required line integral in the counter clockwise direction is

$$\int_C \vec{F}\cdot d\vec{r} = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0, \text{ using } (1).$$

2.10 Green's theorem in a plane:

If R is a closed region of the $x-y$ plane bounds by a simple closed curve C and if M and N are two continuous function of x, y having continuous first order partial derivatives in the region R then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Problem 2.10.1. *Using Green's theorem evaluate* $\int_c \left[\left(xy+y^2\right)dx + x^2dy\right]$, *where C is rounded by* $y = x$ *and* $y = x^2$ *[VTU- July 2019, Jan10,Dec 11, June 17]*

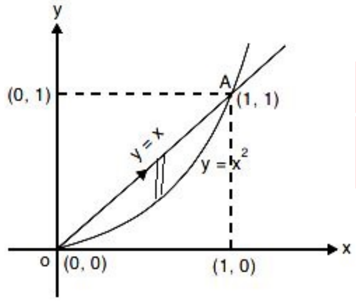
Solution: We shall find the points of intersection of $y = x$ and $y = x^2$.
Equating the expressions of y from both equations

$$x = x^2 \Rightarrow x - x^2 = 0$$
$$x(1-x) = 0$$
$$x = 0,\ 1$$
$$\therefore y = 0,\ 1$$

Hence $(0,0),\ (1,1)$ are the points of intersection.
The given integral is in the form $\oint_C Mdx + Ndy$ where

$M = xy + y^2 \& N = x^2$

The region R bounded by the curves is as shown in the following figure.

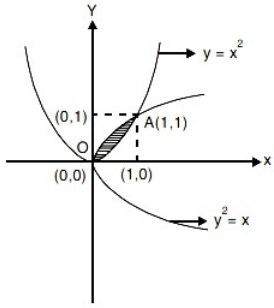


By Greens Theorem,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$
$$= \iint_R \frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(xy + y^2\right)$$
$$= \iint_R 2x - (x + 2y)$$
$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x - 2y) dydx$$
$$= \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx$$
$$= \int_{x=0}^{1} \left[\left(x^2 - x^2\right) - \left(x^3 - x^4\right)\right] dx$$
$$= \int_{x=0}^{1} \left(x^4 - x^3\right) dx$$
$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$
$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

Problem 2.10.2. *Using green's theorem evaluate* $\oint_C \left(3x^2 - 8y^2\right) dx + (4y - 6x) dy$, *where* c *is the boundary of the region enclosed by* $y = \sqrt{x} \& y = x^2$

Solution: The region enclosed by $y = \sqrt{x} \& y = x^2$ is as shown in the figure.



We shall find the points of intersection of the parabola $y=\sqrt{x}$ & $y=x^2$

At the point of intersection,

$$y=\sqrt{x} \& y=x^2$$
$$\sqrt{x}=x^2$$
$$x=x^4$$
$$\left(x-x^4\right)=0$$
$$x\left(1-x^3\right)=0,$$
$$\Rightarrow x=0, x=1$$

When $x=0, y=x^2 \Rightarrow y=0$

$x=1, y=x^2 \Rightarrow y=1$

Hence the points of interaction are $(0,0)\&(1,1)$ The given integral is in the form $\oint_C Mdx+Ndy$ where

$M=3x^2-8y^2 \quad N=4y-6xy$

$$\frac{\partial M}{\partial y}=-16y \quad \frac{\partial N}{\partial x}=-6y$$

By green's theorem,

$$\begin{aligned}\oint_c Mdx+Ndy &= \iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy\\ &=\iint_R(-6y+16y)dxdy\\ &=\int_0^1\int_{y=x^2}^{\sqrt{x}}(-6y+16y)dydx\end{aligned}$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (10y)dydx$$

$$= \int_0^1 \left(10\frac{y^2}{2}\right)\Big|_{x^2}^{\sqrt{x}} \mathrm{d}x$$

$$= \int_0^1 (5y^2)_{x^2}^{\sqrt{x}} \mathrm{d}x$$

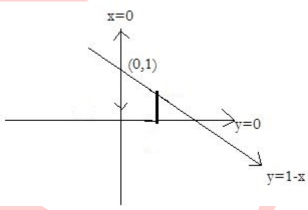
$$= \int_0^1 5\left(x - x^4\right)\mathrm{d}x$$

$$= \left(\frac{5x^2}{2} - \frac{5x^5}{5}\right)_0^1$$

$$= \frac{5}{2} - 1 = \frac{3}{2}$$

Problem 2.10.3. *Using Green's theorem evaluate* $\oint_C \left(3x^2 - 8y^2\right) dx + (4y - 6xy)dy$ *where* C *is boundary of the region defined by* $x = 0, y = 0$ & $x + y = 1$

Solution: The region bounded by $x = 0, y = 0$ & $x + y = 1$ is given by



The given integral is in the form $\oint_C Mdx + Ndy$ where
$M = 3x^2 - 8y^2, \quad N = 4y - 6xy$
$\frac{\partial M}{\partial y} = -16y$ and $\frac{\partial N}{\partial x} = -6y$
By green's theorem,

$$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\int_C \left[\left(xy + y^2\right) dx + x^2 dy\right] = \int_{x=0}^1 \int_{y=0}^{1-x} (-6y) - (-16y)dxdy$$

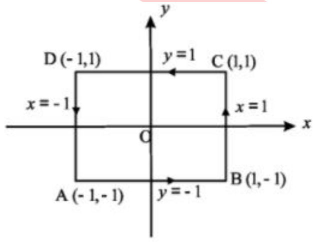
$$= \int_{x=0}^1 \int_{y=0}^{1-x} 10ydxdy$$

$$= \int_{x=0}^1 10\left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= \int_{x=0}^{1} \frac{10}{2}(1-x)^2\,dx$$
$$= 5\left[\frac{(1-x)^3}{-3}\right]_{x=0}^{1}$$
$$= \frac{-5}{3}\left[(1-1)^3-(1-0)^3\right]$$
$$= -5\left[\frac{0-1}{3}\right]$$
$$= \frac{5}{3}$$

Problem 2.10.4. *Use green's theorem to evaluate* $\int_c \left(x^2+xy\right)dx+\left(x^2+y^2\right)dy$, *where* c *is the square formed by the lines* $y=\pm 1, x=\pm 1$

Solution:: The square formed by the lines $y=\pm 1, x=\pm 1$ is in the form :



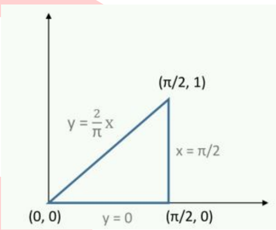
The given integral is in the form $\oint_C Mdx+Ndy$ where
$M=x^2+xy \quad N=x^2+y^2$
By green's theorem, we have

$$\oint_C (Mdx+Ndy)=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$
$$=\int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial}{\partial x}\left(x^2+y^2\right)-\frac{\partial}{\partial y}\left(x^2+xy\right)\right]dxdy$$
$$=\int_{-1}^{1}\int_{-1}^{1}(2x-x)dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x dx dy$$
$$= \int_{-1}^{1} x dx \int_{-1}^{1} dy$$
$$= \int_{-1}^{1} x dx \ [y]_{-1}^{1}$$
$$= \int_{-1}^{1} x dx \ (1+1)$$
$$= \int_{-1}^{1} 2x dx$$
$$= 2\left(\frac{x^2}{2}\right)_{-1}^{1} = 1 - 1 = 0$$

Problem 2.10.5. *Using Green's theorem, evaluate* $\int_C [(y - \sin x)dx + \cos x dy]$ *where* C *is the plane triangle enclosed by the lines* $y = 0, x = \pi/2$ *and* $y = \frac{2}{\pi}x$

Solution:



The given integral is in the form $\oint_C Mdx + Ndy$ where $M = (y - \sin x)$ and $N = \cos x$ By green's theorem, we have

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$
$$\therefore \int_C [(y - \sin x)dx + \cos x dy]$$
$$= \int_{x=0}^{x=\frac{\pi}{2}} \int_{y=0}^{y=\frac{2x}{\pi}} (-\sin x - 1)dydx$$
$$= -\int_0^{\frac{\pi}{2}} (\sin x + 1) \ \ [y]_0^{\frac{2x}{\pi}} dx$$
$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x(\sin x + 1)dx$$

$$= -\frac{2}{\pi}\left\{[x(-\cos x + x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (1) \times (-\cos x + x) dx\right\}$$

(Using Integration by parts)

$$= -\frac{2}{\pi}\left\{\left[\frac{\pi}{2}(-\cos\frac{\pi}{2} + \frac{\pi}{2}) - 0\right] - \int_0^{\frac{\pi}{2}} (1) \times (-\cos x + x) dx\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left[-\sin x + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}}\right\}$$

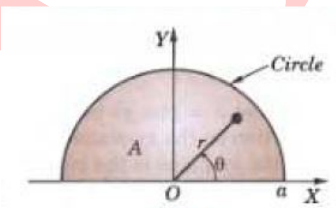
$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left[-\sin\frac{\pi}{2} + \frac{\left(\frac{\pi}{2}\right)^2}{2} - 0\right]\right\}$$

$$= -\frac{\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right)$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Problem 2.10.6. *Apply Green's theorem to evaluate* $\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$, *where* C *is the boundary of the area enclosed by the x-axis and the upper -half of the circle* $x^2 + y^2 = a^2$

Solution::



By Green's theorem

$$\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$$

$$= \iint_A \left[\frac{\partial}{\partial x}\left(x^2 + y^2\right) - \frac{\partial}{\partial y}\left(2x^2 - y^2\right)\right] dxdy$$

$$= 2\iint_A (x + y) dxdy, \quad \text{where } A \text{ is the region given in the figure}$$

Changing to polar coordinates (r, θ), we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$. Also r varies from 0 to a and θ varies from 0 to π

$$\begin{aligned}
\text{Integral } &= 2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta)\cdot r d\theta dr \\
&= 2\int_0^a r^2 dr \times \int_0^\pi (\cos\theta + \sin\theta) d\theta \\
&= 2\left(\frac{r^3}{3}\right)_0^a \times [\sin\theta - \cos\theta]_0^\pi \\
&= 2\frac{a^3}{3} \times [(\sin\pi - \cos\pi) - (\sin 0 - \cos 0)] \\
&= 2\frac{a^3}{3} \times (1+1) = \frac{4a^3}{3}
\end{aligned}$$

2.11 Area using Green's Theorem

Problem 2.11.1. *Apply Green's theorem to prove the area enclosed by a plane curve is* $\frac{1}{2}\int_C xdy - ydx$

Solution: We have Green's theorem that

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \qquad \cdots (1)$$

Now taking $M = -y$ and $N = x$, in equation (1), it gives

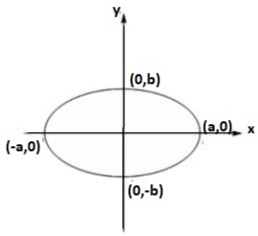
$$\begin{aligned}
\int_C (-y)dx + (x)dy &= \iint_R [1 - (-1)] dxdy \\
&= 2\iint_R dxdy \\
\Rightarrow \frac{1}{2}\oint_C xdy - ydx &= \iint_R dxdy \qquad \cdots (2)
\end{aligned}$$

The RHS of (2) represents the area of the region R bounded by the simple closed curve C.

$\therefore$ the area enclosed by a plane curve $= \boxed{\frac{1}{2}\int xdy - ydx}$.

Problem 2.11.2. *Using Green's theorem find the area of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: The given ellipse is in the form



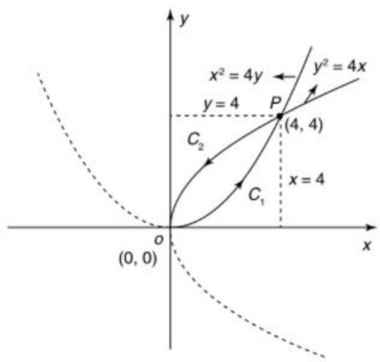
The parametric form of equation of the ellipse is $x = a\cos\theta, y = b\sin\theta$

Then $dx = -a\sin\theta d\theta$ and $dy = b\cos\theta d\theta$

$$A = \frac{1}{2}\oint_c xdy - ydx$$
$$= \frac{1}{2}\int_0^{2\pi}(a\cos\theta)(b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)$$
$$= \frac{1}{2}ab\int_0^{2\pi}(cos^2\theta + \sin^2\theta)d\theta$$
$$= \frac{1}{2}ab\,(2\pi)$$
$$= \pi ab$$

Problem 2.11.3. *Find the area between the parabolas* $y^2 = 4x \& x^2 = 4y$ *with help of green's theorem in a plane.* *(VTU Model 2018)*

Solution:



For intersection points, substitute

$$y^2 = 4x \text{ or } y = 2\sqrt{x} \text{ in } x^2 = 4y$$

$$\Rightarrow x^2 = 4 \times 2\sqrt{x}$$

$$\Rightarrow x^4 = 64x$$

$$\Rightarrow x\left(x^3 - 64\right) = 0$$

$$\Rightarrow x = 0, \quad \text{and } x = 4$$

$$\text{When } x = 0,\ y^2 = 4 \times 0 \Rightarrow y = 0$$

$$\text{When } x = 4,\ y^2 = 4 \times 4 \Rightarrow y = 4$$

Hence the Points of intersection are $(0,0)$ and $(4,4)$. On $C_1 : \quad x^2 = 4y$

$\therefore \quad 2xdx = 4dy \quad \Rightarrow \quad dy = \frac{1}{2}xdx$

and x varies from 0 to 4.

$$I_1 = \int_{C_1} xdy - ydx$$

$$= \int_0^4 x \cdot \frac{1}{2}xdx - \frac{x^2}{4}dx$$

$$= \int_0^4 \left(\frac{x^2}{2} - \frac{x^2}{4}\right) dx$$

$$= \int_0^4 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

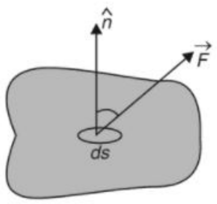
$$= \frac{64}{4 \cdot 3} = \frac{16}{3}$$

On $C_2 : y^2 = 4x \quad \therefore \quad 2ydy = 4dx \Rightarrow dx = \frac{1}{2}ydy$

and y varies from 4 to 0.

$$\therefore I_2 = \int_{C_2} xdy - ydx$$
$$= \int_4^0 \frac{y^2}{4}dy - y \cdot \frac{1}{2}ydy$$
$$= \int_4^0 \left(\frac{y^2}{4} - \frac{y^2}{2}\right) dy$$
$$= \int_4^0 -\frac{y^2}{4}dy$$
$$= \frac{1}{4}\int_0^4 y^2dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{16}{3}$$
$$\therefore \quad \text{Required area} = \frac{1}{2}\left[\frac{16}{3} + \frac{16}{3}\right] = \frac{16}{3}$$

2.12 Surface Integral



If (x, y, z) are the coordinates of a point in space, then the equation $z = f(x, y)$ or $f(x, y, z) = 0$ represents a surface S. If S has a unique normal at each of its points whose direction depends continuously on the points of S, then the surface S is called a smooth surface. Surface integral is a generalization of a double integral. In a surface integral the integrand is integrated along a curved surface.

Consider a vector function $\vec{F}$ defined over a surface S. Let $\mathrm{P(x, y, z)}$ be a point on the surface S and let $\hat{n}$ be the unit outward normal to the surface S at P. Then the normal surface integral of $\vec{F}$ over S is denoted by $\iint_S F.dS$ or $\iint_S F \cdot ndS$.

Other types of surface integrals are $\iint_s (\nabla \times F) \cdot \hat{n}dS$ and $\int fdS$

2.13 Evaluation of a Surface Integral:

A surface integral is evaluated by reducing it to a double integral by projecting the given surface S onto one of the coordinate planes.
Let R be the projection of S onto the xy -plane. Then,

$$dS = \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$

Then,

$$\boxed{\iint_S \vec{F} \cdot \hat{n} dS = \iint_R \vec{F} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}}$$

where $\hat{n}$ is unit outward drawn normal to S.
In a similar way the surface integral can be evaluated by projecting S onto the yz -plane as R_1 and xz -plane as R_2 as follows :

$$\iint_S \vec{F} \cdot \hat{n} dS = \iint_{R_1} \vec{F} \cdot \hat{n} \frac{dydz}{|\hat{n} \cdot \vec{i}|}$$

$$\iint_S \hat{F} \cdot \hat{n} dS = \iint_{R_2} \vec{F} \cdot \hat{n} \frac{dxdz}{|\hat{n} \cdot \hat{j}|}$$

Note : Suppose the velocity of a fluid in xyz space is described by the vector field $\mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{z})$. Let S be a surface in xyz space. The amount of the fluid flowing through the surface per unit time is called the **flux** of fluid through the surface. For this reason, the surface integral $= \iint_S (\vec{F} \cdot \hat{n}) ds$ is often called the **flux integral**.
If $\iint_S (\vec{F} \cdot \hat{n}) ds = 0$, then $\vec{F}$ is said to be a solenoidal vector point function.

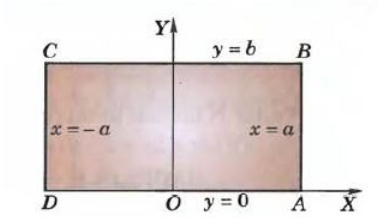
2.14 Stokes Theorem

If S is a surface bounded by a simple closed curve C and if $\vec{F}$ any continuously differentiable vector is function then

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S \mathrm{curl} \vec{F} \cdot \hat{n} ds = \iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n} ds}$$

Problem 2.14.1. *Using Stoke's theorem evaluate* $\int_C F \cdot dR$ *for* $\vec{F} = \left(x^2 + y^2\right) i - 2xyj$ *taken round the rectangle bounded by the lines* $x = \pm a, y = 0 \& y = b$ *(VTU-June17)*

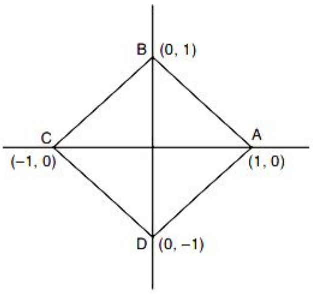
Solution:: Let $ABCD$ be the given rectangle as shown in figure.



$$
\begin{aligned}
\text{CurF} &= \nabla \times \vec{F} \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} \\
&= i(0-0) - j(0-0) + k(-2y-2y) \\
&= -4yk \\
\hat{n} &= k \\
ds &= \frac{dxdy}{|\hat{n}\cdot k|} = dxdy \\
\int_s \text{CurF}\cdot\hat{n}\ \text{ds}) &= \iint(-4y)dxdy \\
&= -4\int_0^b\int_{-a}^{a} ydxdy \\
&= -\ 4\int_0^b x\Big|_{-a}^{a} ydy \\
&= -4(2a)\int_0^b ydy \\
&= -\left[8a\frac{y^2}{2}\right]_0^b \\
&= -4ab^2
\end{aligned}
$$

Problem 2.14.2. *Evaluate* $\int_c xydx + xy^2dy$ *by stoke's theorem where* C *is the square in the* xy *plane with vertices* $(1,0)(-1,0),(0,1),(0,-1)$

Solution:



According to stoke's theorem we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} ds$$

Now,

$$\begin{aligned} \text{curl}\, \vec{F} &= \nabla \times \vec{F} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} \\ &= \left(y^2 - x\right) \hat{k}, \end{aligned}$$

$$\text{Further } ds = \frac{dxdy}{|\hat{n} \cdot \hat{k}|} = dxdy$$

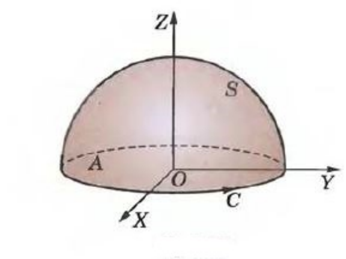
$$\therefore \iint_S \text{curl}\, \vec{F} \cdot \hat{n} ds = \iint \left(y^2 - x\right) dxdy$$

It can be clearly seen from the figure that $-1 \le x \le 1$ and $-1 \le y \le 1$ Now,

$$\begin{aligned} \iint_S \text{curl}\, \vec{F} \cdot \hat{n} ds &= \int_{x=-1}^{1} \int_{y=-1}^{1} \left(y^2 - x\right) dydx \\ &= \int_{x=-1}^{1} \left[\frac{y^3}{3} - xy\right]_{y=-1}^{1} dx \\ &= \int_{x=-1}^{1} \left[\left(\frac{1}{3} + \frac{1}{3}\right) - x(1+1)\right] dx \\ &= \int_{x=-1}^{1} \left(\frac{2}{3} - 2x\right) dx \\ &= \left[\frac{2}{3}x - x^2\right]_{x=-1}^{1} \\ &= \frac{2}{3}(1+1) - (1-1) = \frac{4}{3} \end{aligned}$$

Problem 2.14.3. *Using Stoke's theorem Evaluate* $\int_C \boldsymbol{F} \cdot \boldsymbol{dR}$ *for the vector field* $\boldsymbol{F} = (2x - y)i + yz^2 j - y^2 zk$ *over the upper half of surface of* $x^2 + y^2 + z^2 = 1$ *bounded by its projection on the xy plane*

Solution::



$$\int_C \vec{F} \cdot \overrightarrow{dr} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} dS \text{ (Stoke's theorem)}$$

C is the circle: $x^2 + y^2 = 1, z = 0(xy$ -plane)

Here unit normal vector to the surface of the sphere $\phi = x^2 + y^2 + z^2 = 1$ is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) i + \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right) j + \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right) k$$

$$= 2xi + 2yj + 2zk$$

$$|\nabla\phi| = \sqrt{4(x^2 + y^2 + z^2)}$$

$$= \sqrt{4} \qquad (\because (x^2 + y^2 + z^2) = 1)$$

$$= 2$$

$$\hat{n} = \frac{2xi + 2yj + 2zk}{2} = xi + yj + zk$$

$$|\hat{n} \cdot k| = z$$

$$ds = \frac{dxdy}{|\hat{n}\cdot k|} = \frac{dxdy}{z}$$

$$\text{Now} \quad \text{curl F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)i + 0j + k$$

$$= k$$

$$\therefore \quad \int \text{curl}\,\vec{F}\cdot\hat{n}ds = \iint_S z\cdot\frac{dxdy}{z}$$

$$= \iint_R dxdy$$

$$= \text{ area of circle } \; C : x^2+y^2 = 1$$

$$= \pi$$

Here R is the projection of S on xy -plane

2.15 Question Bank : Module 2-Vector Calculus

2.15.1 Question Bank :The Gradient, Divergence and Curl :

1. Find $grad\phi$ where $\phi = 3x^2y - y^3z^2$ at the point (1, -2, -1) (VTU Jan 2017)
2. If $\vec{F} = \nabla(xy^3z^2)$ find $divF$ and $curlF$ at (1,-1,1) (VTU Model 2018)
3. Find $divF$ and $curlF$ where $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$ Is $\vec{F}$ irrotational?

   (VTU Model 2022, Jan 2020, June 2019, July 2017, Jul 2015. Jul 2013, Jan 2008, Aug 2002)
4. If $F = (x+y+1)i + j - (x+y)k$, then show that $\vec{F}.curl\vec{F} = 0$ (VTU Jan 2014, July 2013)
5. If $\vec{V} = \vec{w}\times\vec{r}$, P.T. $w = \frac{1}{2}(\nabla\times\vec{V})$ or $curl\vec{V} = 2\vec{w}$ where $\vec{w}$ is a constant vector.

   (VTU Jan 2015)

6. If $\vec{r} = xi + yj + zk$, and $|\vec{r}| = r$, find $graddiv\left(\frac{\vec{r}}{r}\right)$ (VTU Jan 2015)

7. If $\phi = x^2 + y^2 + z^2$, and $\vec{F} = x^2 i + y^2 j + z^2 k$, then find $grad\phi$, $div\vec{F}$ and $curl\vec{F}$.

   (VTU July 2014) **Ans:** $2xi + 2yj + 2zk$, $2x + 2y + 2z$, $\vec{0}$.

8. If $\vec{F} = 2xy^3z^4 i + 3x^2y^2z^4 j + 4x^2y^3z^3 k$, find (i) $\nabla \cdot \vec{F}$ (ii) $\nabla \times \vec{F}$ (VTU Jan 2014)

9. $div\vec{F}$ and $curlF$ at the point (1,2,3) if $\vec{F} = grad(x^3y + y^3z + z^3x - x^2y^2z^2)$ (VTU 2007)

10. Find the divergence and curl of the vector $\vec{V} = xyzi + 3x^2yj + (xz^2 - y^2z)k$ at the point (2,-1,1)

    (VTU Jan 2017)

11. If $\vec{F} = x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$, find $\text{Curl}(\text{Curl}\ \vec{F})$ (VTU Model 2022)

12. If $\Phi = x^3y^3z^3$, then find $\nabla\Phi$ at $(1, 2, 1)$

    Ans : $24i + 12j + 24k$

13. If $\vec{V} = 3xzi + 2xyj - yz^2k$, find $div\vec{V}$

    Ans : $3z + 2x - 2yz$.

14. If $\vec{V} = yzi + 3xzj + zk$, find $curl\vec{V}$

    Ans : $-3xi + yj + 2zk$

15. If $\Phi = x^2 - y^2 - z^2 - 2$, find $grad\Phi$ at (1,-1,2)

    Ans : $2i + 2j - 4k$

16. If $F = xy^2i + 2x^2yzj + 3yz^2k$, then find $div(curlF)$

    Ans : 0

17. If $F = (3x^2y - z)i + (xz^3 + y^4)j - 2x^3z^2k$, then find $grad(divF)$ at $(2, -1, 0)$

    Ans : $-6i + 24j - 32k$

18. If $\vec{F} = 3xyz^2i - 4x^2yj - xy^2zk$, then find $\nabla(\nabla \cdot F)$ at $(-1, 2, 1)$
    Ans : $-16i + 7j + 12k$

19. If $A = x^2yi - 2xzj + 2yzk$, find $curl(curlA)$.
    Ans : $2(x + 2)j$

20. If $A = x^2yi + xzj + 2yzk$, find $curl(curlA)$ at (1,1,1)
    Ans : $4j$

21. If $\phi = x^2 + y^2 + z^2$, and $\vec{F} = x^2i + y^2j + z^2k$, then find $grad\phi$, $div\vec{F}$ and $curl\vec{F}$. (VTU July 2014)
    Ans: $2xi + 2yj + 2zk$, $2x + 2y + 2z$, $\vec{0}$.

22. Find $div\vec{F}$ and $curlF$ at the point (1,2,3) if $\vec{F} = x^2yzi + xy^2zj + xyz^2k$
    Ans : 12, $5i - 16j + 9k$

23. $div\vec{F}$ and $curlF$ at the point (1,2,3) if $\vec{F} = 3x^2i + 5xy^2j + 5xyz^3k$
    Ans : 278, $5(27i - 54j + 8k)$

24. If $\vec{V} = \frac{\vec{r}}{r}$, show that $div(\vec{V}) = \frac{2}{r}$ and $curl(\vec{V}) = 0$

25. Find $curl(grad\ f)$ given $f(x, y, z) = x^2 + y^2 - z$
    Ans : 0

26. Find $curl(curl\vec{A})$, given $\vec{A} = x^2yi + y^2zj + z^2yk$ (VTU 2003)
    Ans : $2(x + z)j + 2yk$

27. prove that $\nabla\left(\frac{1}{r^2}\right) = \frac{-2\vec{r}}{r^4}$

28. prove that $\nabla r = \frac{1}{r}\vec{r} = \hat{r}$

29. Find $divF$ and $curlF$ where $\vec{F} = \nabla(xy^3z^2)$ (VTU Model 2018)

2.15.2 Question Bank :Angle between two surfaces :

1. Find the angle between two surfaces $x^2 + y^2 + z^2 = 9$ & $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$
   (VTU Model 2022, June 2019, July 2017, Dec 2010, Jul 2009)

2. Find the angle between two surfaces $x^2 + y^2 + z^2 = 4$ & $z = x^2 + y^2 - 13$ at the point $(2, 1, 2)$
   (VTU Model 2018)

3. Find the constants a & b so that the surface $ax^2 - byz = (a + 2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$. (VTU Model 2018)

4. Find the angle between two surfaces $xlogz = y^2 - 1, x^2y = 2 - z$ at the point (1,1,1)

5. Find the unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$
   Ans : $\frac{1}{\sqrt{11}}(-i - 3j + k)$

6. Find the unit vector normal to the surface $x^2y + y^2z + z^2x = 5$ at the point $(1, -1, 2)$
   Ans : $\frac{2i-3j+5k}{\sqrt{38}}$

7. Find the unit vector normal to the surface $x^2 + y^2 + z^2 = 3$ at the point $(1, 1, 1)$ **Ans :** $\frac{i+j+k}{\sqrt{3}}$

8. Find a unit normal to the surface $xy^2z^3 = 1$ at the point $(1, 1, 1)$.
   Ans : $\frac{i+2j+3k}{\sqrt{14}}$

9. Find the angle between two surfaces $x^2 + y^2 + z^2 = 9$ & $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$ **(VTU Dec 2010, Jul 2009)**
   Ans : $cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

10. Find the angle between the directions of the normal to the surface $x^2yz = 1$ at the points $(-1, 1, 1)$ and $(1, -1, -1)$. **Ans :** $\theta = \pi$

11. Show that the surfaces $5x^2 - 2yz - 9x = 0$ and $4x^2y + z^3 - 4 = 0$ are orthogonal at the point $(1, -1, 2)$.

12. Find the unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$ **Ans :** $\frac{-i+3j+2k}{\sqrt{14}}$

13. Find the angle between two surfaces $xlogz = y^2 - 1, x^2y = 2 - z$ at the point (1,1,1) **Ans :** $cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$

14. Find the constants $a\&b$ so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$
Ans : $a = -6, b = -10$

2.15.3 Question Bank :Directional Derivative :

1. Find the directional derivative of $4xz^3 - 3x^2y^2z$ at (2,-1,2) along $2i - 3j + 6k$. (VTU Jan 2020, Model 2018, Jan 2015) **Ans :** $\frac{376}{7}$

2. Find the directional derivative of $xy^3 + yz^3$ at $(2, -1, 1)$ in the direction of the vector $i + 2j + 2k$
(VTU July 2017)

3. Find the directional derivative of $\Phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of the vector $2i - j - k$ (VTU Aug 2005, Aug 2004, Mar 2000)

4. Find the directional derivative of $f = xy^2 + yz^3$ at the point $(1, 2, -1)$ in the direction of the normal to the surface $xlogz - y^2 = -4$ at the point $(-1, 2, 1)$. (VTU Feb 2004)

5. In which direction the directional derivative of x^2yz^3 is maximum at $(2, 1, -1)$ and find the magnitude of this maximum. (VTU Jan 2009, Dec 2008) **Ans :** $-4i - 4j + 12k, 4\sqrt{11}$

6. Find the directional derivative of $\Phi = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of the vector $2i - j - 2k$ (VTU 2007) **Ans :** $\frac{37}{3}$

7. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$ (Model 2022)

8. If $\Phi = \frac{xz}{x^2+y^2}$, then find the directional derivative at $(1, -1, 1)$ in the direction of $\vec{a} = i - 2j + k$ **Ans :** $\frac{-1}{2\sqrt{6}}$

9. Find the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of $i + 2j + 2k$ **Ans :** $\frac{-11}{3}$

10. Show that the directional derivative of $\Phi = x^3y^2z$ is maximum along the direction of $9i + 3j + k$ at (1,2,3). Also find magnitude of maximum.
    Ans : $\nabla\Phi|_{(} 1, 2, 3) = 4(9i + 3j + k)$ and $|\nabla\Phi| = 4\sqrt{91}$

11. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.
    Ans : $96(i + 3j - 3k)$, $96\sqrt{19}$

12. Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1, 2, 3)$ in the direction of $2i + j - k.$ Ans : $\frac{86}{\sqrt{6}}$

13. Find the directional derivative of $f = x^2 - y^2 + 2z^2$ in the direction of $4i - 2j + k$. Also calculate the magnitude of the maximum directional derivative.
    Ans: $\frac{28}{\sqrt{21}}$, $\sqrt{164}$

14. Find the directional derivative of $\nabla \cdot \nabla\Phi$ at the point (1, -2, 1) in the direction of normal to the surface $x^2yz = 3x + z^2$, where $\Phi = 2x^2y^2z^4$ **Ans :** $\frac{1724}{21}$

15. Find the directional derivative of $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ in the direction $2i - j - k$ Ans : $\frac{16}{\sqrt{6}}$

2.15.4 Question Bank :Solenoidal and irrotational vector fields

1. Find 'a' for which $\vec{F} = (x + 3y)i + (y - 2z)j + (x + az)k$ is solenoidal. (VTU Jan 2017)

2. Prove that $\frac{xi+yj}{x^2+y^2}$ is both solenoidal & irrotational. (VTU Model 2018, Model 2014)

3. Find the constants a, b, c such that the vector field $\vec{f} = (x+2y+az)i+(bx-3y-z)j+(4x+cy+2z)k$ is irrorational. (VTU July 2014)

4. Find the value of the constant a such that the vector field $\vec{F} = (axy+bz^3)i+(3x^2-cz)j+(3xz^2-y)k$ is a conservative force field. Hence find a scalar function Φ such that $\vec{F} = \nabla\Phi$ (VTU June 2019, Model 2018)

5. Find the constants a, b, c such that the vector field $\vec{f} = (x+y+az)i + (x+cy+2z)k+(bx+2y-z)j$ is irrorational. (VTU July 2017). Also find Φ such that $\vec{F} = \nabla\Phi$ (VTU Jul 2015)

6. Find the value of the constant a such that the vector field $\vec{F} = (axy-z^3)i+(a-2)x^2j+(1-a)xz^2k$ is irrotational. Hence find a scalar function Φ such that $\vec{F} = \nabla\Phi$ (VTU Jul 2014)

7. Prove that $\vec{F} = (y^2-z^2+3yz-2x)i+(3xz+2xy)j+(3xy-2xz+2z)k$ is both solenoidal and irrotational. (VTU Jan 2016)

8. Find the constants a and b such that $F = (axy+z^3)i+(3x^2-z)j+(bxz^2-y)k$ is irrotational. Find Φ such that $F = \nabla\Phi$ (VTU Jan 2020, July 2014, Jan 2014, Jun 2012, Feb 2005)

9. Show that $\vec{F} = (x^2-yz)i+(y^2-zx)j+(z^2-xy)k$ is irrotational and hence find its scalar potential. (VTU July 2015, Jan 2013)

10. If $\vec{u} = x^2i+y^2j+z^2k$ and $\vec{v} = yzi+zxj+xyk$ Show that $\vec{u}\times\vec{v}$ is a solenoidal vector.

(VTU July 2017)

11. A vector field is given by $\vec{A} = (x^2+xy^2)i+(y^2+x^2y)j$, show that the field is irrotational and find the scalar potential. (VTU July 2017)

12. Show that $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational. (VTU Model 2022)

13. Define an irrotational vector. Find the constants a, b and c such that $\vec{F} = \left(axy - z^3\right)\hat{\imath}$
is irrotational. (VTU Model 2022)

14. Prove that $F = 3y^4z^2i + 4x^3z^2j - 3x^2y^2k$ is a solenoidal vector .

15. Find a for which $\vec{F} = (x + 3y)i + (y - 2z)j + (x + az)k$ is solenoidal. (VTU Jan 2017)

16. Find the value of m if $\vec{A} = (x + 2y)i + (my + 4y)j + (5z + 6x)k$ is a solenoidal vector. **Ans :** $m = -10$

17. Find the value of a if $\vec{F} = (ax + 3y + 4z)i + (x - 2y + 3z)j + (3x + 2y - z)k$ is a solenoidal vector. **Ans :** $a = 3$

18. Find the value of 'a' if the vector $\vec{F} = \left(2x^2y + yz\right)i + \left(xy^2 - xz^2\right)j + \left(axyz - 2x^2y^2\right)k$ is solenoidal. Ans: $a = -6$

19. Prove that $\vec{r}r^3$ is both solenoidal & irrotational .

20. Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational. (VTU Jan 2016)

21. Prove that $r^n r$ is solenoidal iff $n = -3$ and irrotational for all n.

22. Show that $\vec{F} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational. Find Φ such that $\vec{F} = \nabla\Phi$ **(VTU Aug 2003) Ans** $\Phi = 3x^2y + xz^3 - yz$

23. Show that $\vec{F} = (2x + yz)i + (4y + zx)j - (6z - xy)k$ is both solenoidal and irrotational and also find its scalar potential Φ **Ans :** $\Phi = x^2 + 2y^2 - 3z^2 + xyz$

24. Show that $\vec{F} = (y + z)i + (z + x)j + (x + y)k$ is conservative and also find its scalar potential. **Ans :** $\Phi = xy + yz + zx$

25. If $\nabla\Phi = (y^2 - 2xz^2 - 1)i + 2xyj + 2x^2zk$, find Φ **Ans :** $\Phi = xy^2 + x^2z^2 - x + c$

26. Show that $F = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ is a conservative force field and find its scalar potential.
Ans : $\Phi = x^2y^2 + y^2z^2 + xyz + c$)

27. Prove that $\vec{F} = (-x^2 + yz)i + (4y - z^2x)j + (2xz - 4z)k$ is solenoidal.

28. Show that $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right) i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is irrotational and solenoidal.

29. Determine the constants a and b such that the curl of vector
$\vec{A} = (2xy + 3yz)\hat{i} + \left(x^2 + axz - 4z^2\right)\hat{j} - (3xy + byz)\hat{k}$ is zero.
Ans : $a = 3, b = 8$

2.15.5 Question Bank :Line Integrals, workdone

1. Find the total work done in moving a particle in the force field $F = 3xyi - 5z] + 10xk$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$ (VTU June 2019)

2. If $\vec{F} = \mathrm{xyi} + \mathrm{yz}j + zxk$, Evaluate $\int_c \vec{F}.\overrightarrow{dr}$. Where c is the curve represented by $\mathrm{x} = \mathrm{t}, \mathrm{y} = t^2, \mathrm{z} = t^3, -1 \le t \le 1$ (VTU Jan 2020)

3. Find the total work done by the force $\vec{F} = 3x^2i + (2xz - y)j + zk$, along $x = t, y = \frac{t^2}{4}, z = \frac{3t^3}{8}$ (VTU Model 2018)

4. Find the total work done by the force represented by $\vec{F} = 3xyi - yj + 2zxk$ in moving a particle round the circle $x^2 + y^2 = 4$ [VTU-2010]

5. Find the work done in moving a particle in the force field $\vec{F} = 3x^2i + (2xz - y)j + zk$, along [VTU-17]

   (i) The straight line from (0,0,0) to (2,1,3). (VTU Model 2022)

(ii) The curve defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$

6. If $\vec{F} = 3xyi - y^2j$. Evaluate $\int \vec{F}.dR$ where C is the curve in the xy-plane $y = 2x^2$ from (0, 0) to (1, 2). [VTU 2010]

7. If $\vec{F} = (3x^2 + 6y)i - 14yzj + 20xz^2k$, evaluate $\int \vec{F}.d\vec{r}$ from (0,0,0) to (1,1,1) along the curve given by $x = t, y = t^2, z = t^3$ [VTU 2001]

8. A vector field is given by $\vec{F} = sinyi + x(1 + cosy)j$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2, z = 0$ [VTU Jan 2018]

9. If C is a simple closed curve in the xy-plane not enclosing the origin. Show that $\int_c \vec{F}.dR = 0$ where $\vec{F} = \frac{yi-xj}{(x^2+y^2)}$ [VTU Jan 2018]

10. If $\vec{F} = \left(5xy - 6x^2\right)\hat{\imath} + (2y - -4x)\hat{\jmath}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $C : y = x^3$ in the xy - plane from the point $(1, 1)$ to $(2, 8)$ (VTU Model 2022)

11. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left(x^2 - y^2\right)\mathrm{i} + (\mathrm{xy})\mathbf{j}$, where **c** is the arc of the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$ Ans: $\frac{824}{21}$

12. If $\vec{F} = x^2i + xyj$, Evaluate $\int_C \vec{F} \cdot d\vec{r}$ from (0,0) to (1,1) along (i) the line $y = x$ (ii) the parabola $y = \sqrt{x}$ Ans: (i) $\frac{2}{3}$ (ii) $\frac{7}{12}$

2.15.6 Question Bank :Greens Theorem

1. Using Green's theorem, Evaluate $\oint_c (x^2 + y^2)dx + 3x^2ydy$, where C is the circle $x^2 + y^2 = 4$ traced in the positive sense. (VTU Model 2018)

2. Using Green's theorem evaluate $\oint_c (xy + y^2)dx + x^2dy$, where C is rounded by $y = x$ and $y = x^2$ [VTU- Model 2022, Jan 2020, Jan10,Dec 11, June 17]

3. Using Green's theorem evaluate $\oint_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region enclosed by $y = \sqrt{x}$ & $y = x^2$ [VTU July 2018]

4. Find the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ using Green's theorem in a plane. [VTU: – June 2018]

5. Find the area between the parabolas $y^2 = 4x$ & $x^2 = 4y$ with help of green's theorem in a plane. (VTU Model 2018)

6. Apply Green's theorem to evaluate $\int_C \left[\left(3x - 8y^2\right) dx + (4y - 6xy)dy\right]$, where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$ (VTU Model 2022)

7. Using green's theorem, evaluate $\int_c \left(x^2ydx + x^2dy\right)$, where c is the boundary described counter clockwise of the triangle with vertices $(0, 0), (1, 0), (1, 1)$. Ans: $\frac{5}{12}$

8. Employ Green's theorem in a plane to show that the area enclosed by a plane curve c is $\frac{1}{2}\oint_c xdy - ydx$ and hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Ans: πab sq.units

9. Find the area of the asteroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ by employing green's theorem. Ans : $\frac{3\pi a^2}{8}$ sq .units

10. Using Green's theorem, Evaluate $\int_C \left(xy - x^2\right) dx + x^2ydy$, where c is the closed curve formed by $y = 0, x = 1 \& y = x$ Ans :$\frac{-1}{12}$

2.15.7 Question Bank :Surface Integrals, Stokes Theorem

1. Using Stoke's theorem, evaluate $\int_c \vec{F}.d\vec{R}$, where $\vec{F} = yi + zj + xk$ and C is the boundary of upper half of the sphere $x^2 + y^2 + z^2 = 1$ (VTU Model 2018)

2. Evaluate $\int_s F.dS$ where $\vec{F} = 4xi - 2y^2j + z^2k$, s is the surface bounding the region $x^2 + y^2 = 4, z = 0 \& z = 3$ [VTU-June17]

3. Verify stoke's theorem for $\vec{F} = (x^2 + y^2)i - 2xyj$ taken round the rectangle bounded by the lines $x = \pm a$, $y = 0 \& y = b$ [VTU-Model 2022, Jan 2020, June 2018, June 17]

4. If $\vec{F} = 2xyi + yz^2j + xzk$ and S is the rectangular parallelepiped bounded by $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3$ evaluate $\iint_s \vec{F}.nd\hat{s}$ [VTU-June 2018]

5. Use Stokes theorem to evaluate $\int_c \vec{F}.d\vec{R}$, where $\vec{F} = (2x - y)i - yz^2j - y^2zk$ over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy-plane. [VTU Jan 2018]

6. Using Stoke's theorem, evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (x^2 + y^2) - 2xy$, taken around the rectangle whose vertices are $(a, 0), (a, b), (-a, b), (-a, 0)$. (VTU Model 2022)

7. Evaluate $\int_c xydx + xy^2dy$ by stoke's theorem where c is the square in the x -y plane with vertices (1,0)(-1,0)(0,1)(0,-1). Ans : $\frac{8}{3}$

8. Using Stoke's theorem Evaluate $\int_C^{\square} F.dR$ for the vector field $F = (2x - y)i + yz^2j - y^2zk$ over the upper half of surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane Ans: π

9. Using Stoke's theorem Evaluate $\int_c (ydx + zdy + xdz)$ where c is the curve of interaction of $x^2 + y^2 + z^2 = a^2 \& x + z = a$

10. Using stoke's theorem evaluate $\int_c [(x + y)dx + (2x - z)dy + (y + z)dz]$, where C is the boundary of the triangle with vertices $(2, 0, 0)(0, 3, 0)\&(0, 0, 6)$ Ans : 21

Module 3

Partial Differential Equations(PDE's)

Syllabus:

Importance of partial differential equations for Mechanical Engineering application

Formation of PDE's by elimination of arbitrary constants and functions, Solution of nonhomogeneous PDE by direct integration, Homogeneous PDEs involving derivatives with respect to one independent variable only, Solution of Lagrange's linear PDE.Derivation of one-dimensional heat equation and wave equation.

Self-Study: Solution of the one-dimensional heat equation and wave equation by the method of separation of variables.

Applications: Vibration of a rod/membrane. (RBT Levels: L1, L2 and L3)

3.1 Partial Differential Equations (PDE):

An equation containing the dependent variable z, independent variables x and y and one or more partial derivatives of the dependent variable is called a partial differential equation. In general it may be written in the form

$f(x, y, z, z_x, z_y, ..., z_{xx}, z_{yy}, z_{xy}) = 0$

Note : If $z = z(x, y)$, then we are using following notations to solve the problems.

$\frac{\partial z}{\partial x} = z_x = p$; $\frac{\partial z}{\partial y} = z_y = q$; $\frac{\partial^2 z}{\partial x^2} = z_{xx} = r$; $\frac{\partial^2 z}{\partial x \partial y} = z_{xy} = s$; $\frac{\partial^2 z}{\partial y^2} = z_{yy} = t$

3.2 Formation of PDE by eliminating the arbitrary constants or function :

Consider a relationship of the form $f(x, y, z, a, b) = 0$ where $z = z(x, y)$ and a, b are arbitrary constants, the given relation is differentiated partially w.r.t x, y. Elimination of a, b from all these relations will give the PDE.

Problem 3.2.1. *Form the PDE by eliminating the arbitrary constants from the following.* $ax^2 + by^2 + z^2 = 1$

Solution::

$$ax^2 + by^2 + z^2 = 1 \qquad (1)$$

differentiate (1) partially w.r.t 'x'

$$2ax + 0 + 2zp = 0$$

$$\implies a = \frac{-zp}{x}$$

differentiate (1) partially w.r.t'y'

$$0 + 2by + 2zq = 0$$

$$\implies b = \frac{-zq}{y}$$

Using a, b values in (1),we get

$$-zpx - zqy + z^2 = 1$$

$$\implies zpx + zqy = z^2 - 1$$

This is the required partial differential equation.

Problem 3.2.2. *Form the PDE by eliminating the arbitrary constants a and b from* $(x - a)^2 + (y - b)^2 + z^2 = k^2$

Solution: : $(x - a)^2 + (y - b)^2 + z^2 = k^2 \qquad \ldots(i)$

Differentiating (i) partially w. r. t. x and y,

$(x - a) + z\frac{\partial z}{\partial x} = 0 \Rightarrow (x - a) = -z\frac{\partial z}{\partial x}$

$(y - b) + z\frac{\partial z}{\partial y} = 0 \Rightarrow (y - b) = -z\frac{\partial z}{\partial y}$

Substituting for $(x-a)$ and $(y-b)$

from these in (i), we get

$\left(-z\frac{\partial z}{\partial x}\right)^2 + \left(-z\frac{\partial z}{\partial y}\right)^2 + z^2 = k^2$

i.e. $z^2\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = \mathrm{k}^2$

This is the required partial differential equation.

Problem 3.2.3. *Form the PDE by eliminating the arbitrary constants from* $z = ax + by + cxy$

Solution:: $z = ax + by + cxy \quad \ldots$ (i)

Differentiating (i) partially w.r.to y, we get

$\frac{\partial z}{\partial x} = a + cy \ldots (ii)$

$\frac{\partial z}{\partial y} = b + cx \ldots (iii)$

It is not possible to eliminate a, b, c from relations (i) to (iii).

Partially differentiating (ii) w.r.to y,

$\frac{\partial^2 z}{\partial x \partial y} = c$

Using this in (ii) and (iii)

$\mathrm{a} = \frac{\partial \mathrm{z}}{\partial \mathrm{x}} - \mathrm{y}\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}\partial \mathrm{y}}$

$b = \frac{\partial z}{\partial y} - x\frac{\partial^2 z}{\partial x \partial y}$

Substituting for a b, c in (i), we get

$z = x\left[\frac{\partial z}{\partial x} - y\frac{\partial^2 z}{\partial x \partial y}\right] + y\left[\frac{\partial z}{\partial y} - x\frac{\partial^2 z}{\partial x \partial y}\right] + xy\frac{\partial^2 z}{\partial x \partial y}$

$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - xy\frac{\partial^2 z}{\partial x \partial y}$

This is the required partial differential equation.

Problem 3.2.4. *Form the PDE by eliminating the arbitrary functions from the following.* $z = y^2 + 2f(\frac{1}{x} + logy)$ *(VTU July 2006, 2007)*

Solution: $z = y^2 + 2f(\frac{1}{x} + logy)$.......(1)

differentiate (1) partially w.r.t 'x' $\frac{\partial z}{\partial x} = p = 2f'(\frac{1}{x} + logy)(\frac{-1}{x^2})$.......(2)

differentiate (1) partially w.r.t 'y' $\frac{\partial z}{\partial y} = q = 2y + 2f'(\frac{1}{x} + logy)(\frac{1}{y})$.......(3)

But from eq(2) $2f'(\frac{1}{x} + logy) = -px^2$

Substituting this in (3) we get, $\frac{\partial z}{\partial y} = 2y + -px^2(\frac{1}{y})$

$\therefore (3) \implies px^2 + qy = 2y^2$

This is the required partial differential equation.

Problem 3.2.5. *Form a partial differential equation from* $x^2 + y^2 + (z-c)^2 = a^2$

Solution: $x^2 + y^2 + (z-c)^2 = a^2$ (1)

(1) contains two arbitrary constants a and c.

Differentiating (1) partially w.r.t. x we get

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$x + (z-c)p = 0 \qquad (2)$$

Differentiating (1) partially w.r.t. y we get

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$y + (z-c)q = 0 \qquad (3)$$

Let us eliminate c from (2) and (3):

From (2), $(z-c) = -\frac{x}{p}$

Putting this value of $z-c$ in (3), we get

$y - \frac{x}{p}q = 0$

$py - qx = 0$

This is the required partial differential equation.

Problem 3.2.6. *Form a partial differential by eliminating arbitrary function from* $z = f\left(x^2 + y^2\right)$

Solution: Differentiating z partially w.r.t. x and y,

$p = \frac{\partial z}{\partial x} = f'\left(x^2 + y^2\right).2x$ (1)

$q = \frac{\partial z}{\partial y} = f'\left(x^2 + y^2\right) \cdot 2y$ (2)

$\frac{(1)}{(2)}$ gives

$\mathrm{p/q = x/y}$

or $\mathrm{yp - xq = 0}$ is the required PDE.

Problem 3.2.7. *Form a partial differential by eliminating arbitrary functions from* $z = f(x+ct) + g(x-ct)$

Solution: Differentiating z partially with respect to x and t

$\frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct)$ (1)

$\frac{\partial z}{\partial t} = f'(x+ct)c + g'(x-ct)(-c)$ (2)

$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct)$ (3)

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= f''(x+ct)c^2 + g''(x-ct)c^2 \qquad (4)\\ &= c^2\left[f''(x+ct) + g''(x-ct)\right]\\ &= c^2\frac{\partial^2 z}{\partial x^2}\end{aligned}$$

Thus the pde is $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$

Problem 3.2.8. *Form the PDE by eliminating arbitrary function from :* $x+y+z = f\left(x^2+y^2+z^2\right)$

Solution: $x+y+z = f\left(x^2+y^2+z^2\right)$ (1)

Differentiating (1) partially w.r.t. x and y $1+\frac{\partial z}{\partial x} = f'\left(x^2+y^2+z^2\right)\left[2x+2z\frac{\partial z}{\partial x}\right]$ (2)

and $1+\frac{\partial z}{\partial y} = f'\left(x^2+y^2+z^2\right)\left[2y+2z\frac{\partial z}{\partial y}\right]$ (3)

$(2) \div (3)$, gives

$$\frac{1+\frac{\partial z}{\partial x}}{1+\frac{\partial z}{\partial y}} = \frac{\left[2x+2z\frac{\partial z}{\partial x}\right]}{\left[2y+2z\frac{\partial z}{\partial y}\right]}$$

Re-arranging the terms, we get

$$(y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} = x-y.$$

This is the required pde.

Problem 3.2.9. *Form the PDE by eliminating arbitrary constants from:* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Solution: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ (1)

Differentiating partially w.r.t. x

$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x} = 0,$

or $\frac{x}{a^2} = -\frac{z}{c^2}\frac{\partial z}{\partial x}$
or $\frac{-c^2}{a^2} = \frac{z}{x}\frac{\partial z}{\partial x}$ (2)
Differentiating (2) partially w.r.to x, we get

$$0 = \frac{z}{x}\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\left[\frac{x\frac{\partial z}{\partial x} - z(1)}{x^2}\right]$$

Multiplying by x^2 and rearranging, we get

$$zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0$$

Problem 3.2.10. *Form the PDE by eliminating arbitrary function from:* $z = f(xy/z)$

Solution:: Given $z = f(xy/z)$ (1)

Differentiating (1) partially w.r.t. x and y

$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left[\frac{z(y) - xy\frac{\partial z}{\partial x}}{z^2}\right]$ (2)

$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left[\frac{z(x) - xy\frac{\partial z}{\partial y}}{z^2}\right]$ (3)

$(2) \div (3)$ gives

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{\left[\frac{z(y) - xy\frac{\partial z}{\partial x}}{z^2}\right]}{\left[\frac{z(x) - xy\frac{\partial z}{\partial y}}{z^2}\right]}$$

By rearranging and simplifying we get

$$\Rightarrow x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$$

or

$$\mathrm{xp} = \mathrm{yq}$$

This is the required PDE.

Problem 3.2.11. *Form the P.D.E. of* $z = yf(x) + x\phi(y)$ *where* f *and* ϕ *are arbitrary functions.* *(VTU Model 2018, Jan 2018, July 2017, July 2016, Jan 2016)*

Solution: $z = yf(x) + x\phi(y)$ (1)

Differentiate (1) w.r.t. x and y partially we get

$$\frac{\partial z}{\partial x} = yf'(x) + \phi(y) \qquad (2)$$

$$\text{and} \quad \frac{\partial z}{\partial y} = f(x) + x\phi'(y) \qquad (3)$$

Differentiate (2) w.r.t. x again

$\frac{\partial^2 z}{\partial x^2} = yf''(x)$ (4)

Differentiate (3) w.r.t. y and x again

$\frac{\partial^2 z}{\partial y^2} = x\phi''(y)$ (5)

and $\frac{\partial^2 z}{\partial x \partial y} = f'(x) + \phi'(y)$ (6)

Consider

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = [xyf'(x) + x\phi(y)] + [yf(x) + xy\phi'(y)]$$

$$= [yf(x) + x\phi(y)] + xy\,[f'(x) + \phi'(y)] \quad \therefore \text{ required PDE is}$$

$$= z + xy\frac{\partial^2 z}{\partial x \partial y} \quad \text{by (1) and (6)}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\frac{\partial^2 z}{\partial x \partial y}$$

Problem 3.2.12. *Form the partial differential equation by eliminating the arbitrary functions from* $z = f(y-2x) + g(2y-x)$ *(VTU Dec 2011)*

Solution:: By data, $z = f(y-2x) + g(2y-x)$

$p = \frac{\partial z}{\partial x} = -2f'(y-2x) - g'(2y-x)$

$q = \frac{\partial z}{\partial y} = f'(y-2x) + 2g'(2y-x)$

$r = \frac{\partial^2 z}{\partial x^2} = 4f''(y-2x) + g''(2y-x)$ (1)

$s = \frac{\partial^2 z}{\partial x \partial y} = -2f''(y-2x) - 2g''(2y-x)$ (2

$t = \frac{\partial^2 z}{\partial y^2} = f''(y-2x) + 4g''(2y-x)$ (3)

(1) $\times 2 + (2) \Rightarrow 2r + s = 6f''(y-2x)$

$(2) \times 2 + (3) \Rightarrow 2s + t = -3f''(y-2x)$

Now dividing (4) by (5) we get

$$\frac{2r+s}{2s+t} = -2$$

or

$$2r + 5s + 2t = 0$$

Thus $\quad 2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$ is the required PDE

Problem 3.2.13. *Eliminate the arbitrary function ϕ from*

$$lx + my + nz = \phi\left(x^2 + y^2 + z^2\right)$$

Solution: Differentiate (1) w.r.t. x and y partially we get

$$\ell.1 + n\frac{\partial z}{\partial x} = \phi'\left(x^2 + y^2 + z^2\right) \cdot \left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$m \cdot 1 + n\frac{\partial z}{\partial y} = \phi'\left(x^2 + y^2 + z^2\right) \cdot \left(2y + 2z\frac{\partial z}{\partial y}\right)$$

Dividing (2) by (3) we get

$$\frac{\ell + n\frac{\partial z}{\partial x}}{m + n\frac{\partial z}{\partial y}} = \frac{x + z\frac{\partial z}{\partial x}}{y + z\frac{\partial z}{\partial y}}$$

$$\text{i.e. } \frac{\ell + np}{m + nq} = \frac{x + zp}{y + zq}$$

Rearranging the terms, we get

$$(ny - mz)p + (\ell z - nx)q = (mx - \ell y)$$

3.3 Formation of PDE from $\phi(u, v) = 0$ where u, v are functions of (x, y, z)

NOTE: $z = z(x, y)$

Procedure : Consider $\phi(u, v) = 0 \quad \cdots$ (1)

Differentiate (1) partially w.r.t 'x' by using chain rule,

$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0$

i.e $\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} \quad \cdots$ (2)

Differentiate (1) partially w.r.t 'y' by using chain rule,

$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0$

i.e $\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} \quad \cdots$ (3)

equation $(2) \div (3)$ gives us

$$\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} \quad \cdots (4)$$

which gives required PDE.

Problem 3.3.1. *Form the PDE by eliminating the arbitrary function from the following.* $\phi(xy + z^2, x + y + z) = 0$

Solution: We have, $\phi(u, v) = 0$ (1)

Differentiate (1) partially w.r.t 'x' by using chain rule,

$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0$

i.e $\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}$ (2)

Differentiate (1) partially w.r.t 'y' by using chain rule,

$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0$

i.e $\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}$ (3)

Equation $(2) \div (3)$ gives us

$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}$ (4)

here $u = xy + z^2$ and $v = x + y + z$

now $\frac{\partial u}{\partial x} = y + 2zp$; $\frac{\partial u}{\partial y} = x + 2zq$;

$\frac{\partial v}{\partial x} = 1 + p$; $\frac{\partial v}{\partial y} = 1 + q$;

eqn(4) becomes

$$\frac{y + 2zp}{x + 2zq} = \frac{1 + p}{1 + q}$$

$$(y + 2zp)(1 + q) = (1 + p)(x + 2zq)$$

$$y + yq + 2zp + 2zpq = x + xp + 2zq + 2zpq$$

$$(x - 2z)p + (2z - y)q = (y - x)$$

This is the required PDE.

Problem 3.3.2. *Form a partial differential equation from the equation* $f\left(x + y + z, x^2 + y^2\right.$
0 *(VTU Model 2022)*

Solution: : The given relation is $f\left(x + y + z, x^2 + y^2 + z^2\right) = 0$

The given relation is in the form $f(u, v) = 0 \quad \cdots$ (1)

where $u = x + y + z$ and $v = x^2 + y^2 + z^2$ then

$$\frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p, \quad \frac{\partial u}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q$$

$$\frac{\partial v}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} = 2(x + zp), \quad \frac{\partial v}{\partial y} = 2y + 2z\frac{\partial z}{\partial y} = 2(y + zq)$$

Differentiating eqn (1) partially w. r. to x we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}(1 + p) + \frac{\partial f}{\partial v}2(x + zp) = 0 \qquad \cdots (2)$$

Differentiating eqn (1) partially w. r. t y we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}(1 + q) + \frac{\partial f}{\partial v}2(y + zq) = 0 \qquad \cdots (3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3), we get

$$\begin{vmatrix} (1+p) & 2(x+zp) \\ (1+q) & 2(y+zq) \end{vmatrix} = 0 \text{ or}$$

$$\Rightarrow \quad 2(y + zq)(1 + p) - 2(x + zp)(1 + q) = 0$$

$$\Rightarrow \quad (y + zq)(1 + p) - (x + zp)(1 + q) = 0$$

$$\Rightarrow \quad (y + yp + zq + zpq) - (x + xq + zp + zpq) = 0$$

$\Rightarrow \quad (y - z)p + (z - x)q = x - y$ is the required PDE.

Problem 3.3.3. *Form a partial differential equation from the equation,* $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$

Solution:: Given $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$

Let $\quad u = z^2 - xy, \quad v = \frac{x}{z}$

Then the given equation is of the form $\phi(u, v) = 0$

The elimination of ϕ from this equation gives,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

i.e. $\begin{vmatrix} 2zp - y & \frac{z-px}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$

i.e $\quad (2zp - y)\left(\frac{-xq}{z^2}\right) - \left(\frac{z-px}{z^2}\right)(2zq - x) = 0$

i.e $\quad px^2 - q\left(xy - 2z^2\right) = zx$

Problem 3.3.4. *Form the PDE by eliminating the arbitrary function from the following.*
$f(x^2 + 2yz, y^2 + 2zx) = 0$

Solution: $u = x^2 + 2yz, \quad v = y^2 + 2zx$
and given relation is in the form :$f(u, v) = 0$
The elimination of f from this equation gives,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x + 2y\frac{\partial z}{\partial x} & 2y + 2x\frac{\partial z}{\partial y} \\ 2y\frac{\partial z}{\partial y} + 2z & 2y + 2x\frac{\partial z}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow (y^2 - xz)p + (x^2 - yz)q = z^2 - xy$$

3.4 Solution OF NON-HOMOGENEOUS PDE BY DIRECT INTEGRATION:

In this method ,the solution is obtained by integrating the equation a sufficient number of times until z is found.
In place of arbitrary constants, we must however use arbitrary functions of the variable held fixed. i.e. if we integrate w.r.t x ,then we must add a function of y as arbitrary constant and vice versa.

Problem 3.4.1. *Solve the following partial differential equation:*
$\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

Solution:: The above partial differential equation can be written in the following form: $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y$
By integrating the above equation with respect to x we fiend:

$$\frac{\partial z}{\partial y} = \frac{1}{3}x^3 y + f(y)$$

where $F(y)$ is arbitrary.
Integrating the above equation with respect to y we get: $z = \frac{1}{6}x^3 y^2 + \int f(y)dy +$

$g(x)$ where $g(x)$ is arbitrary.

The above results can be written in the following form:

$$z = \frac{1}{6}x^3y^2 + F(y) + g(x)$$

where $F(y) = \int f(y)dy$

The above equation has two arbitrary functions and is therefore a general solution.

Problem 3.4.2. *Solve :* $\frac{\partial^2 z}{\partial x \partial y} = sinxsiny$ *for which* $\frac{\partial z}{\partial y} = -2siny$ *when* $x = 0$*,and* $z = 0$ *if* y*is an odd multiple of* $\frac{\pi}{2}$*(i.e.* $z = 0$ *if* $y = (2n+1)\frac{\pi}{2}$ *(VTU Model 2022, Jan 2020, Model 2018, July 2017, Jan 2017, Jan 2016, July 2011)*

Solution: $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = sinxsiny$

Integrating w.r.t.x treating y as constant,

$\frac{\partial z}{\partial y} = siny \int sinxdx + f(y)$

$\frac{\partial z}{\partial y} = -sinycosx + f(y)$ (1)

Now Integrating w.r.t.y treating x as constant,

$z = -cosxsinydy + \int f(y)dy + g(x)$

i.e. $z = cosxcosy + F(y) + g(x)$ (2)

where $F(y) = \int f(y)dy$

By data, $\frac{\partial z}{\partial y} = -2siny$ when $x = 0$,

$\therefore$ from (1) $-2siny = -siny + f(y)$

$\therefore f(y) = -siny$

Hence, $F(y) = \int f(y)dy = \int(-siny)dy = cosy$

$\implies z = cosxcosy + cosy + g(x)$

Also by data,

$z = 0$ if $y = (2n+1)\frac{\pi}{2}$

which gives $g(x) = 0, \because cos(2n+1)\frac{\pi}{2} = 0$

Thus the solution (2), of the PDE becomes

$z = cosxcosy + cosy = cosy(cosx + 1)$

Problem 3.4.3. *Solve by direct integration:* $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$ *(VTU Jan 2014)*

Solution:

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$$

Integrating w.r.t. x (treating y as a constant), we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\sin(2x + 3y) + f(y)$$

Integrating w.r.t. x, we get

$$\frac{\partial z}{\partial y} = -\frac{1}{4}\cos(2x + 3y) + x\int f(y)dy + g(y)$$

$$= -\frac{1}{4}\cos(2x + 3y) + x\phi(y) + g(y)$$

Integrating w.r.t. 'y' we get

$$z = -\frac{1}{12}\sin(2x + 3y) + x\int \phi(y)dy + \int g(y)dy$$

$$z = -\frac{1}{12}\sin(2x + 3y) + x\phi_1(y) + \phi_2(y)$$

Problem 3.4.4. *Solve* $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$

Solution: Integrating twice w.r.t. x (keeping y fixed).

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y)$$

Now integrating w.r.t. y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y)$$

$$\text{where} \quad z = \frac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$$

where u, v, w are arbitrary functions.

Problem 3.4.5. *Solve :* $\frac{\partial^2 z}{\partial x^2} = xy$ *by direct integration.*

Solution: Integrating with respect to x, we get

$$\frac{\partial z}{\partial x} = y\frac{x^2}{2} + f(y)$$

Integrating again with respect to x, we get

$$z = y\frac{x^3}{6} + xf(y) + g(y)$$
$$z = \frac{1}{6}x^3y + xf(y) + g(y)$$

Problem 3.4.6. *Solve* $\frac{\partial^2 u}{\partial x \partial y} + 9x^2y^2 = \cos(2x - y)$ *given that* $u = 0$ *when* $y = 0$ *and* $\frac{\partial u}{\partial y} = 0$ *when* $x = 0$

Solution: :

$$\frac{\partial^2 u}{\partial x \partial y} + 9x^2y^2 = \cos(2x - y) \qquad \cdots (1)$$

Integrating with respect to x, we get

$$\frac{\partial u}{\partial y} + 9\frac{x^3}{3}y^2 = \frac{\sin(2x - y)}{2} + f(y) \qquad \cdots (2)$$

Integrating with respect to y, we get

$$u + 3x^3\frac{y^3}{3} = -\frac{\cos(2x - y)}{2(-1)} + \int f(y)dy + g(x)$$
$$u + x^3y^3 = \frac{1}{2}\cos(2x - y) + F(y) + g(x) \qquad \cdots (3)$$

where $F(y) = \int f(y)dy$

Given that $\frac{\partial u}{\partial y} = 0$ when $x = 0$.

Using this in equation (2), we get

$$0 + 0 = \frac{\sin(-y)}{2} + f(y)$$
$$0 = -\frac{1}{2}\sin y + f(y)$$
$$f(y) = \frac{1}{2}\sin y$$
$$F(y) = \int f(y)dy = \frac{1}{2}(-\cos y) = -\frac{1}{2}\cos y$$

Therefore, equation (3) becomes,

$$u + x^3y^3 = \frac{1}{2}\cos(2x - y) - \frac{1}{2}\cos y + g(x)$$

Also we have $u = 0$ when $y = 0$

$$0 + 0 = \frac{1}{2}\cos 2x - \frac{1}{2}(1) + g(x)$$
$$g(x) = \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

Hence, the solution is

$$u + x^3y^3 = \frac{1}{2}\cos(2x - y) - \frac{1}{2}\cos y + \sin^2 x$$
$$i.e. u = -x^3y^3 + \frac{1}{2}\cos(2x - y) - \frac{1}{2}\cos y + \sin^2 x$$

3.5 Solution OF HOMOGENEOUS PDE

Suppose that the dependent variable has been differentiated w.r.to one independent variable only, (say x only). Then the PDE can be treated as an ordinary differential equation(ODE) which can be solved easily. The arbitrary constants in the solution are then replaced by the arbitrary function of other variable (i.e. y) giving the solution of the PDE.

Problem 3.5.1. *Solve* $\frac{\partial^2 z}{\partial x^2} = a^2 z$, *given that when* $x = 0, z = 0$*and* $\frac{\partial z}{\partial x} = asiny$

Solution: Given PDE is $\frac{\partial^2 z}{\partial x^2} = a^2 z$......(1)

ODE :$\frac{d^2 z}{dx^2} = a^2 z$

$i.e.\ (D^2 - a^2)z = 0$

A.E is $m^2 - a^2 = 0$

$\therefore m = \pm a$

The Solution of ODE is :$z = c_1 e^{ax} + c_2 e^{-ax}$

The Solution of PDE is :$z = f(y)e^{ax} + g(y)e^{-ax}$......(1)

Now apply the conditions to find $f(y)$and $g(y)$.

By data ,when $x = 0, z = 0$

(1) becomes $f(y) + g(y) = 0$

Also, when $x = 0, \frac{\partial z}{\partial x} = asiny,$

diff (1) partially w. r. t.x

$\frac{\partial z}{\partial x} = af(y)e^{ax} - ag(y)e^{-ax}$

i.e. $asiny = af(y) - ag(y) \implies siny = f(y) - g(y)$

By solving ,we get $f(y) = \frac{siny}{2}$and $g(y) = \frac{-siny}{2}$

Simplifying ,we get

$z = siny \;\; sinhax$

Problem 3.5.2. *Solve* $\frac{\partial^2 z}{\partial y^2} = z$*, if* $y = 0, z = e^x$ *and* $\frac{\partial z}{\partial y} = e^{-x}$

Solution:: The ODE corresponding to given PDE is :

$(D^2 - 1)z = 0$ where $D = \frac{d}{dy}$

A.E. is $m^2 - 1 = 0$

$m = 1, -1$

Hence Solution is

$z = c_1 e^y + c_2 e^{-y}$

Replacing c_1 and c_2 by the functions of x, we get

$$z = f(x)e^y + g(x)e^{-y} \tag{1}$$

Given that $z = e^x$ when $y = 0$

Using this in (1), we get

$$e^x = f(x) + g(x) \tag{2}$$

On differentiating (1) w.r.to y, we get

$$\frac{\partial z}{\partial y} = f(x)e^y - g(x)e^{-y} \tag{3}$$

Given that $\frac{\partial z}{\partial y} = e^{-x}$ when $y = 0$

Using this in (3), we get

$$e^{-x} = f(x) - g(x) \tag{4}$$

Solving (2) and (4), we get

$e^x + e^{-x} = 2f(x)$ and $e^x - e^{-x} = 2g(x)$

$\Rightarrow f(x) = \frac{e^x + e^{-x}}{2} = \cos hx, \;\; g(x) = \frac{e^x - e^{-x}}{2} = \sin hx$

Hence the Solution (1) becomes

$z = \cos hx \; e^y + \sin hx \; e^{-y}$

Problem 3.5.3. *Solve* $\frac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0, z = e^y$ *and* $\frac{\partial z}{\partial x} = 1$

Solution: The ODE coresponding to given PDE is

$(D^2 + 1)z = 0$, where $D = \frac{d}{dx}$

A.e. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Solution: is $z = c_1 \cos x + c_2 \sin x$ where c_1 and c_2 are constants.

Replacing c_1 and c_2 by functions of other variable , i.e. y, we get

$$z = f(y)\cos x + g(y)\sin x \quad (1)$$

$$\therefore \quad \frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \quad (2)$$

Given that, when $x = 0, z = e^y \Rightarrow e^y = f(y)$.

when $x = 0, \frac{\partial z}{\partial x} = 1 \quad \therefore 1 = g(y)$

Hence the desired solution is $z = e^y \cos x + \sin x$

Problem 3.5.4. *Solve* $\frac{\partial^2 z}{\partial x^2} = a^2 z$ *given that when* $x = 0, \frac{\partial z}{\partial x} = a\sin y$ *and* $\frac{\partial z}{\partial y} = 0$

Solution: $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$

$\Rightarrow \left(D^2 - a^2\right) z = 0, \quad D = \frac{\partial}{\partial x}$

The AE is $m^2 - a^2 = 0 \Rightarrow m = \pm a$

$z = c_1 e^{ax} + c_2 e^{-ax}$

(Since z is a function of x and y, c_1 and c_2 are arbitrary function of y)

$$\text{Therefore, } z = f_1(y)e^{ax} + f_2(y)e^{-ax} \quad (1)$$

Given $\frac{\partial z}{\partial x} = a\sin y$ and $\frac{\partial z}{\partial y} = 0$ when $x = 0$.

Differentiating (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = af_1(y)e^{ax} - af_2(y)e^{-ax}$$

$$a\sin y = af_1(y) - af_2(y)$$

$$\sin y = f_1(y) - f_2(y) \quad (2)$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = f_1'(y)e^{ax} + f_2'(y)e^{-ax}$$

$$0 = f_1'(y) + f_2'(y)$$

Integrating this with respect to y, we get

$$c = f_1(y) + f_2(y) \qquad (3)$$

Adding equations (2) and (3), we get

$$c + \sin y = 2f_1(y) \Rightarrow f_1(y) = \frac{c + \sin y}{2}$$

Subtracting equation (3) from (2), we get

$$\sin y - c = 2f_2(y) \Rightarrow f_2(y) = \frac{\sin y - c}{2}$$

Therefore,

$$z = \left[\frac{c + \sin y}{2}\right] e^{ax} + \left[\frac{\sin y - c}{2}\right] e^{-ax}$$

$$= c\left(\frac{e^{ax} - e^{-ax}}{2}\right) + \sin y\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

$$z = c \sinh ax + \sin y \cosh ax$$

Problem 3.5.5. *Solve* $\frac{\partial^2 z}{\partial y^2} - z = 0$ *given that* $z = \cos x$ *and* $\frac{\partial z}{\partial y} = \sin x$ *when* $y = 0$.

Solution: :

$$\frac{\partial^2 z}{\partial y^2} - z = 0 \Rightarrow \left(D^2 - 1\right) z = 0, \quad \text{Where} \quad D = \frac{\partial}{\partial y}$$

The AE is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$\text{C.F.} = c_1 e^y + c_2 e^{-y}$$

Since z is a function of x and y, c_1 and c_2 are arbitrary function of x

$$z = f_1(x)e^y + f_2(x)e^{-y} \qquad (1)$$

Using $z = \cos x$ when $y = 0$ equation (1) becomes

$$\cos x = f_1(x) + f_2(x) \qquad (2)$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = f_1(x)e^y - f_2(x)e^{-y} \qquad (3)$$

Also we have $\frac{\partial z}{\partial y} = \sin x$ when $y = 0$, (3) becomes,

$$\sin x = f_1(x) - f_2(x) \qquad (4)$$

Adding equations (2) and (4), we get

$$\cos x + \sin x = 2f_1(x) \Rightarrow f_1(x) = \frac{1}{2}(\cos x + \sin x)$$

Subtracting (4) from (2), we get

$$\cos x - \sin x = 2f_2(x) \Rightarrow f_2(x) = \frac{1}{2}(\cos x - \sin x)$$

Therefore, the required solution is

$$z = \frac{1}{2}(\cos x + \sin x)e^y + \frac{1}{2}(\cos x - \sin x)e^{-y}$$

Problem 3.5.6. *Solve* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + 5z = 0$ *given that* $z = e^y$ *and* $\frac{\partial z}{\partial x} = 0$ *when* $x = 0$.

Solution: : The given equation can be written as

$$\left(D^2 - 2D + 5\right) z = 0, \quad \text{where } D = \frac{\partial}{\partial x}$$

The AE is $m^2 - 2m + 5 = 0$

$$m = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$z = e^x \left[c_1 \cos 2x + c_2 \sin 2x\right]$

$$z = e^x \left[f_1(y) \cos 2x + f_2(y) \sin 2x\right] \tag{1}$$

Given that $z = e^y$ when $x = 0$, equation (1) becomes

$$e^y = 1\left[f_1(1) + f_2(y)(0)\right] \Rightarrow f_1(y) = e^y \tag{2}$$

Differentiating equation (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = e^x \left[f_1(y)(-2\sin 2x) + f_2(y)(2\cos 2x)\right] + e^x \left[f_1(y) \cos 2x + f_2(y) \sin 2x\right]$$

Using $\frac{\partial z}{\partial x} = 0$ for $x = 0$ the above equation becomes

$$0 = 1\left[f_1(y)(0) + f_2(y)(2)\right] + 1\left[f_1(y)(1) + f_2(y)(0)\right]$$

$$0 = 2f_2(y) + f_1(y)$$

$$f_2(y) = -\frac{1}{2}f_1(y) = -\frac{1}{2}e^y \quad \text{from equation (2)}$$

Therefore, the solution becomes,

$$z = e^x \left[e^y \cos 2x - \frac{1}{2}e^y \sin 2x\right]$$

$$z = e^{x+y} \left[\cos 2x - \frac{1}{2} \sin 2x\right]$$

Problem 3.5.7. *Solve* $\frac{\partial^2 z}{\partial y^2} - 5\frac{\partial z}{\partial y} + 6z = 0$ *given that* $z = x$ *and* $\frac{\partial z}{\partial y} = 0$ *when* $y = 0$.

Solution: The given equation can be written as

$$\left(D^2 - 5D + 6\right) z = 0, \quad \text{where } D = \frac{\partial}{\partial y}$$

The AE is $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$

$$z = c_1 e^{2y} + c_2 e^{3y}$$

$$z = f_1(x)e^{2y} + f_2(x)e^{3y} \tag{1}$$

Using the condition $z = x$ when $y = 0$, equation (1) becomes

$$x = f_1(x) + f_2(x) \tag{2}$$

Differentiating equation (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = 2f_1(x)e^{2y} + 3f_2(x)e^{3y}$$

Using $\frac{\partial z}{\partial y} = 0$ for $y = 0$ the above equation becomes

$$0 = 2f_1(x) + 3f_2(x) \tag{3}$$

Multiplying equation (2) by 2 and subtracting from equation (3), we get

$$-x = f_2(x)$$

$$(3) \Rightarrow f_1(x) = \frac{-3}{2} f_2(x) = \frac{-3}{2}(-x) = \frac{3x}{2}$$

Therefore, the solution becomes

$$z = \frac{3x}{2} e^{2y} - xe^{3y}$$

Problem 3.5.8. *Solve* $\frac{\partial^2 z}{\partial x^2} - 6\frac{\partial z}{\partial x} + 9z = 0$ *given that* $z = 0$ *and* $\frac{\partial z}{\partial x} = e^y$ *when* $x = 0$.

Solution: : The given equation can be written as

$$\left(D^2 - 6D + 9\right) z = 0, \quad \text{where } D = \frac{\partial}{\partial x}$$

The AE is $m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$

$$z = \left(c_1 + c_2 x\right) e^{3x}$$

$$z = f_1(y)e^{3x} + f_2(y)xe^{3x}$$

Using $z = 0$ when $x = 0$, equation (1) becomes

$$0 = f_1(y) + 0 \Rightarrow f_1(y) = 0$$

Therefore,

$$z = f_2(y)xe^{3x}$$

Differentiating with respect to x, we get

$$\frac{\partial z}{\partial x} = f_2(y)\left[x\left(3e^{3x}\right) + 1 \cdot e^{3x}\right]$$

Using $\frac{\partial z}{\partial x} = e^y$ when $x = 0$, the above equation becomes

$$e^y = f_2(y)[0 + 1] \Rightarrow f_2(y) = e^y$$

Therefore, the solution becomes

$$z = e^y x e^{3x} = x e^{3x+y}$$

3.6 Lagranges Linear PDE

A linear partial differential equation of first order of the form $Pp + Qq = R$ where P, Q and R are functions of x, y and z and $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

Its solution is in the form $F(u, v) = 0$ where F is an arbitrary function and $u(x, y, u) = c_1$ and $v(x, y, u) = c_2$ form a solution of the auxiliary system of equations,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The above Auxiliary equations can be solved using any one of the following methods :

1) Methods of grouping: By grouping any two of three ratios, it may be possible to get an ordinary differential equation containing only two variables, eventhough $P; Q; R$ are in general, functions of x, y, z. By solving this equation, we can get a solution of the simultaneous equations. By this method, we may be able to get two independent solutions, by using different groupings. **2) Methods of multipliers:** If we can find a set of three quantities $1,\ \mathrm{m}, \mathrm{n}$ which may be constants or functions of the variables $\mathrm{x}, \mathrm{y}, \mathrm{z}$, such that $lP + mQ + nR = 0$, then the solution of the simultaneous equation is found out as follows.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

Since $lP + mQ + nR = 0, l dx + m dy + n dz = 0$.

If $l dx + m dy + n dz = 0$ is an exact differential of some function $u(x, y, z)$, then we get $du = 0$. Integrating this, we get $u = a$, which is a solution of

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
Similarly, if we can find another set of independent multipliers l', m', n', we can get another independent Solution $v = b$.

Problem 3.6.1. *Find the general* **Solution:** *of the partial differential equation* $y^2zp + x^2zq = y^2x$

Solution:: The auxiliary system of equations is, $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$ Taking the first two members we have $x^2dx = y^2dy$ which on integration given $x^3 - y^3 = c_1$.
Again taking the first and third members, we have $xdx = zdz$
which on integration given $x^2 - z^2 = c^2$
Hence, the general Solution is $F(x^3 - y^3, x^2 - z^2) = 0$

Problem 3.6.2. *Solve the equation* $(x - 2z)p + (2z - y)q = y - x$.

Solution:: Given: $(x - 2z)p + (2z - y)q = y - x$.
This is of Lagrange's type of PDE where $P = x - 2z, Q = 2z - y, R = y - x$.
The subsidiary equations are $\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x}$. (1)
Using the multipliers $1, 1, 1$, each ratio in $(1) = \frac{dx+dy+dz}{0}$.

$$dx + dy + dz = 0$$

Integrating, we get $x + y + z = a$ (2)
Using the multipliers y, x, 2z,
each ratio in $(1) = \frac{ydx+xdy+2zdz}{0}$.

$$d(xy) + 2zdz = 0$$

Integrating, we get $xy + z^2 = b$
Therefore the general solution of the given equation is $f\left(x + y + z, xy + z^2\right) = 0$.

Problem 3.6.3. *Solve* $(mz - ny)p + (nx - lz)q = ly - mx$

Solution: Given equation is of the form $Pp + Qq = R$ Auxiliary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} = k(\text{ say })$$

Now consider,

$$xdx + ydy + zdz = k[x(mz - ny) + y(nx - lz) + z(ly - mx)] = 0$$

i.e. $\quad xdx + ydy + zdz = 0$

On integration we get,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c \text{ or } x^2 + y^2 + z^2 = 2c = c_1$$

Consider,

$$ldx + mdy + ndz = k[l(mz - ny) + m(nx - lz) + n(ly - mx)]$$

i.e. $\quad ldx + mdy + ndz = 0$

On integration we get

$$lx + my + nz = c_2$$

$\therefore$ general solution is

$$\phi\left(x^2 + y^2 + z^2, lx + my + nz\right) = 0$$

Problem 3.6.4. *Solve* $p\cot x + qcoty = cotz$

Solution: Given equation is of the form $Pp + Qq = R$.

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{or} \quad \frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$$\text{or } \tan xdx = \tan ydy = \text{tanz}\, dz$$

Consider $\quad \tan xdx = \tan ydy$

On integration we get,

$\log\sec x = \log\sec y + c$

or $\log\sec x - \log\sec y = c$ or $\log\frac{\sec x}{\sec y} = c$

or $\sec x\cos y = e^c = c_1$

Similarly consider

$\tan ydy = \text{tanz}\, dz$

On integration we get

$\log\sec y = \log\sec z + c'$

$\log\dfrac{\sec y}{\sec z} = c'$ or $\sec y\cos z = e^{c'} = c_2$

$\therefore$ the general solution is

$$f\left(c_1, c_2\right) = 0 \Rightarrow f(\sec x\cos y, \sec y\cos z) = 0$$

Problem 3.6.5. *Solve* $\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$ *(VTU Feb. 2004, 2006, Dec. 2011, June 2012)*

Solution: Given equation is of the form $Pp + Qq = R$ $\therefore$ The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} = k(say) \quad (1)$$

Now consider

$$dx - dy = k\left[(x^2 - yz) - (y^2 - zx)\right] \quad [\text{ using } (1)$$

$$= k[(x - y)(x + y + z)]$$

$$\Rightarrow \frac{dx - dy}{(x - y)(x + y + z)} = k$$

Similarly

$$\Rightarrow \frac{dy - dz}{(y - z)(x + y + z)} = k, \frac{dz - dx}{(z - x)(x + y + z)} = k$$

$$\Rightarrow \frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} = \frac{dz - dx}{(z - x)(x + y + z)}$$

$$\Rightarrow \frac{dx - dy}{(x - y)} = \frac{dy - dz}{(y - z)} = \frac{dz - dx}{(z - x)} \quad (2)$$

Consider first two fractions of (2),

$$\frac{dx - dy}{(x - y)} = \frac{dy - dz}{(y - z)}$$

On integration,

$$\log(x - y) = \log(y - z) + c$$

$$\Rightarrow \quad \log\frac{x - y}{y - z} = c \text{ or } \frac{x - y}{y - z} = e^c = c_1$$

Consider last two fractions of (2),

$$\frac{dy - dz}{(y - z)} = \frac{dz - dx}{(z - x)}$$

On integration we get,

$$\log(y - z) = \log(z - x) + c'$$

$$\Rightarrow \quad \log\left(\frac{y - z}{z - x}\right) = c' \Rightarrow \frac{y - z}{z - x} = e^{c'} = c_2$$

$\therefore$ general solution is

$$\phi\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$$

Problem 3.6.6. *Solve* $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ *(VTU Model 2022, Aug. 2002, Feb. 2003, Jan. 2007)*

Solution: Given equation is of the form $Pp+Qq=R$ Auxiliary equations are

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}=k(say) \qquad (1)$$

Let us consider,

$$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}=k[(y-z)+(z-x)+(x-y)]=0 \quad \text{[using (1)}$$

$$\text{or } x^{-2}dx+y^{-2}dy+z^{-2}dz=0$$

On integration, we get

$$-x^{-1}-y^{-1}-z^{-1}=c \text{ or } \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-c=c_1$$

Now consider,

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=k[x(y-z)+y(z-x)+z(x-y)]=0 \quad \text{[using (1)}$$

On integration we get

$$\log x+\log y+\log z=c'$$

$$\Rightarrow \log(xyz)=c'$$

$$\Rightarrow xyz=e^{c'}=c_2$$

∴ General solution is

$$\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz\right)=0$$

Problem 3.6.7. *Solve* $x\left(y^2-z^2\right)p+y\left(z^2-x^2\right)q=z\left(x^2-y^2\right)$ *(VTU Model 2022, Aug. 2001, 2004*

Solution: Given equation is of the form $Pp+Qq=R$ Auxiliary equations are

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)}=k(say) \qquad (1)$$

We shall consider,

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=k\left[\left(y^2-z^2\right)+\left(z^2-x^2\right)+\left(x^2-y^2\right)\right]=0 \quad \text{using (1)}$$

i.e. $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

On integration we get

$$\log x+\log y+\log z=c \quad \text{or} \quad \log xyz=c$$

$$\text{or } \log xyz=c \quad xyz=e^c=c_1$$

Now consider,

$$xdx+ydy+zdz=k\left[x^2\left(y^2-z^2\right)+y^2\left(z^2-x^2\right)+z^2\left(x^2-y^2\right)\right]=0$$

i.e. $xdx+ydy+zdz=0$

On integration,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c'$$

$$\Rightarrow x^2+y^2+z^2 = 2c'=c_2$$

$\therefore$ general solution is

$$f\left(xyz, x^2+y^2+z^2\right)=0$$

Problem 3.6.8. *Solve* $xp-yq=y^2-x^2$ *(VTU June 2013, Dec. 2013, Jan. 2014*

Solution: Given equation is of the form $Pp+Qq=R$

$$\therefore \quad P=x, Q=-y, \quad R=y^2-x^2$$

Auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \text{ i.e. } \frac{dx}{x}=\frac{dy}{-y}=\frac{dz}{y^2-x^2}=k(\text{ sny })$$

Consider first two fractions,

$$\frac{dx}{x}=\frac{dy}{-y}$$

On integration we get

$$\log x=-\log y+c \text{ or } xy=e^c=c_1$$

$$\text{Consider } xdx+ydy=k\left[x^2-y^2\right]$$

$$\frac{xdx+ydy}{x^2-y^2}=k=\frac{-dz}{x^2-y^2}$$

$$\Rightarrow xdx+ydy=-dz$$

On integration

$$\frac{x^2}{2}+\frac{y^2}{2}=-z+c'$$

$$x^2+y^2+2z=2c'=c_1$$

$\therefore$ General solution is

$$f\left(xy, x^2+y^2+2z\right)=0$$

Problem 3.6.9. *Solve* $\left(y^2+z^2\right)p+x(yq-z)=0$ *(VTU March 2001, Aug. 2005, Dec. 2014)*

Solution: Given equation can be written as $\left(y^2+z^2\right)p+xyq=xz$ which is of the form $Pp+Qq=R$

The auxiliary equations are

$$\frac{dx}{y^2+z^2}=\frac{dy}{xy}=\frac{dz}{xz}=k$$

Consider last two fractions,

$$\frac{dy}{xy}=\frac{dz}{xz} \text{ or } \frac{dy}{y}=\frac{dz}{z}$$

On integration we get

$$\log y=\log z+c \text{ or } \log\frac{y}{z}=c \text{ or } \frac{y}{z}=e^c=c_1$$

Using the multiplies, $x, -y, -z$ in (1), we have

$$xdx-ydy-zdz=k\left[x\left(y^2+z^2\right)-y(xy)-z(xz)\right]$$

$$\text{i.e.}\quad xdx-ydy-zdz=0$$

On integration we get

$$\frac{x^2}{2}-\frac{y^2}{2}-\frac{z^2}{2}=c' \quad \text{or} \quad x^2-y^2-z^2=2c'=c_2$$

$\therefore$ General solution is

$$\phi\left(y/z, x^2-y^2-z^2\right)=0$$

Problem 3.6.10. *Solve* $(y+z)p+(z+x)q=x+y$ *(VTU Feb. 2005*

Solution: Given equation is of the form $Pp+Qq=R$ Auxiliary equations are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}=k(say)$$

Now consider,

$$dx-dy=k[(y+z)-(z+x)]=(-k)(x-y)$$

$$\Rightarrow\frac{dx-dy}{x-y}=(-k)$$

Similarly

$$\Rightarrow\quad\frac{dy-dz}{y-z}=(-k),\frac{dz-dx}{z-x}=(-k)$$

$$\Rightarrow\quad\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$$

Consider first two fractions of (2),

$$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

On integration we get,

$$\log(x-y)=\log(y-z)+c$$

$$\Rightarrow\quad\log\left(\frac{x-y}{y-z}\right)=c\quad\text{or}\quad\frac{x-y}{y-z}=e^c=c_1$$

Consider last two fractions of (2),

$$\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$$

On integration we get,

$$\log(y-z)=\log(z-x)+c'$$

$$\Rightarrow \quad \log\left(\frac{y-z}{z-x}\right)=c' \quad \text{or} \quad \frac{y-z}{z-x}=e^{c'}=c_2$$

$\therefore$ general solution is

$$\phi\left(\frac{x-y}{y-z},\frac{y-z}{z-x}\right)=0$$

Problem 3.6.11. *Solve* $\left(x^2-y^2-z^2\right)p+2xyq=2xz$ *(VTU July 2007)*

Solution: Given equation is of the form $Pp+Qq=R$ Auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}=k(\text{ say })$$

Consider last two fractions

$$\frac{dy}{2xy}=\frac{dz}{2xz}\Rightarrow\frac{dy}{y}=\frac{dz}{z}$$

On integration we get

$$\log y=\log z+c \quad \text{or}$$

$$\log\left(\frac{y}{z}\right)=c \quad \frac{y}{z}=e^c=c_1$$

Consider,

$$xdx+ydy+zdz=k\left[x\left(x^2-y^2-z^2\right)+y(2xy)+z(2xz)\right] \quad [\text{using }(1)]$$

$$\Rightarrow \quad \frac{xdx+ydy+zdz}{x\left(x^2+y^2+z^2\right)}=k$$

Also $\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}=k=\frac{dz}{2xz}$ by (1)

$\Rightarrow \quad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$

On integration we get,

$$\left(x^2+y^2+z^2\right)=\log z+c'$$

$$\Rightarrow \quad \log\left(\frac{x^2+y^2+z^2}{z}\right)=c' \quad \text{or} \quad \frac{x^2+y^2+z^2}{z}=e^c=c_2$$

$\therefore$ general solution is

$$\phi\left(\frac{y}{z},\frac{x^2+y^2+z^2}{z}\right)=0$$

Problem 3.6.12. *Solve* $x\left(y^2+z\right)p - y\left(x^2+z\right)q = z\left(x^2-y^2\right)$ *(VTU June 2014)*

Solution: Given equation is of the form $Pp + Qq = R$

Auxiliary equations are

$$\frac{dx}{x\left(y^2+z\right)} = \frac{dy}{-y\left(x^2+z\right)} = \frac{dz}{z\left(x^2-y^2\right)} = k$$

Consider,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = k\left[\left(y^2+z\right) - \left(x^2+z\right) + \left(x^2-y^2\right)\right] = 0 \quad \text{[using}$$

or $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = 0$

On integration we get,

$$\log x + \log y + \log z = 0$$

$$\Rightarrow \quad \log(xyz) = c \text{ or } xyz = e^c = c_1$$

Now consider,

$$xdx + ydy - dz = k\left[x^2\left(y^2+z\right) - y^2\left(x^2+z\right) - z\left(x^2-y^2\right)\right] = 0$$

On integration we get

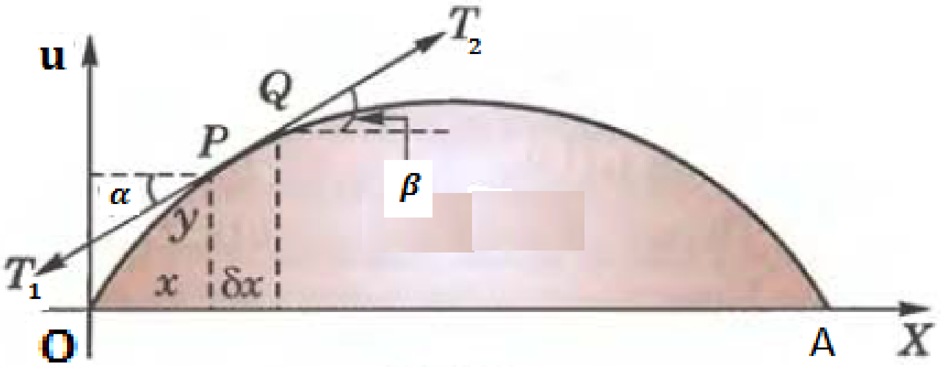
$$\frac{x^2}{2} + \frac{y^2}{2} - z = c' \text{ or } x^2 + y^2 - 2z = 2c' = c_2$$

$\therefore$ general solution is $\phi\left(xyz, x^2+y^2-2z\right) = 0$

3.7 Derivation of one dimensional Wave Equation :

(VTU Model 2018, Jan 2017, July 2016)

Consider a uniform elastic string of length ℓ stretched tightly between the points O and A and displaced slightly from its equilibrium position OA. Taking the end O as the origin, OA as the axis and a perpendicular line through O as the u-axis, we shall find the displacement u as a function of the distance x and time t.



Assumptions

(i) String is perfectly flexible and does not offer resistance to bending.

(ii) String is homogeneous (i.e. mass per unit length is a constant).

(iii) The tension caused by stretching the string is so large that the forces due to weight of the string(i.e. effect of gravity) can be neglected.

(iv) Motion takes places in the XY plane and each particle of the string moves perpendicular to the equilibrium position OA of the string. (i.e. there is no horizontal vibration)

Let ρ be the mass per unit length of the string. Consider the motion of an element PQ of length δx. Let T_1 and T_2 be the tensions at the points P and Q of the string. Since the string does not offer resistance to bending(by assumption (i)), the tensions T_1 and T_2 are tangential to the curve of the string respectively at P and Q.

Since there is no motion in the horizontal direction(assumtion (iv)), we have

$$T_1 cos\alpha = T_2 cos\beta = T(constant) \qquad (*)$$

Mass of element PQ is $\rho\delta x$.

The vertical components of T_1 and T_2 are $-T_1 sin\alpha$ and $T_2 sin\beta$.

(where negative sign is used because T_1 is directed downwards).

Hence the resultant force acting vertically upwards is ,

$$F = T_2 sin\beta - T_1 sin\alpha \qquad (3.1)$$

By Newton's second law of motion, the equation of motion in the vertical direction is,

Force = mass $\times$ acceleration for the segmentPQ

$$= \rho\, \delta\, x \times \frac{\partial^2 u}{\partial t^2} \tag{3.2}$$

(since ρ is the mass per unit length, mass of the segment PQ of length δx is $\rho\delta x$)

Equating (3.1) and (3.2),

$$\rho\delta x \times \frac{\partial^2 u}{\partial t^2} = T_2 sin\beta - T_1 sin\alpha$$

Dividing by T we get ,

$$\frac{\rho\delta x \times \frac{\partial^2 u}{\partial t^2}}{T} = \frac{T_2 sin\beta}{T} - \frac{T_1 sin\alpha}{T}$$

$$= \frac{T_2 sin\beta}{T_2 cos\beta} - \frac{T_1 sin\alpha}{T_1 cos\alpha} \quad (\because from(*) T_1 cos\alpha - T_2 cos\beta = T)$$

$$= tan\beta - tan\alpha$$

$$= \left(\frac{\partial u}{\partial x}\right)_Q - \left(\frac{\partial u}{\partial x}\right)_P$$

($\because$ $tan\alpha$ and $tan\beta$ are respectively the slopes of the string at P and Q, i.e. $tan\alpha = \left(\frac{\partial u}{\partial x}\right)_P$, $tan\beta = \left(\frac{\partial u}{\partial x}\right)_Q$,

Here we use partial derivatives because u also depends on time t.)

$$\frac{\rho\delta x \times \frac{\partial^2 u}{\partial t^2}}{T} = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x$$

dividing both sides by δx and taking limit as $\delta x \to 0$ we get,

$$\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = lim_{\delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}$$

$$= \frac{\partial^2 u}{\partial x^2}$$

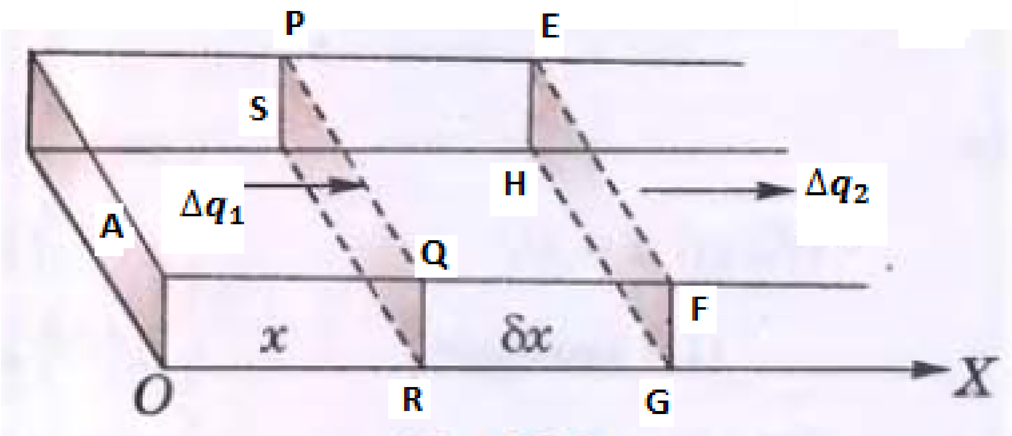
$$i.e \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

$$= c^2 \frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{T}{\rho} \tag{3.3}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one - dimensional wave equation. The notation c^2 (in stead of c) for the physical constant $\frac{T}{\rho}$ has been chosen to indicate that this constant is positive.

3.8 Derivation of one dimensional Heat Equation

(VTU Model 2018, Jan 2018,July 2017, Jan 2017, July 2016) Consider a homogeneous bar of constant cross-sectional area **A**. Let ρ be the density (i.e. mass per unit volume) , s be the specific heat and K be the thermal conductivity of the material . Let the sides be insulated so that the stream lines of heat flow are parallel and perpendicular to the area **A**. Let one end of the bar be taken as origin **O** and the direction of the heat flow be along positive x - axis. Let $u(x, t)$ be the temperature of the bar at time $t > 0$ at a distance x from the end O.



We use the following empirical laws in respect of heat flow.

(i) Heat flows from a higher temperature to a lower temperature (i.e. heat flows in the direction of decreasing temperature).

(ii) The amount of heat gained or lost by the body is proportional to the mass of the body and to the temperature change, where constant of proportionality is the specific heat s.

(iii) The rate of heat flow across an area is proportional to the area and the rate of change of temperature with respect to its distance normal to the area, where constant of proportionality is the thermal conductivity K. (i.e. if q is the quantity of heat that flows across a slab of area **A** and thickness δx in one second , where the difference of temperature at faces is δu , then $q \propto Adu/dx$
i.e. $q = -KA\frac{du}{dx}$ (where negative sign is due to empirical law (i))

Consider an element of bar between the planes PQRS and EFGH at a distance x and $x + \delta x$ from the end O. Let δu be the change in the temperature in this element of thickness δx of the bar.
The volume of this element = Area $\times$ thickness = $A\delta x$
The mass of the element = density $\times$ volume = $\rho A\delta x$
Now from empirical law (ii),
The quantity of heat in this element = $s\times$ mass $\times$ temperature = $s\ \rho\ A\ \delta x\ \delta u$, where s is the specific heat.
Hence the rate of increase of heat in this slab element is,

$$\delta q = s\rho A\delta x\frac{\partial u}{\partial t} \tag{1}$$

If δq_1 is the rate of inflow of heat and δq_2 is the rate of outflow of heat for the slab element, then from empirical law (iii) we have
$\delta q_1 = -KA\left(\frac{\partial u}{\partial x}\right)_x and \delta q_2 = -KA\left(\frac{\partial u}{\partial x}\right)_{(x+\delta x)}$
Thus the rate of increase of heat is

$$\delta q = \delta q_1 - \delta q_2$$

$$i.e.\delta q = -KA\left(\frac{\partial u}{\partial x}\right)_x + KA\left(\frac{\partial u}{\partial x}\right)_{(x+\delta x)} \tag{2}$$

From equations (1) and (2) we can write ,

$$s\rho A\delta x\frac{\partial u}{\partial t} = -KA\left(\frac{\partial u}{\partial x}\right)_x + KA\left(\frac{\partial u}{\partial x}\right)_{(x+\delta x)}$$

$$= KA\left[\left(\frac{\partial u}{\partial x}\right)_{(x+\delta x)} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\text{(or)}\frac{\partial u}{\partial t} = \frac{K}{\rho s}\frac{\left(\frac{\partial u}{\partial x}\right)_{(x+\delta x)} - \left(\frac{\partial u}{\partial t}\right)_x)}{\delta x}$$

Taking limit as $\delta x \to 0$, the RHS of above equation becomes ,

$$\frac{\partial u}{\partial t} = \frac{K}{\rho s}\, lim_{\delta x\to 0}\frac{\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)\right]}{\delta x}$$

$$= \frac{K}{\rho s}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{K}{\rho s}\frac{\partial^2 u}{\partial x^2}$$

Denoting $c^2 = \frac{K}{\rho s}$ we get , $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ This equation is called one dimensional heat equation.

3.9 Question Bank : Module 3 - Partial Differential Equations

3.9.1 Question Bank :Formation of PDE by eliminating arbitrary constants

Form the PDE by eliminating the arbitrary constants from

1) Form the partial differential equation by eliminating the arbitrary constants from $(x-a)^2 + (y-b)^2 = 4$ (VTU Model 2022)

2) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (VTU Model 2018)

3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (VTU Jan 2017) **Ans**:$xp^2 - zp + xzr = 0$

4) Find the PDE of the family of all spheres whose centre's lie on the plane $z = 0$and have a constant radius 'r' (VTU July 2014)

5) Form the partial differential equation by eliminating the arbitrary constants from $(x-a)^2+(y-b)^2+z^2=c^2$ (VTU Model 2018, Jan 2010)

6) $z=e^{ax+by}f(ax-by)$ **Ans**:$bp+aq=2abz$

7) $z=alog(x^2+y^2)+b$ **Ans**:$py-qx=0$

8) $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ **Ans**:$2z=px+qy$

9) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (VTU Jan 2017)

10) Find the PDE of the family of all spheres whose centers lie on the plane $z=0$and have a constant radius 'r'

Ans:$z^2(p^2+q^2+1)=r^2$

11) Find the PDE of the family of all spheres whose centers lie on the z-axis and have a constant radius 'r'

Ans:$xq-yp=0$

3.9.2 Question Bank :Formation of PDE by eliminating arbitrary functions

Form the PDE by eliminating the arbitrary functions from the following.

1) $z=y^2+2f(\frac{1}{x}+logy)$
(VTU Jan 2020, July 2018, June 2015, 2006, 2007)

2) $lx+my+nz=\phi(x^2+y^2+z^2)$ (VTU Jan 2017)
Ans:$(mz-ny)p+(nx-lz)q=(ly-mx)$

3) $z=yf(x)+x\phi(y)$ (VTU Model 2018, July 2018,Jan 2018, July 2017, July 2016, Jan 2016) **Ans**: $xys+z=px+qy$

4) $z=f(x+ct)+g(x-ct)$ (VTU Model 2022, Jan 2017, 2009, 2005)
Ans:$z_{tt}=c^2z_{xx}$

5) $\phi(x+y+z,x^2+y^2+z^2)=0$ (VTU Model 2022, June 2019, July 2017, Jan 2017, 2007)

6) $f(x^2+2yz, y^2+2zx)=0$ (VTU July 2016)
Ans:$p(y^2-zx)+q(x^2-yz)=(z^2-xy)$

7) $f(\frac{xy}{z}, z)=0$ (VTU July 2016)

8) $z=f(y+2x)+g(y-3x)$ (VTU July 2015)

9) $z=f\left(\frac{xy}{z}\right)$ (VTU July 2017, July 2013)

10) $z=f(y)+\phi(x+y)$ (VTU Jan 2013)

11) $z=f(y-2x)+g(2y-x)$ (VTU Jan 2012)

12) $\phi(xy+z^2, x+y+z)=0,$ (VTU July 2017, Jan 2015, 2006)

13) $f(x^2+y^2, z-xy)=0$ (VTU Model 2018, 2007)**Ans**:$py-qx=y^2-x^2$

14) $ax+by+cz=f(x^2+y^2+z^2)$ (VTU Model 2022)

15) $\phi(x+y+z, x^2+y^2-z^2)=0$ **Ans**:$p(y+z)-q(x+z)=(x-y)$

16) $f(x^2-xy, \frac{x}{z})=0$ **Ans**:$x^2p+(2z^2-xy)q=xz$

17) $z=f(x+y)+g(2x+y)$ **Ans**:$r-3s+2t=0$

18) $ax+by+cz=f(x^2+y^2+z^2)$ (VTU Model 2022)
Ans:$(bz-cy)p+(cx-az)q=(ay-bx)$

3.9.3 Question Bank : Solution by Direct integration

Solve by Direct integration

1) $\frac{\partial^2 z}{\partial x^2}=xy$ subject to the condition that $z_x=log(1+y)$ when $x=1$ and $z=0$ when $x=0$ (VTU Model 2022, June 2018)

2) $\frac{\partial^2 z}{\partial x^2}=xy$ subject to the condition that $z=1$ and $z_x=y$ when $x=0$ (VTU Jan 2018)

3) $\frac{\partial^2 z}{\partial x \partial y} = sinxsiny$ for which $\frac{\partial z}{\partial y} = -2siny$ when $x = 0$,and $z = 0$ if $y$is an odd multiple of $\frac{\pi}{2}$
(VTU Model 2022, Jan 2020, Model 2018, July 2017, Jan 2017, Jan 2016, July 2011)

4) Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = cos(2x + 3y)$. (VTU June 2019, Jan 2014)
Ans:$z = \frac{-1}{12} sin(2x + 3y) + xF(y) + G(y) + h(x)$.

5) $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ subject to the conditions $\frac{\partial z}{\partial x} = logx$ when $y = 1$ and $z = 0$ when $x = 1$ (VTU July 2016)

6) $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} cosx$ subject to the conditions, (i)$u(x, 0) = 0$ (ii) $\frac{\partial u}{\partial t}(0, t) = 0$. Also show that $u \to sinx$ as $t \to \infty$.

7) $\frac{\partial^2 z}{\partial x \partial y} = sinxcosy$ for which $\frac{\partial z}{\partial y} = -2cosy$ when $x = 0$,and $z = 0$ if y is $n\pi$

8) Solve $\frac{\partial^2 z}{\partial x \partial y} + 9x^2 y^2 = cos(2x - y)$ Given that $z = 0$ when $y = 0$ and $\frac{\partial z}{\partial y} = 0$ when $x = 0$. (VTU July 2015)

9) Solve $\frac{\partial^2 z}{\partial x \partial y} = sinxcosy$ for which $\frac{\partial z}{\partial y} = -2cosy$ when $x = 0$,and $z = 0$ if $y$is $n\pi$ **Ans**:$z = -siny(cosx + 1)$

3.9.4 Question Bank : Solution of homogeneous PDE

Solve the following Partial Differential Equations.

1) Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$, given that when $x = 0, z = 0$ and $\frac{\partial z}{\partial x} = asiny$ (Jan 2020, VTU July 2017)

2) Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0, z = e^y$ and $\frac{\partial z}{\partial x} = 1$ [VTU June 2019, July 2012]

3) Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0$ subject to $z = 1$ and $\frac{\partial z}{\partial x} = y$, when $x = 0$ (VTU Model 2022, Jan 2018, July 2016)

4) Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + 2z = 0$ subject to $z = e^y$ and $\frac{\partial z}{\partial x} = 0$, when $x = 0$ (VTU July 2016)

5) Solve $\frac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$ (VTU Model 2018, July 2017, Jan 2017)

6) Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$, given that when $x = 0$, $\frac{\partial z}{\partial x} = asiny$ and $\frac{\partial z}{\partial y} = 0$ **Ans** : $z = kcoshax + sinysinhax$

7) Solve $\frac{\partial^2 z}{\partial y^2} + z = 0$, given that when $y = 0, z = cosx$ and $\frac{\partial z}{\partial y} = sinx$ **Ans**:$z = cos(x - y)$

8) Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0, z = e^y$and $\frac{\partial z}{\partial x} = 1$ (VTU July 2012) **Ans**:$z = e^y cosx + sinx$

9) Solve $\frac{\partial^2 u}{\partial x^2} + u = 0$, given when $u(0, y) = \sqrt{e}$and $\frac{\partial u}{\partial x}(0, y) = 1$ **Ans**:$u = \sqrt{e}cosx + sinx$

10) Solve $\frac{\partial^2 z}{\partial x^2} = z$, given that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$ (VTU Model 2022)

11) Solve $\frac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$ (VTU Model 2018, July 2017, Jan 2017)

12) Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$, given that when $x = 0, z = 0$ and $\frac{\partial z}{\partial x} = asiny$ (VTU July 2017, Jan 2017)

13) Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0$ subject to $z = 1$ and $\frac{\partial z}{\partial x} = y$, when $x = 0$ (VTU Model 2022, Jan 2018, July 2016)

14) Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + 2z = 0$ subject to $z = e^y$ and $\frac{\partial z}{\partial x} = 0$, when $x = 0$ (VTU July 2016)

3.9.5 Question Bank :Lagranges Linear PDE

1) Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ (VTU Model 2018, June 2012, Dec 2011, 2007, 2003)

2) Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ (VTU Model 2018, 2007)

3) Find the general solution of $px + qy = z$.

4) Solve $ptanx + qtany = tanz$

5) Solve $(y - z)p + (z - x)q = x - y$

6) Solve $(mz - ny)p + (nx - lz)q = ly - mx$. (VTU June 2019, 2010)

7) Solve $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ (VTU Model 2022, June 2012, Dec 2011, 2007, 2003)

8) Solve $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$ (VTU Model 2022, June 2010, 2004)

9) Solve $x(y - z)p + y(z - x)q = z(x - y)$ (VTU Jan 2020, June 2009)

10) Solve $(y^2 + z^2)p + xyq = xz$ (VTU 2014)

11) Solve $xp - yq = y^2 - x^2$ (VTU 2014, 2013)

3.9.6 Question Bank :Derivation of Wave and Heat Equations

1) With usual notations, derive one-dimensional wave Equation (VTU Model 2022, Jan 2020, Model 2018, Jan 2017, July 2016)

2) With usual notations, derive one-dimensional heat Equation (VTU Model 2022, June 2019, Model 2018, Jan 2018,July 2017, Jan 2017, July 2016)

Module 4

Numerical Methods I

Syllabus

Importance of numerical methods for discrete data in the field of Mechanical Engineering

Solution of algebraic and transcendental equations: Regula-Falsi and Newton-Raphson methods (only formulae). Problems. Finite differences, Interpolation using Newton's forward and backward difference formulae, Newton's divided difference formula and Lagrange's interpolation formula (All formulae without proof). Problems. Numerical integration: Trapezoidal, Simpson's (1/3)rd and (3/8)th rules(without proof). Problems.

Self-Study: Bisection method, Lagrange's inverse Interpolation.

Applications: Finding approximate solutions to solve mechanical engineering problems involving numerical data. (RBT Levels: L1, L2 and L3)

4.1 Forward and Backward Differences

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots\ldots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ are respectively, called the first forward differences where Δ is the forward difference operator.

Thus the first forward differences are

$$\Delta y_r = y_{r+1} - y_r, \quad r = 0, 1, 2, 3, \ldots$$

The difference of first forward differences is called second forward differences. i.e,

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0, 1, 2, 3, \ldots$$

Similarly, the other higher order differences namely the third; fourth, etc. are obtained and tabulated. Such a tabular arrangement is called forward difference table:

In general,

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r$$

defines the n^{th} forward differences.

Following table shows how the forward differences of all orders can be formed.

The Forward Difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

The differences $y_1 - y_0, y_2 - y_1, \ldots\ldots., y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots\ldots.\nabla y_n$ respectively are called first backward differences where ∇ is the backward difference operator.

Similarly, we define higher order backward differences as,

$$\nabla y_r = y_r - y_{r-1}$$

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

Backward Difference Table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

4.2 Interpolation:

If $y_0, y_1, y_2, \ldots, y_n$ be the set of values of an unknown function $y = f(x)$ corresponding to the values of x : $\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$ the process of finding the values of y for any given value of x between x_0 and x_n is called interpolation. Also the process of finding the values of y outside the given range of x is called extrapolation.

4.3 Newton's Forward Interpolation Formula [NFIF]

Let $y = f(x)$ be a function which takes values $y_0 = f(x_0), y_1 = f(x_0 + h), y_2 = f(x_0 + 2h), \ldots, y_n$ corresponding to various equi-spaced values of x with spacing h, say $\mathrm{x}_0, x_1 = \mathrm{x}_0 + \mathrm{h}, x_2 = \mathrm{x}_0 + 2\,\mathrm{h}, \ldots, x_n = x_0 + nh$.
Suppose, we wish to evaluate the function $y = f(x)$ for a value $x_p = x_0 + ph$ where p is any real number, then for any real number p,

$$y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots$$

This formula is mainly used for interpolating the values of y **near the beginning** of a set of tabular values.

4.4 Newton's Backward Interpolation Formula [NBIF]

The value of $y = f(x)$ at $x_p = x_n + ph$ is approximately given by

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

This formula is mainly used for interpolating the values of y **near the end** of a set of tabular values (second half).

4.5 Step by step working procedure for the problems :

Step1: We construct the difference table in accordance with the interpolation formula.

Step 2: We compute the value of p where $p = \frac{x_p - x_0}{h}$ in case of forward interpolation formula, x_0 being the first value of x, h being the step length, and $p = \frac{x_p - x_n}{h}$ in case of backward interpolation formula, x_n being the last value of x and h being the step length.

Step 3 : The value of p along with the value of the finite differences is substituted in the Interpolation formula which results in the value of y at the desired value of x.

Problem 4.5.1. *The population of a town in the decimal census was as given below. Estimate the population for the year* 1895

year	*1891*	*1901*	*1911*	*1921*	*1931*
Population(in thousands)	*46*	*66*	*81*	*93*	*101*

Solution:: Here $x_0 = 1891, h = 10, \quad x_p = 1895$

$\Rightarrow \quad p = \frac{x_p - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$

The difference table is given by

x	y	Δy	Δ²y	Δ³y	Δ⁴y
1891	46				
		20			
1901	66		−5		
		15		2	
1911	81		−3		−3
		12		−1	
1921	93		−4		
		8			
1931	101				

$$y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2}(-5) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(2) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

$$= 54.8528 \text{ thousands}$$

Problem 4.5.2. *Given* $f(40) = 184, f(50) = 204, f(60) = 226, f(70) = 250, f(80) = 276$, $f(90) = 304$, *find* $f(85)$ *using suitable interpolation formula.* *[VTU: June12/Jan16]*

Solution:: Here we have to find y at $x = 85$ since the value $x = 85$ is in the second half of the table near $x = 90$, NBIF is appropriate and the backward differences are tabulated as below.

x	y					
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

We have Newton's backward difference interpolation formula

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \ldots$$

Here, $x_n = 90, h = 10, \nabla y_n = 28, \nabla^2 y_n = 2, x_p = 85$

$$p = \frac{x_p - x_n}{h}$$

$$p = \frac{85 - 90}{10} = -0.5$$

$$\text{y} = 304 + (-0.5)28 + \frac{(-0.5)((-0.5)+1)}{2!}2 = 289.75$$

Thus $f(85) = 289.75$

Problem 4.5.3. *In the following table, values of y are consecutive terms of a series of which 23.6 is the 6th term.*

x :	*3*	*4*	*5*	*6*	*7*	*8*	*9*
y:	*4.8*	*8.4*	*14.5*	*23.6*	*36.2*	*52.8*	*73.9*

Find the first and tenth terms of the series.

Solution:: To find first term (i.e. y when $x = 1$), let us use Newton's forward interpolation formula.

Here, $x_0 = 3, \quad h = 1, \quad x_p = 1$

$\therefore \quad p = \frac{x_p - x_0}{h} = -2$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

$$y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

$$Y(1) = 4.8 + (-2)\times 3.6 + \frac{(-2)(-3)}{2}(2.5) + \frac{(-2)(-3)(-4)}{6}(0.5)$$
$$= 3.1$$

To obtain tenth term, we use Newton's Backward interpolation formula

$$x_n = 9, h = 1, x_p = x_n + ph = 10$$
$$10 = 9 + p \Rightarrow p = 1$$

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

$$y_{10} = 73.9 + 1(21.1) + \frac{1(2)}{2!}(4.5) + \frac{1(2)(3)}{3!}(0.5)$$
$$= 73.9 + 21.1 + 4.5 + 0.5 = 100$$

Problem 4.5.4. *From the following table find the number of students who have obtained (a) less than 45 marks (b) between 40 and 45 marks. [VTU- July15/Jan17]*

marks	30 − 40	40 − 50	50 − 60	60 − 70	70 − 80
No. of students	31	42	51	35	31

Solution: : Let us first prepare a new table with the following data :

x: marks	40	50	60	70	80
y: No. of students with marks < x	31	31+42=73	73+51=124	124+35=159	190

Now We shall first find y_{45}, number of students with marks less than 45.

$$x_0 = 40, h = 10, x_p = 45$$
$$x_0 + hp = 40 + 10p = 45 \Rightarrow p = .5$$

Marks less than (x)	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42	9	−25	37
50	73	51	−16	12	
60	124	35	−4		
70	159	31			
80	190				

By Newton's forward difference formula,

$$
\begin{aligned}
y(x_p) &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 \\
&\quad + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \\
&= 31 + (.5)(42) + \frac{(.5)(.5-1)}{2}(9) + \frac{(.5)(.5-1)(.5-2)}{6}(-25) \\
&\quad + \frac{(.5)(.5-1)(.5-2)(.5-3)}{24}(37) \\
&= 47.8672 \approx 48
\end{aligned}
$$

Hence no. of students getting marks less than $45 = 48$

By given data, no. of students getting marks less than $40 = 31$

Hence no. of students getting marks between 40 and $45 = 48 - 31 = 17$

Problem 4.5.5. *Find the cubic polynomial which takes the following values:*

$$
\begin{array}{rcccc}
x: & 0 & 1 & 2 & 3 \\
f(x): & 1 & 2 & 1 & 10
\end{array}
$$

Here, $h = 1$. Hence using the formula, $x_p = x_0 + ph$, and choosing $x_0 = 0$, we get $p = \frac{x_p - x_0}{h} = \frac{x-0}{1} = x$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		−2	
		−1		12
2	1		10	
		9		
3	10			

$\therefore$ By Newton's forward difference formula,

$$
\begin{aligned}
y &= y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 \\
&= 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12) \\
&= 2x^3 - 7x^2 + 6x + 1
\end{aligned}
$$

Problem 4.5.6. *Using newtons backward interpolation formula find the interpolating polynomial from the following table and hence find* $f(12.5)$.

x	*10*	*11*	*12*	*13*
y	*22*	*24*	*28*	*34*

Solution::The Backward difference table is :

x	$f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
10	22			
		2		
11	24		2	
		4		0
12	28		2	
		6		
13	34			

$P = \frac{x-x_n}{h} = \frac{x-13}{1} = x - 13$

By Newton's backward interpolation formula,

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$y = 34 + (x-13)6 + \frac{(x-13)(x-13+1)2}{2}$$

$$y = 34 + 6x - 78 + (x-13)(x-12)$$

$$y = 34 + 6x - 78 + x^2 - 25x + 156$$

$$y = x^2 - 19x + 112$$

is the interpolating polynomial for the data. Now

$$f(12.5) = (12.5)^2 - 19(12.5) + 112$$

$$= 268.25 - 237.5$$

$$\therefore f(12.5) = 30.75$$

4.6 Lagrange's interpolation formula and inverse interpolation formula

Let $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of x :

$x_0, x_1, x_2, \ldots, x_n$ at unequal intervals then

$$\begin{aligned}
y(x) = &\frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}y_0 \\
&+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1 \\
&+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2+\ldots \\
&+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)\ldots(x_3-x_n)}y_3+\ldots \\
&+\vdots \\
&+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n
\end{aligned}$$

Problem 4.6.1. *Use Lagrange's interpolation formula to find y at* $x = 10$ *given[VTU-Jan17]*

x	*5*	*6*	*9*	*11*
y	*12*	*13*	*14*	*16*

Solution:

$$\text{Let } x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$$

$$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$$

Lagrange's interpolation formula :

$$\begin{aligned}
y(x) = &\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 \\
&+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 \\
&+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 \\
&+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3
\end{aligned}$$

When $x = 10$

$$y = \frac{4(1)(-1)12}{(-1)(-4)(-6)} + \frac{5(1)(-1)13}{(1)(-3)(-5)} + \frac{5(4)(-1)14}{(4)(3)(-2)} + \frac{5(4)(1)16}{(6)(5)(2)}$$

$$= 14.666$$

Thus y at $x = 10$ is 14.67

Problem 4.6.2. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1492	2366	5202

Evaluate $f(9)$, using Lagrange's formula.

Solution:

$$x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$$

$$y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202$$

Lagrange's interpolation formula :

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

Putting $x = 9$ and substituting the given values in Lagrange's formula, we get

$$
\begin{aligned}
f(9) =& \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\
&+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\
&+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\
&+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\
&+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 \\
&= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810
\end{aligned}
$$

Problem 4.6.3. *Find the polynomial f(x) by using Lagrange's formula and hence find* $f(3)$ *for*

$x:$	0	1	2	5
$f(x):$	2	3	12	147

Solution:

$$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 5$$

$$y_0 = 2,\ y_1 = 3,\ y_2 = 12,\ y_3 = 147$$

Lagrange's interpolation formula :

$$
\begin{aligned}
y(x) =& \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 \\
&+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3
\end{aligned}
$$

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$f(x) = x^3 + x^2 - x + 2$ (on simplification)

$$f(3) = 27 + 9 - 3 + 2 = 35$$

Problem 4.6.4. *Using Lagrange formula, calculate $f(3)$ from the following table.*

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solution:: Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$

$y_0 = f(x_0) = 1$, $y_1 = f(x_1) = 14$, $y_2 = f(x_2) = 15$, $y_3 = f(x_3) = 5$, $y_4 = f(x_4) = 6$, $y_5 = f(x_5) = 19$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} y_0$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} y_3$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} y_4$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} y_5$$

Here $x = 3$ then

$$f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1$$
$$+\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14$$
$$+\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15$$
$$+\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5$$
$$+\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6$$
$$+\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$
$$= \frac{12}{240} - \frac{18}{60} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{40} \times 19$$
$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95 = 10$$
$$f(3) = 10$$

4.7 Divided Differences :

Let $\mathrm{f}(\mathrm{x}_0), \mathrm{f}(\mathrm{x}_1), \mathrm{f}(\mathrm{x}_2), \ldots, \mathrm{f}(\mathrm{x}_n)$ be the values of an unknown function $\mathrm{y} = \mathrm{f(x)}$ corresponding to the values of $\mathrm{x} : \mathrm{x_0, x_1, x_2, \ldots, x_n}$ at unequal intervals.

The first order divided differences are defined as follows

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0};$$
$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1};$$
$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}; \cdots$$

In general $f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

The second order divided differences are defined as follows

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$
$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$
$$\vdots$$

In general $\mathrm{f}(\mathrm{x_{n-2}, x_{n-1}, x_n}) = \dfrac{\mathrm{f}(\mathrm{x_{n-1}, x_n}) - \mathrm{f}(\mathrm{x_{n-2}, x_{n-1}})}{\mathrm{x_n - x_{n-2}}}$

Similarly the other higher order divided differences are defined.

4.8 Newton's divided difference formula :

Newton's divided difference formula :

$$
\begin{aligned}
y(x) = f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) \\
&+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\
&+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \cdots \\
&\vdots \\
&+ (x - x_0)(x - x_1) \ldots (x - x_{n-1}) f(x_0, x_1, x_2, \ldots, x_n)
\end{aligned}
$$

Problem 4.8.1. *Use Newton's divided difference formula to find* f(4) *given the data*

x	*0*	*2*	*3*	*6*
f(x)	-4	*2*	*14*	*158*

Solution: Here,

$$x_0 = 0, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 6$$

$$f(x_0) = -4, \quad f(x_1) = 2, \quad f(x_2) = 14, \quad f(x_3) = 158$$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2 + 4}{2 - 0} = 3$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{14 - 2}{3 - 2} = 12$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{158 - 14}{6 - 3} = 48$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{12 - 3}{3 - 0} = 3$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{48 - 12}{6 - 2} = 9$$

Third order Divided Differences are give by,

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) &= \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \\
&= \frac{9 - 3}{6 - 0} = 1
\end{aligned}
$$

Newton's divided difference formula is given by:

$$y(x) = f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$
$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$y(4) = -4 + (4 - 0)3 + (4 - 0)(4 - 2)3 + (4 - 0)(4 - 2)(4 - 3)1$$
$$= 40$$

Problem 4.8.2. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Calculate $f(9)$, *using Newton's divided difference formula.*

Solution: : Here $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$f(x_0) = 150$, $f(x_1) = 392$, $f(x_2) = 1452$, $f(x_3) = 2366$, $f(x_4) = 5202$

First order Divided Differences are give by,

$$\mathrm{f}(\mathrm{x}_0, \mathrm{x}_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{392 - 150}{7 - 5} = 121$$
$$\mathrm{f}(\mathrm{x}_1, \mathrm{x}_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1452 - 392}{11 - 7} = 265$$
$$\mathrm{f}(\mathrm{x}_2, \mathrm{x}_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{2366 - 1452}{13 - 11} = 457$$
$$\mathrm{f}(\mathrm{x}_3, \mathrm{x}_4) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{5202 - 2366}{17 - 13} = 709$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{265 - 121}{11 - 5} = 24$$
$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{457 - 265}{13 - 7} = 32$$
$$f(x_2, x_3, x_4) = \frac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2} = \frac{709 - 457}{17 - 11} = 42$$

Third order Divided Differences are give by,

$$\mathrm{f}(\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = \frac{32 - 24}{13 - 5} = 1$$
$$\mathrm{f}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) = \frac{42 - 32}{17 - 7} = 1$$

Newton's divided difference formula is given by:

$$\begin{aligned} y(x) = f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) \\ &+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \\ &+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

Taking $x = 9$ in the above divided difference formula, we obtain

$$\begin{aligned} f(9) &= 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 \\ &+ (9-5)(9-7)(9-11) \times 1 = 150 + 484 + 192 - 16 = 810 \end{aligned}$$

Problem 4.8.3. *Determine* $f(x)$ *as a polynomial in* x *for the following data:*

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

Solution: Here

$$\begin{aligned} x_0 &= -4, \quad f(x_0) = 1245 \\ x_1 &= -1, \quad f(x_1) = 33 \\ x_2 &= 0, \quad f(x_2) = 5 \\ x_3 &= 2, \quad f(x_3) = 9 \\ x_4 &= 5, \quad f(x_4) = 1335 \end{aligned}$$

First order Divided Differences are give by,

$$\begin{aligned} f(x_0, x_1) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{33 - 1245}{-1 + 4} = -404 \\ f(x_1, x_2) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{5 - 33}{0 + 1} = -28 \\ f(x_2, x_3) &= \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{9 - 5}{2 - 0} = 2 \\ f(x_3, x_4) &= \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{1335 - 9}{5 - 2} = 442 \end{aligned}$$

Second order Divided Differences are give by,

$$\begin{aligned} f(x_0, x_1, x_2) &= \frac{-28 - (-404)}{0 - (-4)} = 94 \\ f(x_1, x_2, x_3) &= \frac{2 - (-28)}{2 - (-1)} = 10 \\ f(x_2, x_3, x_4) &= \frac{442 - 2}{5 - 0} = 88 \end{aligned}$$

Third order Divided Differences are give by,

$$f\left(x_0, x_1, x_2, x_3\right) = \frac{10 - 94}{2 - (-4)} = -14$$

$$f\left(x_1, x_2, x_3, x_4\right) = \frac{88 - 10}{5 - 0} = 13$$

4th order Divided Difference is given by,

$$f\left(x_0, x_1, x_2, x_3, x_4\right) = \frac{13 + 14}{5 - (-4)} = 3$$

Applying Newton's divided difference formula,

$$\begin{aligned} y(x) = f(x) = f\left(x_0\right) + \left(x - x_0\right) f\left(x_0, x_1\right) \\ + \left(x - x_0\right)\left(x - x_1\right) f\left(x_0, x_1, x_2\right) \\ + \left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right) f\left(x_0, x_1, x_2, x_3\right) \\ + \left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right)\left(x - x_3\right) f\left(x_0, x_1, x_2, x_3, x_4\right) \end{aligned}$$

$$\begin{aligned} f(x) =& 1245 + (x+4)(-404) + (x+4)(x+1)(94) + \\ & + (x+4)(x+1)(x-0)(-14) + (x+4)(x+1)x(x-2)(3) \\ & = 1245 - 404x - 1616 + (94)\left[x^2 + 5x + 4\right] \\ & - 14x\left[x^2 + 5x + 4\right] + 3x\left[\left(x^2 + 5x + 4\right)(x-2)\right] \end{aligned}$$

$$\begin{aligned} f(x) &= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 \\ & - 70x^2 - 56x + 3x\left[x^3 - 2x^2 + 5x^2 - 10x + 4x - 8\right] \end{aligned}$$

$$\begin{aligned} f(x) &= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 \\ & - 70x^2 - 56x + 3x^4 - 6x^3 + 15x^3 - 30x^2 + 12x^2 - 24x \\ & = 3x^4 - 5x^3 + 6x^2 - 14x + 5 \end{aligned}$$

Problem 4.8.4. *By means of Newton's divided difference formula, find f(8) from the following data-*

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: : $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_4 = 10$, $x_5 = 11$, $x_6 = 13$

$f(x_0) = 48$, $f(x_1) = 100$, $f(x_2 = 294$, $f(x_3) = 900$, $f(x_4) = 1210$, $f(x_5) = 2028$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{100-48}{5-4} = 52$$
$$f(x_1, x_2) = \frac{294-100}{7-5} = 97$$
$$f(x_2, x_3) = \frac{900-294}{10-7} = 202$$
$$f(x_3, x_4) = \frac{1210-900}{11-10} = 310$$
$$f(x_4, x_5) = \frac{2028-1210}{13-11} = 409$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{97-52}{7-4} = 15$$
$$f(x_1, x_2, x_3) = \frac{202-97}{10-5} = 21$$
$$f(x_2, x_3, x_4) = \frac{310-202}{11-7} = 27$$
$$f(x_3, x_4, x_5) = \frac{409-310}{13-10} = 33$$

Third order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3) = \frac{21-15}{10-4} = 1$$
$$f(x_1, x_2, x_3, x_4) = \frac{27-21}{11-5} = 1$$
$$f(x_2, x_3, x_4, x_5) = \frac{33-27}{13-7} = 1$$

4th order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3, x_4) = \frac{1-1}{11-4} = 0$$
$$f(x_1, x_2, x_3, x_4, x_5) = \frac{1-1}{13-5} = 0$$

5th order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

Applying Newton's divided difference formula,

$$\begin{aligned} y(x) = f(x) = f\left(x_0\right) &+ \left(x-x_0\right) f\left(x_0, x_1\right) \\ &+ \left(x-x_0\right)\left(x-x_1\right) f\left(x_0, x_1, x_2\right) \\ &+ \left(x-x_0\right)\left(x-x_1\right)\left(x-x_2\right) f\left(x_0, x_1, x_2, x_3\right) \\ &+ \left(x-x_0\right)\left(x-x_1\right)\left(x-x_2\right)\left(x-x_3\right) f\left(x_0, x_1, x_2, x_3, x_4\right) \end{aligned}$$

Substituting the $x = 8$ and divided differences in the above equation,

$$\begin{aligned} f(8) =& 48 + (8-4)(52) + (8-4)(8-5)(15) \\ &+ (8-4)(8-5)(8-7)(1) \\ &+ (8-4)(8-5)(8-7)(8-10)(0) \\ f(8) =& 48 + (4)(52) + (4)(3)(15) + (4)(3)(1)(1) + 0 \\ f(8) =& 448 \end{aligned}$$

4.9 Numerical Solution of algebraic and transcendental equations:

- Algebraic equation is an equation in the form of a polynomial having a finite number of terms.
  Example: $x^3 - 4x - 9 = 0, x^4 + x^3 = 80$

- A transcendental equation is an equation containing a transcendental function of the variable(s) being solved for.
  Example: $xe^x - 2 = 0,\ tanx = 2x,\ xlogx - 1.2 = 0$

- Numerical method of finding approximate roots of the given function is a repetitive type of process known as **iteration process**.

4.10 Newton Raphson Method

Let the given equation be $f(x) = 0$

We first find an interval (a,b) such that f(a) and f(b) are of opposite signs. Then we select a real number in (a,b) as the initial approximation to the required root.

Use the Newton Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

repeatedly, until we obtain the answer with the desired accuracy.

Problem 4.10.1. *Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.*

Solution:: Let $f(x) = x^4 - x - 10$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$f(1) = -10$ (Here value is -ve)

$f(2) = 16 - 2 - 10 = 4$ (Here value is +ve)

$\therefore$ a root of $f(x) = 0$ lies between $a = 1$ and $b = 2$.

Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton's Raphson formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (*)$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{2^4 - 4 - 10}{4 \times 2^3 - 1}$$

$$= 2 - \frac{4}{31} = 1.871$$

Put $n = 1,$ in (*), the second approximation x_2 is given by,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1.871 - \frac{f(1.871)}{f'(1.871)}$$
$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1}$$
$$= 1.871 - \frac{0.3835}{25.199} = 1.856$$

Put $n = 2,$ in (*), the third approximation x_3 is given by,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$
$$= 1.856 - \frac{0.010}{24.574} - 1.856$$

Since $x_2 = x_3 = 1.856$, the required root is $x = 1.856$

Problem 4.10.2. *Using Newton-Raphson method, find the real root of the equation* $x^3 - 2x - 5 = 0$ *correct to 5 decimal places.*

Solution::

$f(x) = x^3 - 2x - 5, \qquad f'(x) = 3x^2 - 2$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$f(1) = -6$ (Here value is -ve)

$f(2) = -1$ (Here value is -ve)

$f(3) = 16$ (Here value is +ve)

Hence root lies between $(a, b) = (2, 3)$.

Let us take initial point $x_0 = 2.5$

Newton's Raphson formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} \qquad (*)$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$x_1 = x_0 - \frac{x_0^3 - 2x_0 - 5}{3x_0^2 - 2}$$
$$= 2.5 - \frac{2.5^3 - 2(2.5) - 5}{3(2.5)^2 - 2}$$
$$= 2.16418$$

Put $n = 1$, in (*), the first approximation x_2 is given by,

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$
$$= 2.16418 - \frac{(2.16418)^3 - 2(2.16418) - 5}{3(2.16418)^2 - 2}$$
$$= 2.09714$$

Similarly, with $n = 2, 3, 4$ in (*) we get

$$x_2 = 2.09714$$
$$x_3 = 2.09455$$
$$x_4 = 2.09455$$

Since $x_3 = x_4 = 2.09455$, the required root is $x = 2.09455$

Problem 4.10.3. *Using Newton-Raphson method, find the real root of the equation* $3x = \cos x + 1$ *correct to four decimal places.*

Solution: Let $f(x) = 3x - \cos x - 1$ (1)

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

Note : Make sure your calculator is in **radian mode**

$f(0) = 3(0) - cos(0) - 1 = -2,$ (Here value is -ve)

$f(1) = 3(1) - cos(1) - 1 = 1.4597,$ (Here value is +ve)

Hence interval (a,b)= (0,1)

$\therefore$ a root of $f(x) = 0$ lies between 0 and 1.

Let us take $x_0 = 0.6$

Also differentiating (1) we get,

$$f'(x) = 3 + \sin x$$

Newton's Raphson formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$
$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad (*)$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$
$$= \frac{0.6 \sin 6 + \cos 0.6 + 1}{3 + \sin 0.6} = .6071$$

Put $n = 1$ in (*), then second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$
$$= \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$
$$= 0.6071$$

Clearly $x_1 = x_2$.

Hence, desired root is 0.6071 correct to 4 decimal places.

Problem 4.10.4. *Using Newton Raphson method, find a real root of* $x \sin x + \cos x$ *near* $x = \pi$ *correct to 3 decimal places.*

Solution: Given $x_0 = \pi = 3.1416$

$$f(x) = x \sin x + \cos x$$
$$f'(x) = x \cos x + \sin x - \sin x$$
$$= xcosx$$

The iteration formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n} \qquad (*)$$

Note : Make sure your calculator is in **radian mode**

Put $n = 0,$ in (*), the first approximation x_1 is given by,

$$x_1 = x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0}$$

$$= \pi - \frac{\pi \sin \pi + \cos \pi}{\pi \cos \pi} \qquad (\because x_0 = \pi)$$

$$= 2.8233$$

Put $n = 1,$ in (*),

$$x_2 = x_1 - \frac{x_1 \sin x_1 + \cos x_1}{x_1 \cos x_1}$$

$$= 2.8233 - \frac{2.8233 \sin(2.8233) + \cos(2.8233)}{2.8233 \cos(2.8233)} \qquad = 2.7986$$

similarly Putting $n = 2,$ in (*), and calculating we get

$$x_3 = 2.7984$$

since x_2 and x_3 are same upto three decimal places, we stop the procedure and required root is $x = 2.798$

Problem 4.10.5. *Using Newton Raphson method, find a real root of* $x \log_{10} x - 1.2$ *correct to four decimal places.*

Solution: Here initial appoximation is not given.

$$f(x) = x \log_{10} x - 1.2 \qquad (1)$$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$f(1) = -1.2$ (Here value is -ve)

$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979$ (Here value is -ve)

$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136$ (Here value is +ve)

Hence a root of $f(x) = 0$ lies between a=2 and b=3

Let us take $x_0 = 2$ Also,

$$f'(x) = x\log_{10} x - 1.2$$
$$= \log_{10} x + x\frac{d}{dx}(\log_{10} x)$$
$$= \log_{10} x + x\frac{d}{dx}\left(\frac{log_e x}{log_e 10}\right)$$
$$= \log_{10} x + x\frac{1}{x}\ \log_{10} e$$
$$= \log_{10} x + 0.43429$$

Newton's formula gives,

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}$$
$$= x_n - \frac{x_n \log_{10} x_n - 12}{\log_{10} x_n + 0.43429}$$
$$= \frac{0.43429x_n + 1.2}{\log_{10} x_n + 0.43429} \qquad (*)$$

Put $n = 0,$, the first approximation is

$$x_1 = \frac{0.43429x_0 + 1.2}{\log_{10} x_0 + 0.43429}$$
$$= \frac{0.43429(2) + 1.2}{\log_{10} 2 + 0.43429}$$
$$= 2.8132$$

Similarly, Putting $n = 1$ in (*), we get

$$x_2 = \frac{0.43429x_1 + 1.2}{\log_{10} x_1 + 0.43429}$$
$$= \frac{0.43429(2.8132) + 1.2}{\log_{10}(2.8132) + 0.43429}$$
$$= 2.7411$$

Similarly, Putting $n = 2$ in (*), we get

$$x_3 = \frac{0.43429x_2 + 1.2}{\log_{10} x_2 + 0.43429}$$
$$= \frac{0.43429(2.7411) + 1.2}{\log_{10}(2.7411) + 0.43429}$$
$$= 2.7406$$

Similarly, Putting $n = 3$ in (*), we get

$$x_4 = \frac{0.43429x_3 + 1.2}{\log_{10} x_3 + 0.43429}$$
$$= \frac{0.43429(2.7406) + 1.2}{\log_{10}(2.7406) + 0.43429}$$
$$= 2.7406$$

Clearly, $x_3 = x_4$. Hence, the required root is **2.7406** correct to four decimal places.

4.11 Regula Falsi Method

- Let the given equation be $f(x) = 0$

- We first find an interval (a,b) such that f(a) and f(b) are of opposite signs.

- First approximation to the root is given by,
$$\boxed{x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}} \quad (*)$$

- Find $f(x_1)$. If $f(x_1) \neq 0$ then either $f(a)$ and $f(x_1)$ are of opposite signs or $f(x_1)$ and $f(b)$ are of opposite signs.

- If $f(a)$ and $f(x_1)$ are of opposite signs, then next approximation is given by replacing (a, b) by (a, x_1) in equation (*) on the other hand if $f(x_1)$ and $f(b)$ are of opposite signs, then next approximation is given by replacing (a, b) by (x_1, b) in equation (*).

- This process is repeated until the root is obtained with desired accuracy. At each step, the method produces a sequence of shrinking intervals which contains a root.

- Suppose $f(x_1)$ and $f(b)$ are of opposite signs, then
$$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)}$$
Same process is used to find x_3, and so on.

Problem 4.11.1. *Use the Regula-falsi method to find a real root of the equation,* $x^3 - 2x - 5 = 0$ *correct to 2 decimal places.*

Solution:: Let $f(x) = x^3 - 2x - 5$

$f(0) = -5$

$f(1) = -6,$

$f(2) = 2^3 - 2(2) - 5 = -1$ (negative)

$f(3) = 3^3 - 2(3) - 5 = 16$ (positive)

Since $f(2)$ and $f(3)$ are of opposite signs, a real root lies in (2, 3).

Let us take $a = 2$ and $b = 3$. The first approximation to root is x_1 and is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$
$$= \frac{(2(16) - 3(-1))}{(16 - (-1))}$$
$$= 2.058$$

Now $f(2.058) = (2.0583)^3 - 2(2.058) - 5 = -0.4$ (negative)

The root lies between 2.058 and 3

Taking $a = 2.058$ and $b = 3$. The second approximation to the root is given by

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{(2.058)\,f(3) - 3f(2.058)}{f(3) - f(2.058)}$$
$$= \frac{(2.058)16 - 3(-0.4)}{16 - (-0.4)}$$
$$= 2.081$$

Now $f(2.081) = (2.081)^3 - 2(2.081) - 5 = -0.15$ (negative)

The root lies between 2.081 and 3

Take $a = 2.081$ and $b = 3$ The third approximation to the root is given by

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{(2.081)\, f(3) - 3f(2.081)}{f(3) - f(2.081)}$$
$$= \frac{(2.081)(16) - 3(-0.15)}{16 - (-0.15)} = 2.089$$

Now $f(2.089) = (2.089)^3 - 2(2.089) - 5 = -0.0617$ (negative)

The root lies between 2.089 and 3

Take $a = 2.089$ and $b = 3$ The 4th approximation to the root is given by

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{(2.089)\, f(3) - 3f(2.089)}{f(3) - f(2.089)}$$
$$= \frac{(2.089)(16) - 3(-0.0617)}{16 - (-0.0617)} = 2.092$$

x_3 and x_4 are almost equal.

Thus the required approximate root correct to 2 decimal places is 2.09

Problem 4.11.2. *Use the Regula-falsi method to find a real root of the equation* $\cos x = xe^x$, *which lies in* $(0, 1)$. *Carryout 3 iterations. Write the answer correct to 5 decimal places.*

Solution:: Let $f(x) = \cos x - xe^x = 0$

Given $(a, b) = (0, 1)$

So that $f(a) = f(0) = 1$, and $f(b) = f(1) = \cos 1 - e = -2.17798$

The first approximation to root (i.e. x_1) is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1}$$
$$= 0.31467$$

Now $f(x_1) = f(0.31467) = 0.51987$

i.e., the root lies between $(a, b) = (0.31467, 1)$.

$f(a) = f(0.31467) = 0.51987$ and $f(b) = f(1) = -2.17798$

The second approximation to the root (i.e. x_2) is given by

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.31467(-2.17798) - 1(0.51987)}{-2.17798 - 0.51987}$$
$$= 0.44673$$

Now $f(x_2) = f(0.44673) = 0.20356$

the root lies between $(a, b) = (0.44673, 1)$ $f(a) = f(0.44673) = 0.20356$ and $f(b) = f(1) = -2.17798$

The third approximation to the root (i.e. x_3) is given by

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.44673(-2.17798) - 1(0.20356)}{-2.17798 - 0.20356}$$
$$= 0.49402$$

After 3 iterations, the root of the equation $\cos x = xe^x$ is $x = 0.49402$ correct to 5 decimal places.

Problem 4.11.3. *Use the Regula-falsi method to find a real root of the equation,* $x \log_{10} x - 1.2 = 0$ *which lies in* $(2, 3)$. *[VTU: Dec 2010, July 2016]*

Solution:: Let $f(x) = x \log_{10} x - 1.2$ Here $(a, b) = (2, 3)$ $f(a) = f(2) = 2 \log_{10} 2 - 1.2 = -0.59794 \quad (-ve)$

$f(b) = f(3) = 3 \log_{10} 3 - 1.2 = 0.23136 \quad (+ve)$

By method of false position, we have

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$
$$= \frac{2(0.23136) - 3(-0.59794)}{(0.23136) - (-0.59794)} = 2.72102$$

Now, $f(2.72102) = (2.72102) \log_{10} 2.72102 - 1.2 = -0.01709 \ (-ve)$

since, $f(2.72102)$ and $f(3)$ are of opposite sign, so the root lies between $a = 2.72102$ and $b = 3$

$f(a) = f(2.72102) = -0.01709$ and $f(b) = f(3) = 0.23136$

$$\begin{aligned} x_2 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{2.72102f(3) - 3f(2.72102)}{f(3) - f(2.72102)} \\ &= \frac{2.72102(0.23136) - 3(-0.01709)}{0.23136 - (-0.01709)} \\ &= 2.74021 \end{aligned}$$

Now, $f(2.74021) = (2.74021)\log_{10} 2.74021 - 1.2 = -0.00038 \quad (-ve)$
since $f(2.74021)$ and $f(3)$ are of opposite sign, so the root lies between $a = 2.74021$ and $b = 3$
$f(a) = f(2.74021) = -0.00038$ and $f(b) = f(3) = 0.23136$

$$\begin{aligned} x_3 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{2.74021f(3) - 3f(2.74021)}{f(3) - f(2.74021)} \\ &= \frac{2.74021(0.23136) - 3(-0.00038)}{(0.23136) - (-0.00038)} \\ &= 2.74064 \end{aligned}$$

Now, $f(2.74064) = (2.74064)\log_{10} 2.74064 - 1.2 = -0.000005 \quad (-ve)$
since $f(2.74064) \approx 0$, we conclude that $x = 2.7406$ is a root correct to 4 decimal places.

Problem 4.11.4. *Using Regula Falsi Method, compute the real root of the equation $xe^x - 2 = 0$ correct up to three decimals places.* *[VTU Jan 2017]*

Solution:: Here, $f(x) = xe^x - 2$
$f(0) = -2(--\text{ve})$
$f(1) = 1 \times e^1 - 2 = 0.718 \ (+\text{ve})$
Try to obtain a smaller interval.
Put $x = 0.5$, then $f(0.5) = 0.5 \times e^5 - 2 = -1.17(-\text{ve})$

As the value of $f(0.5)$ is -ve and $f(1)$ is positive, root lies in $(a, b) = (0.5, 1)$ and $f(a) = f(0.5) = -1.17$ and $f(b) = f(1) = 0.718$ By method of false position, we have

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.5(0.718) - 1(-1.17)}{0.718 - (-1.17)}$$
$$= 0.8098$$

Now, $f(x_1) = f(0.8098) = 0.8098e^{0.8098} - 2 = -0.18$ (negative)

Root lies between $a = 0.8098$ and $b = 1$

$f(a) = f(0.8098) = -0.18$ and $f(b) = f(1) = 0.718$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8098(0.718) - 1(-0.18)}{0.718 - (-0.18)}$$
$$= 0.8479$$

Now, $f(x_2) = f(0.8479) = 0.8479 - e^{0.8479} - 2 = -0.02$ (negative)

Root lies between $a = 0.8479$ and $b = 1$

$f(a) = f(0.8479) = -0.02$ and $f(b) = f(1) = 0.718$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8479(0.718) - 1(-0.02)}{0.718 - (-0.02)}$$
$$= 0.8519$$

Now, $f(x_3) = 0.8519e^{0.8519} - 2 = -0.0031$ (negative)

Root lies between $a = 0.8519$ and $b = 1$

$f(a) = f(0.8519) = -0.0031$ and $f(b) = f(1) = 0.718$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8519(0.718) - 1(-0.0031)}{0.718 - (-0.0031)}$$
$$= 0.8525$$

Now, $f(x_4) = f(0.8525) = 0.8525 - e^{0.8525} - 2 = -0.0005$ (negative)

Root lies between $a = 0.8525$ and $b = 1$

$f(a) = f(0.8479) = -0.0005$ and $f(b) = f(1) = 0.718$

$$x_5 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0.8525(0.718) - 1(-0.0005)}{0.718 - (-0.0005)}$$
$$= 0.8525$$

Now, $x_4 = x_5 = 0.8525$. Hence approximate root correct to 4 decimal of place is , $x = 0.8525$

4.12 Numerical Integration :

Let $\mathbf{I} = \int_a^b y dx$, where y takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots, x_n$ Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \frac{b-a}{n}$ so that $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$

$$\therefore \quad \mathbf{I} = \int_{x_0}^{x_0+nh} f(x)dx$$

Numerical Integration is the process of obtaining approximately the value of the definite integral $\mathbf{I} = \int_a^b ydx$ without actually integrating the function but only using the values of y at some points of x equally spaced over $[a, b] = [x_0, x_0 + nh]$.

4.13 Three rules to obtain the value of the definite integral:

- **Trapezoidal rule :**

$$I = \frac{h}{2}\left[(y_0 + y_n) + 2\left(y_1 + y_2 + y_3 + \ldots + y_{n-1}\right)\right]$$

- **Simpson's $(1/3)^{th}$ rule:**

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4\left(y_1 + y_3 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{n-2}\right)\right]$$

While using this formula, the given interval of integration must be divided into an even number of sub-intervals(i.e. n=even)

- **Simpson's $(3/8)^{th}$ rule:**

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3\left(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}\right) + 2\left(y_3 + y_6 + \ldots + y_{n-3}\right)\right]$$

While using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

4.14 Working procedure for problems:

Step 1: Given the definite integral $I = \int_a^b y dx$ for evaluation, first divide the interval $[a, b]$ into n equal parts (strips) of width $h = \frac{(b-a)}{n}$

Step 2: Prepare a table consisting the values of x and the corresponding computed values of y

Step 3: Substitute values from this table into the appropriate rule to obtain the approximate value of the given definite integral.

Note: Number of ordinates = $n + 1$ where n is the number of sub intervals.

Problem 4.14.1. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using*

(i) Simpson's $1/3$ *rule,*

(ii) Simpson's $3/8$ *rule,*

Solution:: Let $n = 6$. Hence $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$. Thus Divide the interval $(0, 6)$ into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.05884	0.0385	0.027
$= y$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Simpson's $1/3$ rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{3}\left[(y_0 + y_6) + 4\,(y_1 + y_3 + y_5) + 2\,(y_2 + y_4)\right]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)]$$

$$= 1.3662$$

(ii) By Simpson's $3/8$ rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{8}\left[(y_0 + y_6) + 3\,(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)]$$

$$= 1.3571$$

Actual value : Let us evaluate the integral explicitly.

$$\int_0^6 \frac{1}{1+x^2} = \left[tan^{-1}x\right]_0^6$$

$$= tan^{-1}(6) - tan^{-1}(0) = 1.4056$$

(Note : Keep the Calculator in Radian Mode)

Problem 4.14.2. *Use Simpson's $1/3rd$ rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates.* *(V.T.U., 2011)*

Solution:: Divide the interval $(0, 0.6)$ into $n = 6$ parts each of width $h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$.

The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
$y = e^{-x^2}$	1 y_0	0.9900 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's $1/3$ rd rule, we have

$$\int_0^{0.6} e^{-x^2} dx = \frac{h}{3}\left[(y_0 + y_6) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{0.1}{3}[(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788 + 2(0.9608 + 0.8521)]$$

$$= \frac{0.1}{3}[1.6977 + 10.7308 + 3.6258]$$

$$= \frac{0.1}{3}(16.0543) = 0.5351.$$

Problem 4.14.3. *Compute the value of* $\int_{0.2}^{1.4}(\sin x - \log x + e^x)\,dx$ *using Simpson's* $\frac{3}{8}$*th rule.*

Solution: Let $y = \sin x - \log_e x + e^x$, $n = 6$. Then $h = \frac{1.4-0.2}{6} = 0.2$

(Note : Keep the Calculator in radian mode)

The values of y are as given below :

x :	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y :	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ th rule, we have

$$\int_{0.2}^{1.4} y dx = \frac{3h}{8}\left[(y_0 + y_6) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3}{8}(0.2)[7.7336 + 2(3.1660) + 3(13.3247)] = 4.053$$

Hence $\int_{0.2}^{1.4}(\sin x - \log_e x + e^x)\,dx = 4.053$.

Problem 4.14.4. *Calculate the value of the integral*

$$\int_4^{5.2} \log x dx \text{ by}$$

(a) Simpson's $1/3$ *rule (b) Simpson's* $3/8$ *rule Compare the answers with exact solution.*

Solution:: Here $x_0 = 4, x_n = 5.2$

Let $n = 6$. Then $h = \frac{x_n - x_0}{n} = 0.2$.

Taking $h = 0.2$, divide the range of integration $(4, 5.2)$ into six equal parts.The values of $\log x$ for each point of sub-division are given below:

X	$f(x) = \log x = ln(x)$
$x_0 = 4$	$f(0) = 1.3862944$
$x_0 + h = 4.2$	$f(1) = 1.4350845$
$x_0 + 2h = 4.4$	$f(2) = 1.4816045$
$x_0 + 3h = 4.6$	$f(3) = 1.5260563$
$x_0 + 4h = 4.8$	$f(4) = 1.5686159$
$x_0 + 5h = 5.0$	$f(5) = 1.6094379$
$x_0 + 6h = 5.2$	$f(6) = 1.6486586$

(a) By Simpson's $1/3$ rule, we have

$$\int_4^{52} \log x dx = \frac{h}{3}\left[f(0) + f(6) + 4\{f(1) + f(3) + f(5)\}\right.$$
$$\left. + 2\{f(2) + f(4)\}\right]$$
$$= \frac{0.2}{3}[3.034953 + 4(4.5705787) + 2(3.0502204)]$$
$$= \frac{0.2}{3}[3.034953 + 18.282315 + 6.1004408]$$
$$= \frac{0.2}{3} \times 27.417709 = 1.8278472.$$

(b) By Simpson's $3/8$ rule, we have

$$\int_4^{5.2} \log x dx = \frac{3h}{8}[f(0) + f(6) + 3\{f(1) + f(2) + f(4) + f(5)\} + 2f(3)]$$
$$= \frac{3(0.2)}{8}[3.034953 + 3(6.0947428) + 2(1.5260563)]$$
$$= \frac{0.6}{8}[3.034953 + 18.284228 + 3.0521126]$$
$$= \frac{0.6}{8} \times 24.371294 = 1.827847$$

Exact value of the integral is

$$\int_4^{52} \log x dx = [x(\log x - 1)]_4^{\pi.2}$$
$$= [5.2(\log 5.2 - 1) - 4(\log 4 - 1)] = 3.3730249 - 1.5451774 = 1.8278475$$

Problem 4.14.5. *Evaluate* $\int_0^{\pi/2} \sin x dx$ *by Simpson's rule by taking* $h = \pi/20$. *Compare with exact value.*

Solution: :Here $x_0 = 0$, $x_n = \frac{\pi}{2}$

Given that $h = \frac{\pi}{20}$. $\therefore n = \frac{x_n - x_0}{h} = 10$

x	0	$\pi/20$	$2\pi/20$	$3\pi/20$	$4\pi/20$	$5\pi/20$	$6\pi/20$	$7\pi/20$	$8\pi/20$	$9\pi/20$
x	0	0.1571	0.3142	0.4712	0.6283	0.7854	0.9425	1.0996	1.2566	1.4137
y	0	0.1565	0.3091	0.4540	0.5878	0.7071	0.8090	0.8910	0.9510	0.9877

By simpson's 1/3 rule,

$$\int_{x_0}^{x_{10}} ydx = \frac{h}{3}\left[(y_0 + y_{10}) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right)\right.$$

$$\left.+2\left(y_2 + y_4 + y_6 + y_8\right)\right]$$

$$=\frac{0.1571}{3}[(0+1) + 4(0.1565 + 0.4540 + 0.7071 + 0.8910 + 0.9877)$$

$$+ 2(0.3091 + 0.5878 + 0.8090 + 0.9510)]$$

$$=\frac{0.1571}{3} \times 19.0990 = 1.000150967 = 1.00015$$

Exact Value is

$\int_0^{\pi/2} \sin x dx = (-\cos x)_0^{-12} = 0 - (-1) = 1$

Thus, Simpson's rule gives a better approximation.

4.15 Question Bank : Module 4 - Numerical Methods I

4.15.1 Question Bank :Regula Falsi and Newton Raphson Methods

1. Using Newton-Raphson method, find the real root of the equation $3x = \cos x + 1$ correct to four decimal places. (VTU Model 2022)

2. Find a real root of $x^3 - 9x + 1 = 0$ in $(2, 3)$ by the Regula-Falsi method in four iterations (VTU Model 2022)

3. Use Regula Falsi method to find a real root of the equation $x^3 - 2x - 5 = 0$, correct to 3 decimal places. [VTU - June 2018]

4. Find a real root of $xe^x - cosx = 0$ correct to three decimal places lying in the interval $(0.5, 0.6)$, using Regula-Falsi method. [VTU-Model 2018]

5. Use the Regula-falsi method to find a real root of the equation $\cos x = xe^x$, which lies in $(0, 1)$. Write the answer correct to 4 decimal places. (VTU Model 2022)

6. Use the Regula-falsi method to find a real root of $xlog_{10}x - 1.2 = 0$ [VTU:Jan 2020, June 2019, Model 2018, Dec 2010/July16]

7. Use the Regula-falsi method to find a real root of $cosx = 3x - 1$ [VTU: Dec 2012/July 13/July 15] Ans: 0.607

8. Use the Regula-falsi method to find a real root of $xe^x - 2 = 0$ [VTU- Jan17]

9. Use the Regula-falsi method to find a real root of $xe^x - 3 = 0$, correct to three decimal places. [VTU Jan 2018]

10. Use the regula falsi method to find the forth root of 12 correct to 3 decimal places [VTU: Dec2015/Jan17]

11. Use the Regula Falsi method to find the root of the equation $2x - log_{10}x = 7$ which lies between 3.5 and 4 [VTU-June 12/June17] Ans: 3.7893

12. Find a real root of the equation $x^3 + x^2 + 3x + 4 = 0$ near $x = -1$, correct to four decimal places using Newton- Raphson method. [VTU – Model 2018]

13. Use Newton Raphson method to find a real root of $xsinx + cosx = 0$, near $x = \pi$. Carryout the iterations upto four decimal places of accuracy. [VTU – Model 2022, Jan 2020, June 2019, Model 2018, Dec13/Dec14/Jan15/June17]

14. Find the real root of the equation $xe^x - 2 = 0$ correct to three decimal places using Newton- Raphson method. [VTU- Jan17]

15. Using Newton-Raphson method, find the root that lies near $x = 4.5$ of the equation $tanx = x$ correct to four decimal places. [VTU- Jan 2018, Jan17]

16. Using Newton-Raphson method find the value of cube root of 18 correct to 2 decimals, assuming 2.5 as the initial approximation. [VTU – June17]

17. Find the real root of the equation $xlog_{10}x - 1.2 = 0$ using Newton-Raphson method [VTU – June 2018]

18. Compute one positive root of $2x - \log_{10} x = 7$ by Newton-Raphson method correct to four decimal places.
    Ans : $\mathbf{3.7892}$

19. Use Newton-Raphson method to find a root of the equation $x^3 - 3x - 5 = 0$
    Ans : $\mathbf{2.279}$

20. Find the negative root of the equation $x^3 - 4x + 9 = 0$ from Regula Falsi method. Ans: $-\mathbf{2.7065}$

21. Find the root of the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by the method of false position. Ans : $\mathbf{2.7473}$

22. Find the root of the equation $4\sin x = e^x$, that lies between 0 and 0.5. Correct to 4 places of decimals, using Regula-Falsi method. Ans :$\mathbf{0.3706}$

23. Find the root of the equation $xe^x - 3 = 0$, that lies between 1 and 2, correct to 3 places of decimals, using the method of false position. Ans : $\mathbf{1.049}$

24. Use the Newton-Raphson method to find a root of the equation $xe^x - 2 = 0$ correct to 3 decimal places. (VTU 2005) Ans :$\mathbf{0.853}$

25. Use the Newton-Raphson method to find a root of the equation $cosx = xe^x$ correct to 3 decimal places. Ans : $\mathbf{0.518}$

4.15.2 Question Bank :Newtons Forward and Backward Interpolation formulas

Using Newton's forward interpolation find y at $x = 5$ from the data

1.

x	4	6	8	10
y	1	3	8	16

(VTU Model 2022)

2. Using Newton's backward interpolation formula find the value of y when $x = 6$ from the given table

x	1	2	3	4	5
y	1	-1	1	-1	1

(VTU Model 2022)

3. Given f(40) = 184, f (50)=204, f (60) = 226, f (70) = 250, f (80) = 276, f (90) = 304, find $f(42)$ using Newton's forward interpolation formula. [VTU Jan 2020]

4. Given f(40) = 184, f (50)=204, f (60) = 226, f (70) = 250, f (80) = 276, f (90) = 304, find $f(42)$ and $f(85)$ using suitable interpolation formulae. [VTU: – Model 2018, June 2018, June12/Jan16]

5. Given $sin45° = 0.7071, sin50° = 0.7660, sin55° = 0.8192, sin60° = 0.8660$ Find $sin57°$ using an appropriate interpolation formula. [VTU: – June 2018]

6. From the following table of half-yearly Premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age	45	50	55	60	65
Premium (in Rupees)	114.84	96.16	83.32	74.48	68.48

[VTU Jan 2018]

7. From the following table find the number of students who have obtained (a) less than 45 marks (b) between 40 and 45 marks.

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

[VTU- June 2019, July15/Jan17]

8. From the following table find the number of students who have obtained less than 70 marks.

Marks	0-19	20-39	40-59	60-79	80-99
No. of students	41	62	65	50	17

[VTU – Jan 17]

9. Estimate the probable number of persons with daily income 20 to 25 Rs from the following table.

Income per day(Rs)	Under 10	10- 20	20- 30	30- 40	40- 50
No. of person	20	45	115	210	115

[VTU: Jan14]

10. Estimate the probable number of getting wages below Rs 35 from the following data :

Wages (in Rs)	0-10	10- 20	20- 30	30- 40
Frequency	9	30	35	42

[VTU Jan 2018]

11. The population of the town is given by the table. Using Newton's forward and backward interpolation formula, calculate the increase in the population from the year 1955 to 1985

Year	1951	1961	1971	1981	1991
Population in 1000	19.96	39.65	58.81	77.21	94.61

[VTU-Jan16/June17]

12. Using Newton's backward interpolation formula find the interpolating polynomial for the function given by the following data. Hence find f(12.5)

x	10	11	12	13
f(x)	22	24	28	34

[VTU - Jan17]

13. Using Newton's forward interpolation formula find y at x=160 for the following data.

x	100	150	200	250	300	350	400
f(x)	10.63	13.03	15.04	16.81	18.42	19.90	21.27

[VTU: June13]

14. Use an appropriate interpolation formula to compute using the following data:

x	1.7	1.8	1.9	2.0	2.1	2.2
f(x)	5.474	6.050	6.686	7.389	8.166	9.025

[VTU Model 2018]

15. Estimate the value of $f(22)$ from the following available data:

x	20	25	30	35	40	45
y	354	332	291	260	231	204

Ans : 352.22304

16. The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

$x:$	100	150	200	250	300	350	400
$y:$	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Use Newton's forward formula to find y when $x = 218$ ft. Ans : 15.6993 nautical miles.

17. Find the number of men getting wages between Rs. 10 and Rs. 15 from following table

Wages in R s :	$0-10$	$10-20$	$20-30$	$30-40$
Frequency :	9	30	35	42

Ans : 15

18. Estimate the value of $f(42)$ from the following available data:

$x:$	20	25	30	35	40	45
$f(x):$	354	332	291	260	231	204

Ans :219

19. The area A of a circle of diameter d is given for the following values :

$d:$	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105. Ans : 8666

20. Find a cubic polynomial in x which takes on the values $-3, 3, 11, 27, 57$ and 107, when x $= 0, 1, 2, 3, 4$ and 5 respectively. Ans: $y = x^3 - 2x^2 + 7x - 3$

21. Find the cubic polynomial which takes the following values.

x	0	1	2	3
y	1	2	1	10

Ans: $2x^3 - 7x^2 + 6x + 1$

4.15.3 Question Bank :Newton's Divided Difference interpolation Formula

1. Using Newton's divided difference formula find f(9) from the following data

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

[VTU Model 2022, Jan 2020]

2. Construct the interpolation polynomial for the data given below using Newton's divided difference formula.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

Hence find $f(9)$. [VTU- Model 2018, June 2018, June17] Ans: $f(x) = 2x^3 - 3x^2 + 5x - 4$, $f(3) = 38$ and $f(9) = 1256$

3. Find the equation of the polynomial which passes through the points $(4, -43)$ $(7, 83)(9, 327)$ and $(12, 1053)$ using Newton's divided difference formula [VTU- Jan17] Ans: $f(x) = x^3 - 4x^2 - 7x - 15$

4. Using Newton's divided difference formula find f(82), f(98) from the following data

x	80	85	90	95	100
f(x)	5026	5674	6362	7088	7854

[VTU: Dec12]

5. Given,

x	-4	-1	0	2	5
y	1245	33	5	9	1335

Determine f(x) as a polynomial in x for the following data using Newton's difference formula. [VTU- Jan17] Ans: $y = 3x^4 - 5x^3 + 6x^2 - 14x + 5$

6. Using Newton's divided difference formula, find a polynomial for the data :

x	3	7	9	10
F(x)	168	120	72	63

Hence find y at $x = 8$ [VTU- Jan 2018]

7. Using Newton's divided difference formula, fit an interpolating polynomial for the data given below and hence find y at x = 2

x	0	1	4	8	10
f(x)	-5	-14	-125	-21	355

[VTU: June13]

8. Fit an interpolating polynomial for the data $u_{10} = 355, u_0 = -5, u_8 = -21, u_1 = -14$ $u_4 = -125$ by using Newton's general interpolation formula and hence find u_2 Ans: $f(x) = 2x^3 - 17x^2 + 6x - 5, u_2 = -45$

9. Using Newton's divided difference formula find $f(8), f(15)$ from the following data

X	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Ans: $f(8) = 448, f(15) = 3150$

4.15.4 Question Bank :Lagrange's interpolation formula

1. Using Lagrange's interpolation formula, fit a polynomial which passes through the points $(-1, 0), (1, 2), (2, 9)$ and $(3, 8)$ and hence estimate the value of y when $x = 2.2$ (VTU Model 2022)

2. Use Lagrange's interpolation formula to find y at x = 10 given

x	5	6	9	11
y	12	13	14	16

[VTU- Jan17]

3. Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
f(x)	-12	0	6	12

Hence estimate y at x = 2 [VTU: – June 2018, Jan16]

4. Using Lagrange's formula find the interpolating polynomial that approximates to the function described by the following table.

x	0	1	2	5
f (x)	2	3	12	147

[VTU: Jan 2018, July16]

5. Using Lagrange's formula find y at $x = 4$ by using the following data.

x	0	1	2	5
f (x)	2	3	12	147

[VTU: June 2019],

6. The following table gives the premium payable at ages in years completed. Interpolate the premium payable at age 35 completed, using Lagrange's formula.

Age completed	25	30	40	60
Premium in Rs	50	55	70	95

[VTU- Jan17]

7. Use Lagrange's interpolation formula to find $f(5)$. Given [VTU- July15]

x	0	1	2	3	4
f(x)	3	6	11	18	27

8. Using Lagrange's interpolation formula to fit a polynomial for the following data: [VTU-Model 2018]

x	2	10	17
y	1	3	4

9. Use Lagrange's interpolation formula to find $\mathbf{f}(4)$ given Ans: $f(4) = 40$]

x	0	2	3	6
$f(x)$	-4	2	14	158

10. Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
f(x)	-12	0	6	12

Hence estimate y at $x = 2$ [VTU: Jan16] [Ans: $f(x) = x^3 - 7x^2 + 18x - 12; f(2) = 4$]

11. Using Lagrange's interpolation method, find the value of $f(x)$ at $x = 5$ given the values [Ans: $f(5) = 69.4$]

x	1	3	4	6
f(x)	3	9	30	132

12. Using Lagrange's interpolation formula to fit a polynomial for the following data: [VTU-Model 2018]

x	2	10	17
y	1	3	4

4.15.5 Question Bank :Numerical Integration

1. Evaluate $\int_0^5 \frac{1}{4x+5}dx$ by dividing the interval into 10 equal parts. [Ans: 1.61] (Model 2022)

2. Evaluate $\int_0^6 \frac{dx}{(1+x^2)}$ by using Simpson's (3/8)th rule, taking 7 ordinates. [VTU June 2019]

3. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}d\theta$ by using simpson's $(1/3)^{\text{th}}$ rule by dividing $\left[0, \frac{\pi}{2}\right]$ into 6 equal parts. (VTU Model 2022)

4. Use Simpson's (1/3)th rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking 6 sub intervals. [VTU-Jan 2018, Jan17]

5. Evaluate $\int_0^1 \frac{dx}{(1+x)}$ by using Simpson's (3/8)th rule taking seven ordinates and hence deduce the value of $log_e 2$ [VTU: Jan17]

6. Evaluate $\int_0^1 \frac{xdx}{(1+x^2)}$ by using Simpson's one third rule taking seven ordinates and hence find $log_e 2$ [VTU: Model 2018, July 2018, Dec13/Jan16/Jan17]

7. Evaluate $\int_4^{5.2} log_e x dx$ taking 6 equal strips by applying Simpson's (3/8)th rule. [VTU : Model 2022, Jan 2020, June 2019, June13, Jan17]

8. Use Simpson's (3/8)th rule to obtain the approximate value of $\int_{0.2}^{1.4}(sinx - logx + e^x)dx$ by considering 6 parts. [VTU: Jan 2018, Jan17]

9. Using Simpson's (3/8)th rule, Evaluate $\int_0^3 \frac{dx}{(1+x)^2}$ taking 4 equidistant ordinates. [VTU: Model 2018]

10. Evaluate $\int_0^1 \frac{dx}{(1+x^2)}$ by using Simpson's (1/3)th rule, Simpson's (3/8)th rule [VTU: Model 2022, Jan 2020, Dec12/Jan14]

11. Evaluate $\int_0^1 \frac{xdx}{(1+x^2)}$ by using Simpson's (3/8)th rule, dividing the interval into 3 equal parts, and hence find $log_e\sqrt{2}$ [VTU: July15]

12. Using simpson's 1/3 rule with 7 ordinates, evaluate $\int_2^8 \frac{1}{(log_{10}x)}dx$ [VTU: – June 2018]

13. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{sinx}dx$ using Simpson's $\frac{1}{3}$- rule by taking 10 equal parts. [VTU Model 2019]

14. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cosxdx$ using Simpson's (3/8)th rule by dividing the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ into 6 equal parts. [VTU Model 2019]

15. Evaluate $\int_0^1 \frac{\mathrm{dx}}{1+\mathrm{x}^2}$ by using Simpson's $(1/3)^{\text{th}}$ rule taking four equal strips and hence deduct an approximate value of Π. [Ans: 0.7854]

16. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}d\theta$ by using simpson's $(1/3)^{\text{th}}$ rule by dividing $\left[0, \frac{\pi}{2}\right]$ into 6 equal parts. Ans: $= \mathbf{1.1873}$

17. Use Simpson's $(1/3)^{\text{th}}$ rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking 6 sub intervals. [VTU-Jan17] [Ans: $\mathbf{0.5351}$]

1. Use Simpson's $(1/3)^{\text{th}}$ rule to find $\int_2^8 \frac{1}{\log_{10} x}dx$ by taking 7 ordinates. [Ans: 9.7203]

2. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using (i) Simpson's $1/3$ rule (ii) Simpson's $(3/8)^{\text{th}}$ rule taking seven ordinates and hence deduce the value of $\log_e 2$ [VTU: Jan17] [Ans: $\mathbf{0.6932, 0.693}$]

3. Evaluate $\int_0^1 \frac{xdx}{1+x^2}$ by using Weddle's rule taking seven ordinates and hence find $\log_e 2$ [Ans: 0.3466, $\log_e 2 = \mathbf{0.6932}$] [VTU: Dec13/Jan16/Jan17]

4. Evaluate $\int_4^{5.2} \log_e xdx$ taking 6 equal strips by applying using (i) Simpson's $(1/3)^{\text{th}}$ rule, (ii) Simpsons $3/8^{\text{th}}$ rule and (iii) weddles rule [Ans: $\mathbf{1.8278, 1.8278, 1.8279}$] [VTU: June13/Jan17]

5. Use Simpson's $(3/8)^{\text{th}}$ rule to obtain the approximate value of $\int_0^{0.3} \left(1 - 8x^3\right)^{\frac{1}{2}} dx$ by considering 3 equal intervals. [Ans: $\mathbf{0.2916}$]

6. Use Simpson's $(3/8)^{\text{th}}$ rule to obtain the approximate value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x$ by considering 6 parts. [VTU: Jan17] [Ans: 4.053]

10. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $(1/3)^{\text{th}}$ rule. Simpson's $(3/8)^{\text{th}}$ rule, Weddle'srule. [VTU: Dec12/Jan14] [Ans: 1.3662, 1.3571, 1.3735]

7. Evaluate $\int_0^1 \frac{xdx}{1+x^2}$ by using Simpson's $(3/8)^{\text{th}}$ rule, dividing the interval into 3 equal parts, and hence find $\log_e \sqrt{2}$ [VTU: July15]

8. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Simpson's $(1/3)^{\text{th}}$ rule, (ii) Simpsons $3/8^{\text{th}}$ rule and (iii) weddles rule . Compare the results with actual value. [VTU 2013] Ans: (i) $\mathbf{1.3662}$, (ii) $\mathbf{1.3571}$ (iii) $\mathbf{1.3735}$

13. Evaluate $\int_0^5 \frac{1}{4x+5}dx$ by dividing the interval into 10 equal parts. [Ans: 1.61]

14. Evaluate $\int_0^\pi \sin x dx$ using 11 ordinates. [Ans: 2.0009]

15. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx$ using simpson's $1/3$ rule taking 10 equal parts. [VTU Model 2019] [Ans: 1.3028]

1. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$ by dividing the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ into 6 equal parts. [VTU Model 2019]

Module 5

Numerical Methods II

Syllabus

Introduction to various numerical techniques for handling Mechanical Engineering applications

Numerical Solution of Ordinary Differential Equations (ODEs): Numerical solution of ordinary differential equations of first order and first degree - Taylor's series method, Modified Euler's method, Runge-Kutta method of fourth order and Milne's predictorcorrector formula (No derivations of formulae). Problems.

Self-Study: Adam-Bashforth method.

Applications: Finding approximate solutions to solve mechanical engineering problems.

5.1 Numerical solution of first order differential equations :

Consider the first order differential equation,

$$\frac{dy}{dx} = f(x, y), \text{ with initial condition, } y(x_0) = y_(0),$$

Such problems in which all the initial conditions are given at the initial point only, are called **initial value problems**.

Let $y(x_0)$, $y(x_1)$,..., $y(x_n)$ be the solution values at the equidistant points x_0, x_1, ..., x_n. Computation of the approximate values to these solution values is known as

Numerical solution of the Differential equation.

Analytical methods, when available, generally enable to find the value of y for all values of x. Numerical methods, on the other hand, lead to the values of y corresponding only to some finite set of values of x. Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution involves some error and is an approximate solution.

5.2 Taylors series method :

Taylor's series is a numerical method used to approximate the value of a function f(x) at a specific point x, by using a series of terms that are derived from the function's derivatives evaluated at that point.

Consider the initial value problem $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_(0)$ then the Taylor's series expansion of $y(x)$ at $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0{}' + \frac{(x - x_0)^2}{2!}y_0{}'' + \frac{(x - x_0)^3}{3!}y_0{}''' + \frac{(x - x_0)^4}{4!}y_0{}^{IV} + \cdots$$

Problem 5.2.1. *Find* $y(0.1)$ *for* $y' = x^2y - 1, y(0) = 1$*, using Taylor Series method.*

Solution::

Given $y' = x^2y - 1, y(0) = 1, y(0.2) = ?$

Here, $x_0 = 0, y_0 = 1$

Differentiating successively, we get

$$y' = x^2y - 1$$

$$y'' = 2xy + x^2y'$$

$$y''' = 2y + 4xy' + x^2y''$$

$$y^{IV} = 6y' + 6xy'' + x^2y'''$$

Now substituting, we get

$$y_0' = x_0^2 y_0 - 1 = -1$$
$$y_0'' = 2x_0 y_0 + x_0^2 y_0' = 0$$
$$y_0''' = 2y_0 + 4x_0 y_0' + x_0^2 y_0'' = 2$$
$$y_0^{IV} = 6y_0' + 6x_0 y_0'' + x_0^2 y_0''' = -6$$

Putting these values in Taylor's Series, we have

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{IV} + \dots$$

$$Y(0.1) = 1 + 0.1\cdot(-1) + \frac{(0.1)^2}{2!}\cdot(0) + \frac{(0.1)^3}{3!}\cdot(2) + \frac{(0.1)^4}{4!}\cdot(-6) + \dots$$
$$= 1 - 0.1 + 0 + 0.00033 + 0 + \dots$$
$$= 0.90031$$
$$\therefore y(0.1) = 0.90031$$

Problem 5.2.2. *Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x, y(0) = 0$. Compare the numerical solution obtained with the exact solution.*

Solution: We have $y' = 2y + 3e^x$; $x_0 = 0, y_0 = 0$

Hence $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0, y = 0$ we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{kx} = 2y''' + 3e^x, \qquad y^{it}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots$$
$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \cdots$$
$$= 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{15}{8}x^4 + \cdots$$

Hence $y(0.2) = 3(0.2)+4.5(0.2)^2+3.5(0.2)^3+1.875(0.2)^4+\cdots = 0.8110$

Exact solution : Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x

Its I.F. being e_{-2x}, the solution is

$$ye^{-2x} = \int 3e^x e^{-2x}dx + c = -3e^{-x} + c \text{ or } y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$,

$$\therefore c = 3.$$

Thus the exact solution is $y = 3\left(e^{2x} - e^x\right)$ When $x = 0.2, y = 3\left(e^{0.4} - e^{0.2}\right) = 0.8112$ Comparing (i) and (ii), it is clear that (i) approximates to the exact value up to three decimal places.

Problem 5.2.3. *Solve $y' = x + y, y(0) = 1$ by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.*

Solution:: Differentiating successively, we get

$$y' = x + y \quad y'(0) = 1 \quad [\because y(0) = 1]$$
$$y'' = 1 + y' \quad y''(0) = 2$$
$$y''' = y'' \quad y'''(0) = 2$$
$$y''' = y''' \quad y'''(0) = 2, \text{ etc.}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \cdots$$

Here $x_0 = 0, y_0 = 1$

$$\therefore \quad y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(4)\cdots$$

Thus

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}\cdots$$
$$= 1.1103$$

and

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \cdots$$
$$= 1.2427$$

5.3 Modified Euler's Method :

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \;\; y(x_0) = y_0,$$

To find y_1 i.e. $y(x_1)$ where $x_1 = x_0 + h$, we proceed as follows.

Step 1: We first obtain an initial approximation for $y_1 = y(x_1)$ by applying Euler's formula ${y_1}^{(0)} = y_0 + hf(x_0, y_0)$

Step 2: Since the accuracy is very poor in Euler's formula, the formula is modified and is given by,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$
$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$
$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

In general

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right]$$

and called it as modified Euler's formulae.

Problem 5.3.1. *Given that* $dy/dx = \log_{10}(x + y), y(0) = 1$*, find* $y(0.1)$ *and* $y(0.2)$ *using modified Euler's method.*

Solution: :

Given differential equation is,

$$\frac{dy}{dx} = \log_{10}(x + y) = f(x, y)$$

with initial condition, $x_0 = 0, y_0 = 1$ and $x_1 = 0.1$

Taking, $h = 0.1$, such that

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

and $x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$ **First Stage:**

By Using Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$
$$= y_0 + h\left[\log_{10}(x_0 + y_0)\right]$$
$$= 1 + 0.1\left[\log_{10}(0 + 1)\right] = 1 \quad \text{at } x_1 = x_0 + h = 0.1$$

Applying Euler's modified formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(n)}\right)\right]$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right] = 1.0020$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right] = 1.0021 \text{ and}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(2)}\right)\right] = 1.0021$$

clearly, $y_1^{(3)} = y_1^{(2)} = 1.0021 = y_1($ Improved value $)$ at $x_1 = 0.1$

Second Stage: Taking $y_1 = 1.0021$ and $x_1 = 0.1$

By Using Euler's formula,

$$y_2 = y_1 + hf\left(x_1, y_1\right)$$

$$\Rightarrow \quad y_2 = y_1 + h\left[\log_{10}\left(x_1 + y_1\right)\right]$$

$$= 1.0021 + 0.1\left[\log_{10}(0.1 + 1.0021)\right]$$

$$= 1.0063, \text{ at } x_2 = 0.2$$

Applying Euler's modified formula,

$$y_2^{(n+1)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(n)}\right)\right]$$

$${y_2}^{(1)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2\right)\right]$$

$$= 1.0124$$

$${y_2}^{(2)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, {y_2}^{(1)}\right)\right]$$

$$= 1.0125$$

And

$${y_2}^{(3)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(2)}\right)\right] = 1.0125$$

Clearly,

$${y_2}^{(3)} = {y_2}^{(2)} = 1.0125 = y_2$$

at $x_2 = 0.2$

Hence the required value of y at $x = 0.2$ is 1.0125 .

Problem 5.3.2. *Using modified Euler's method find* y *at* $\mathrm{x} = 0.2$ *given* $\frac{dy}{dx} = 3x + \frac{y}{2}$ *with* $\mathrm{y}(0) = 1$ *taking* $\mathrm{h} = 0.1$. *Perform two iteration at each step.*

Solution: : Given differential equation is,

$$\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y)$$

with initial condition, $x_0 = 0, y_0 = 1$. Let $x_1 = 0.1$

Taking, $h = 0.1$, such that

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

and $x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$

Stage 1: By Using Euler's formula

$$\begin{aligned} y_1^{(0)} &= y_0 + hf\left(x_0, y_0\right) \\ &= y_0 + h\left[3x_0 + \frac{y_0}{2}\right] \\ &= 1.05 \end{aligned}$$

Applying Euler's modified formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(n)}\right)\right]$$

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right] \\ &= y_0 + \frac{h}{2}\left\{\left[3x_0 + \frac{y_0}{2}\right] + \left[3x_1 + \frac{y_1^{(0)}}{2}\right]\right\} \\ &= 1.0020 \\ y_1^{(2)} &= y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right] \\ &= y_0 + \frac{h}{2}\left\{\left[3x_0 + \frac{y_0}{2}\right] + \left[3x_1 + \frac{y_1^{(1)}}{2}\right]\right\} \\ &= 1.0667 \end{aligned}$$

Thus $y_1 = y(0.1) = 1.0667$

Stage 2: By Using Euler's formula

$$\begin{aligned} y_2^{(0)} &= y_1 + hf\left(x_1, y_1\right) \\ &= y_1 + h\left[3x_1 + \frac{y_1}{2}\right] \\ &= 1.1 \end{aligned}$$

Applying Euler's modified formula,

$$y_2^{(n+1)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(n)}\right)\right]$$

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_1, y_1^{(0)}\right)\right] \\ &= y_1 + \frac{h}{2}\left\{\left[3x_1 + \frac{y_1}{2}\right] + \left[3x_2 + \frac{y_2^{(0)}}{2}\right]\right\} \\ &= 1.1671 \\ y_2^{(2)} &= y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(1)}\right)\right] \\ &= y_1 + \frac{h}{2}\left\{\left[3x_1 + \frac{y_1}{2}\right] + \left[3x_2 + \frac{y_2^{(1)}}{2}\right]\right\} \\ &= 1.1675 \end{aligned}$$

Thus $y_2 = y(0.2) = 1.1675$

Problem 5.3.3. *Using Euler's modified method, obtain a solution of the equation*

$$dy/dx = x + |\sqrt{y}|$$

with initial conditions $y = 1$ *at* $x = 0$*, for the range* $0x0.6$ *in steps of 0.2 .*

Solution:: The various calculations in this method are arranged as follows:

x	$x+\lvert\sqrt{y}\rvert=y'$	Mean slope	Old $y+0.2$ (mean slope) = new y
0.0	$0+1=1$	$-$	$1+0.2(1)=1.2$
0.2	$0.2+\lvert\sqrt{(1.2)}\rvert$ $=1.2954$	$\frac{1}{2}(1+1.2954)$ $=1.1477$	$1+0.2(1.1477)=1.2295$
0.2	$0.2+\lvert\sqrt{(1.2295)}\rvert$ $=1.3088$	$\frac{1}{2}(1+1.3088)$ $=1.1544$	$1+0.2(1.1544)=1.2309$
0.2	$0.2+\lvert\sqrt{(1.2309)}\rvert$ $=1.3094$	$\frac{1}{2}(1+1.3094)$ $=1.1547$	$1+0.2(1.1547)=1.2309$
0.2	1.3094	$-$	$1.2309+0.2(1.3094)=1.4927$
0.4	$0.4+\lvert\sqrt{(1.4927)}\rvert$ $=1.6218$	$\frac{1}{2}(1.3094+1.6218)$ $=1.4654$	$1.2309+0.2(1.4654)=1.5240$
0.4	$0.4+\lvert\sqrt{(1.524)}\rvert$ $=1.6345$	$\frac{1}{2}(1.3094+1.6345)$ $=1.4718$	$1.2309+0.2(1.4718)=1.5253$
0.4	$0.4+\lvert\sqrt{(1.5253)}\rvert$ $=1.6350$	$\frac{1}{2}(1.3094+1.6350)$ $=1.4721$	$1.2309+0.2(1.4721)=1.5253$
0.4	1.6350	$-$	$1.5253+0.2(1.635)=1.8523$
0.6	$0.6+\lvert\sqrt{(1.8523)}\rvert$ $=1.9610$	$\frac{1}{2}(1.635+1.961)$ $=1.798$	$1.5253+0.2(1.798)=1.8849$
0.6	$0.6+\lvert\sqrt{(1.8849)}\rvert$ $=1.9729$	$\frac{1}{2}(1.635+1.9729)$ $=1.8040$	$1.5253+0.2(1.804)=1.8861$
0.6	$0.6+\lvert\sqrt{(1.8861)}\rvert$ $=1.9734$	$\frac{1}{2}(1.635+1.9734)$ $=1.8042$	$1.5253+0.2(1.8042)=1.8861$

Hence $y(0.6)=1.8861$ approximately.

Problem 5.3.4. *Using modified Euler's method find* $y(0.1)$ *correct to four decimal places solving the equation* $\frac{dy}{dx}=x-y^2, y(0)=1$ *taking* $h=0.1$.

Try this Yourself !

Answers: $y_1^{(0)} = 0.9, y_1^{(1)} = 0.9145, y_1^{(2)} = 0.9132, y_1^{(3)} = 0.9133,$
Thus $\mathrm{y}(0.1) = 0.9133$

5.4 R.K. Method of order 4:

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y\left(x_0\right) = y_0$$

We need to find $y\left(x_1\right) = y_1$ With the help of formula

$$y_1 = y_0 + k$$

where

$$\mathrm{k} = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

and

$$\begin{aligned}
k_1 &= hf\left(x_0, y_0\right) \\
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
k_4 &= hf\left(x_0 + h, y_0 + k_3\right)
\end{aligned}$$

Problem 5.4.1. *Given* $\frac{dy}{dx} = 3x + \frac{y}{2}$ $\;y(0) = 1$ *compute* $y(0.2)$ *by taking* $h = 0.2$ *using Runge-kutta method of fourth order.*

Solution:: By data $f(x, y) = 3x + \frac{y}{2}, x_0 = 0, y_0 = 1, h = 0.2$
We shall first compute k_1, k_2, k_3, k_4

$$\begin{aligned}
k_1 &= hf\left(x_0, y_0\right) \\
&= (0.2)f(0, 1) \\
&= (0.2)\left[(3)(0) + \frac{1}{2}\right] \\
&= 0.1
\end{aligned}$$

$$\begin{aligned} k_2 &= hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) \\ &= (0.2)f\left[0+\frac{0.2}{2}, 1+\frac{0.1}{2}\right] \\ &= (0.2)f(0.1, 1.05) \\ &= (0.2)\left[3(0.1)+\frac{1.05}{2}\right] \\ &= 0.165 \end{aligned}$$

$$\begin{aligned} k_3 &= hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) \\ &= (0.2)f\left[0+\frac{0.2}{2}, 1+\frac{0.165}{2}\right] \\ &= (0.2)f(0.1, 1.0825) \\ &= (0.2)\left[3(0.1)+\frac{1.0825}{2}\right] \\ &= 0.16825 \end{aligned}$$

$$\begin{aligned} k_4 =& hf\left(x_0+h, y_0+k_3\right) = (0.2)f\left[0+\frac{0.2}{2}, 1+\frac{0.16825}{2}\right] \\ =& (0.2)f(0.2, 1.16825) = (0.2)\left[3(0.2)+\frac{1.16825}{2}\right] \\ =& 0.236825 \end{aligned}$$

We have,

$$y\left(x_0+h\right) = y_0+\frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

Therefore,

$$y(0.2) = 1+\frac{1}{6}[0.1+2(0.165+2(0.16825)+0.236825)] = 1.1672208$$

Problem 5.4.2. *Using the Runge-Kutta method of fourth order, solve* $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ *with* $y(0) = 1$ *at* $x = 0.2, 0.4$.

Solution:: We have $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

To find $y(0.2)$ Hence

$$x_0 = 0, y_0 = 1, h = 0.2$$

$$k_1 = hf(x_0, y_0) = 0.2f(0,1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196.$

To find $y(0.4)$:

Here $x_1 = 0.2, y_1 = 1.196, h = 0.2.$

$$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

Problem 5.4.3. *Apply the Runge-Kutta method to find the approximate value of* y *for* $x = 0.2$*, in steps of 0.1 , if* $dy/dx = x + y^2, y = 1$ *where* $x = 0.$

Solution:: Given $f(x, y) = x + y^2$. Here we take $h = 0.1$ and carry out the calculations in two steps. Step I. $x0 = 0, y0 = 1, h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.1f(0,1) = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1f(0.05, 1.1) = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1f(0.05, 1.1152) = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1168) = 0.1347$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\therefore \quad = \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165$
Step II.

$$x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$$

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1165) = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1f(0.15, 1.194) = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1f(0.2, 1.1576) = 0.1823$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$

Problem 5.4.4. *Using the Runge-Kutta method of order 4 , find y for* $x = 0.1, 0.2$ *given that* $dy/dx = xy + y^2, y(0) = 1$.

Solution:: We have $f(x, y) = xy + y^2$.

To find $y(0.1)$:

Here $x_0 = 0, y_0 = 1, h = 0.1$.

$$\therefore \quad k_1 = hf(x_0, y_0) = (0.1) \times f(0, 1) = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) \times f(0.05, 1.05) = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) \times f(0.05, 1.0577) = 0.1172$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = (0.1) \times f(0.1, 1.1172) = 0.13598$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$

To find $y(0.2)$: Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1$

$$k_1 = hf\left(x_1, y_1\right) = (0.1) \times f(0.1, 1.1169) = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) \times f(0.15, 1.1848) = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) \times f(0.15, 1.1959) = 0.1609$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right) = (0.1) \times f(0.2, 1.2778) = 0.1888$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773.$

5.5 Predictor–Corrector methods

To solve a differential equation over an interval (x_n, x_{n+1}), using previous single-step methods, only the values of y at the beginning of interval is required. However, in the following methods, four prior values are needed for finding the value of y_{n+1} at a given value of x. Also the solution at y_{n+1} is obtained in two stages. This method of refining an initially crude estimate of y_{n+1} by means of a more accurate formula is known as **Predictor–Corrector method**. A Predictor formula is used to predict the value of y_{n+1} and then a Corrector Formula is applied to calculate a still better approximation of y_{n+1}. Now we study one such method namely **Milne's method**.

5.6 Milne's method

Milne's predictor formula :

$$y_4^{(p)} = y_0 \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

Milne's corrector formula :

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

Problem 5.6.1. *Given that* $dy/dx = x - y^2$ *and the data* $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. *Compute y at* $x = 0.8$ *by applying Milne's method*

Solution:: Given, $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary conditions $y = 0$ at $x = 0$.

$$\therefore \quad x_0 = 0.0, \quad y_0 = 0.0000, \quad f_0 = 0.0000$$
$$x_1 = 0.2, \quad y_1 = 0.020, \quad f_1 = 0.1996$$

and

$$x_2 = 0.4, \quad y_2 = 0.0795, \quad f_2 = 0.3937$$
$$x_3 = 0.6, \quad y_3 = 0.1762, \quad f_3 = 0.5689$$

Using the predictor, $y_4^{(p)} = y_0 \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$x = 0.8, \quad y_4^{(p)} = 0.3049, \quad f_4 = 0.7070$

and the corrector,

$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$, yields

$y_4^{(c)} = 0.3046,$

Problem 5.6.2. *Using Milne's method find* $y(4.5)$ *given* $5xy' + y^2 - 2 = 0$ *given* $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$; $y(4.4) = 1.0187$.

Solution:: We have $y' = \left(2 - y^2\right)/5x = f(x, y)$

Then the starting values of the Milne's method are

$$\begin{aligned}
&x_0 = 0, \quad y_0 = 1, \quad f_0 = \tfrac{2-12}{5\times 4} = 0.05\\
&x_1 = 4.1, \quad y_1 = 1.0049, \quad f_1 = 0.0485\\
&x_2 = 4.2, \quad y_2 = 1.0097, \quad f_2 = 0.0467\\
&x_3 = 4.3, \quad y_3 = 1.0143, \quad f_3 = 0.0452\\
&x_4 = 4.4, \quad y_4 = 1.0187, \quad f_4 = 0.0437
\end{aligned}$$

Since y_5 is required, we use the predictor

$$y_5^{(p)} = y_1 + \tfrac{4h}{3}\left(2f_2 - f_3 + 2f_4'\right) \qquad \langle h$$

$$x = 4.5, y_5^{(p)} = 1.0049 + \tfrac{4(0.1)}{3}(2 \times 2.0467 - 0.0452 + 2 \times 0.0437) = 1.023$$

$$f_5 = \frac{2 - y_5^2}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424$$

Now using the corrector $y_5^{(c)} = y_3 + \frac{h}{3}\left(f_3 + 4f_4 + f_5\right)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}(0.0452 + 4 \times 0.0437 + 0.0424) = 1.023$$

Hence $y(4.5) = 1.023$ Given that $dy/dx = xy + y^2, y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$. Find y at $x = 0.4$ using Milne's method. **Solution:**: We have $\quad f(x, y) = xy + y^2$.

Now the starting values for the Milne's method are:

x	y	$f = y'$
$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the predictor

$$y_4^{(p)} = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$x_4 = 0.4 \quad y_4^{(p)} = 1.8344 \quad f_4 = 4.0988$$

and the corrector,

$$\begin{aligned}
y_4^{(c)} &= y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)\\
y_4^{(c)} &= 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]\\
&= 1.8397 \quad f_4 = 4.1159.
\end{aligned}$$

Again using the corrector,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159]$$
$$= 1.8391 \quad f_4 = 4.1182$$

Again using the corrector,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$
$$= 1.8392 \text{ which is same as (i)}$$

Hence $y(0.4) = 1.8392$.

5.7 Question Bank : Module 5 - Numerical Methods II

5.7.1 Question Bank :Taylor's series method

1. Use Taylor's series method to find y at $x = 0.1, 0.2, 0.3$ considering terms up to the third degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$ [VTU-Dec 2018, Dec 2012]

2. From Taylor's series method, find $y(0.1)$ considering up to fourth degree term if $y(x)$ satisfies the equation $\frac{dy}{dx} = x - y^2, y(0) = 1$ [VTU-Jan 2015]

3. Employ Taylor's series method to find y at $x = 0.1$ and 0.2 correct to four places of decimal in step size of 0.1 given the linear differential equation $\frac{dy}{dx} - 2y = 3e^x$ whose solution passes through origin. Also find $y(0.1)$ and $y(0.2)$ by analytical method. [VTU-Jan 2018, Jan 2014,July 2013]

4. Using Taylor's series method for $y' = \sqrt{x^2 + y}$, $y(0) = 0.8$, find $y(0.1)$. consider up to third order derivative terms. [VTU-July 2017]

5. Applying Taylor's series method, find y at $x = 0.1$. Given $\frac{dy}{dx} = x + y^2, y(0) = 1$. [VTU-Jan 2014]

6. Using Taylor's series method solve $\frac{dy}{dx} = x^2y + 1, y(0) = 0$. Find the third order solution at $x = 0.4$ [VTU-June 2012]

7. Using Taylor's series method find $y(0.1)$, Given $\frac{dy}{dx} = x^2y - 1, y(0) = 1$ [VTU- Jan 2021, July 2017, June 2012, July 2011]

8. Use Taylor's series method to solve $\frac{dy}{dx} = x^2y - 1, y(0) = 1$ at $x = 0.2$. Consider upto 4th degree terms. [VTU Model 2022, July 2017, july 2016, Jan 2016, Dec 2012]

9. Find $y(0.1)$ correct to 6 decimal places by using Taylor's series method. Given $dy = (xy + 1)dx, y(0) = 1.1$ [VTU-Jun 2010]

10. Use Taylors series method to find $y(4.1)$. Given that $(x^2 + y)y' = 1$ and $y(4) = 4$ [VTU Jan 2018]

11. Use Taylors series method to find $y(0.1)$. Given that $y' + y + 2x = 0$ and $y(0) = -1$. Consider up to third order derivative term. [VTU July 2019]

12. Use Taylors series method to find $y(0.1)$ from $y' = 3x + y^2$ and $y(0) = 1$. Consider up to fourth derivative term. [VTU Model 2022, July 2019]

13. Use Taylors series method to find $y(0.1)$ and $y(0.2)$. Given $y' = x + y, y(0) = 1$ (VTU Sept 2020)

14. Use Taylors series method to find $y(1.1)$ from $y' = e^x - y, y(0) = 2$ (VTU Jan 2021, Sept 2020)

15. Use Taylors series method to find $y(0.2)$ taking h=0.1. Given $y' = e^x - y^2, y(0) = 1$ (VTU Model 2022)

5.7.2 Question Bank :Modified Euler's method

1. Using modified Euler's method find $y(20.2)$ and $y(20.4)$ given that $\frac{dy}{dx} = log_{10}\left(\frac{x}{y}\right)$ With $y(20) = 5$ taking $h = 0.2$. [VTU-July 2017]

2. Apply modified Euler's method to solve the following initial value problems by considering the accuracy up to two approximations in every step $\frac{dy}{dx} = x +$

$|\sqrt{y}|$ in the range $0 \leq x \leq 0.4$ by taking $h = 0.2$ given that $y = 1$ at $x = 0$ initially. [VTU-Jan 2021, Dec 2018, July 2017]

3. Given $\frac{dy}{dx} = \frac{1}{1+x^2} - 2y^2, y(0) = 0$. Find $y(0.5)$, by taking $h = 0.25$ using Euler's modified method. [VTU-July 2017, Dec 2011]

4. Solve by using modified Euler's method to obtain $y(1.2)$. Given $y' = \frac{y+x}{y-x}$, $y(1) = 2$. [VTU-July 2015]

5. Solve by using modified Euler's method to obtain y at x=0.2 taking h=0.1. Given $y' = \frac{y-x}{y+x}$, $y(0) = 1$. Carryout 3 iterations. [VTU- Model 2022, Dec 2018, Dec 2012]

6. Solve by using modified Euler's method to find $y(0.4)$ by taking $h = 0.2$. Given
$\frac{dy}{dx} = log(x + y), y(0) = 2$ [VTU-Jan 2014, 2015]

7. Determine the value of y when $x = 0.1$ given that $y(0) = 1$ and $y'' = x^2 + y^2$, using modified Euler's method. Take $h = 0.05$ [VTU-Jan 2014]

8. Given $\frac{dy}{dx} = x + y, y(0) = 1$. compute y(0.2) by using modified Euler's method. [VTU-Jan 2013]

9. Use modified Euler's method to find y at x=0.1.Given $\frac{dy}{dx} = 3x + \frac{y}{2}, y(0) = 1$, $h = 0.1$ [VTU Sept 2020, Jan 2018]

10. $\frac{dy}{dx} = 1 + \frac{y}{x}$ with $y(1) = 2$. Find the value of y at x=1.2 using Eulers Modified method. (VTU - Dec 2018)

11. Find $y(0.2)$ by using modified Euler's method, given that given that $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Take $h = 0.1$ and carry out two modifications at each step. [VTU-July 2019]

12. Use modified Euler's method to find y at $x = 0.4$. Given $\frac{dy}{dx} = x + \sin y, y(0) = 1, h = 0.2$ (VTU Jan 2020)

13. Use modified Euler's method to find y at $x = 1.2$. Given $\frac{dy}{dx} = 1 + \frac{y}{2}, y(1) = 2, h = 0.2$ (VTU Sept. 2020)

14. Use modified Euler's method to find y at $x = 0.1$. Given $\frac{dy}{dx} = x^2 + y, y(0) = 1, h = 0.05$ (VTU Model 2022, Jan 2021)

5.7.3 Question Bank :Runge-kutta method

1. Apply Runge-kutta method of order 4 to find an approximate value of y for $x = 0.2$ in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$ [VTU-June, July 2014]

2. Use fourth order Runge-kutta method to solve $(x + y)\frac{dy}{dx} = 1, y(0.4) = 1$ at $x = 0.5$ correct to four decimal places. [VTU-July 2013]

3. Use fourth order Runge-kutta method to find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$ taking $h = 0.2$. [VTU-Dec 2018, Jan 2018, Jan 2014, Dec 2012, Dec 2011]

4. Solve $(y^2 - x^2)dx = (y^2 + x^2)dy$ for $x = 0.1$. Given that $y = 1$ at $x = 0$ initially, by applying Runge-kutta method of order 4. [VTU- Model 2022, July 2017, Dec 2012, June 2012, July 2011]

5. Use fourth order Runge-kutta method to solve $\frac{dy}{dx} = x + y$ at $x = 0.2$ given that $y(0) = 1$ [VTU-Jan 2013]

6. Use fourth order Runge-kutta method to solve $\frac{dy}{dx} = -xy^2, y(0) = 2$.Compute $y(0.2)$by taking $h = 0.1$ [VTU-June 2012]

7. Use fourth order Runge-kutta method to solve the following initial value problem
$\frac{dy}{dx} = 3x + \frac{y}{2}, y(0) = 1$ .compute $y(0.2)$ with $h = 0.1$ [VTU-Model 2022, Jan 2021, Dec 2015, Jan 2016, July 2016]

8. Given $\frac{dy}{dx} = x + y, y(0) = 1.$ compute y(0.2) by using fourth order Runge-kutta method. (take h=0.2) [VTU-Jan 2013]

9. Use fourth order Runge-kutta method to solve the following initial value problem
$10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ .compute $y(0.2)$ with $h = 0.1$ [VTU Dec 2010]

10. Use Runge-kutta method to find $y(0.1)$ from $\frac{dy}{dx} = x^2 + y$ given that $y(0) = -1$ [VTU- July 2019]

11. Use fourth order Runge-kutta method to solve the following initial value problem
$\frac{dy}{dx} + y + xy^2 = 0, y(0) = 1$ Find y at $x = 0.1$ (Jan 2020)

12. Use fourth order Runge-kutta method to solve the following initial value problem
$\frac{dy}{dx} = 3e^x + 2y, y(0) = 1$ Find y at $x = 0.1, h = 0.1$ (VTU : Model 2022, Sept 2020)

5.7.4 Question Bank :Milne's Method

1. Applying Milne's Predictor - Corrector method, to find y(1.4), from $\frac{dy}{dx} = x^2+\frac{y}{2}$, given that $y(1) = 2, y(1.1) = 2.2156, y(1.2) = 2.4549, y(1.3) = 2.7514$ (VTU - Model 2022)

2. Applying Milne's Predictor-Corrector method, find y(0.8), from $\frac{dy}{dx} = x^3+y$, given that $y(0) = 2, y(0.2) = 2.073, y(0.4) = 2.452, y(0.6) = 3.023$ (VTU - Model 2022)

3. Using Milne's Predictor-Corrector method, find $y(4.5)$, given $\frac{dy}{dx} = \frac{2-y^2}{5x}$ and $y(4.1) = 1.0049, \quad y(4.2) = 1.0097$ $y(4.3) = 1.0143, \quad y(4.4) = 1.0187$ (VTU - Model 2022)

4. Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762$. Compute y at $x = 0.8$ by applying Milne's method. [VTU-Jan 2021, Sept 2020, July 2015, Jan 2016, June 2010]

5. The following table gives the solution of $5xy' + y^2 - 2 = 0$. find the value of y at $x = 4.5$ using Milne's predictor and corrector formulae. Use the corrector formula twice.

X	4	4.1	4.2	4.3	4.4
Y	1	1.0049	1.0097	1.0143	1.0187

[VTU-July 2017]

6. Given $\frac{dy}{dx} = xy + y^2$; y(0)=1 , y(0.1)=1.1169 , y(0.2) =1.2773, y(0.3)=1.5049 find y(0.4) correct to three decimal places using Milne's predictor corrector method . Apply the corrector formula twice. [VTU Jan 2014, Dec 2012, June 2012]

7. If $\frac{dy}{dx} = 2e^x - y, y(0) = 2, y(0.1) = 2.010, y(0.20 = 2.040, y(0.3) = 2.090$ find $y(0.4)$ correct to 4 decimal places by Milnes predictor corrector method. Apply the corrector formula twice. [VTU Jan 2018, July 2013]

8. Given $2\frac{dy}{dx} = (1 + x^2)y^2$ and $y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12, y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's method. [VTU- Model 2022, Dec 2011]

9. Given $\frac{dy}{dx} = \frac{x+y}{2}$, Given that $y(0) = 2, y(0.5) = 2.636, y(1) = 3.595$ and $y(1.5) = 4.968$ Find the value of y at $x = 2$ using Milnes Predictor and Corrector Formulae.

10. Given $\frac{dy}{dx} = \frac{1}{x+y}$, Given that $y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755$ and $y(0.6) = 2.2493$. Find the value of y at $x = 0.8$ using Milnes Predictor and Corrector Formulae. [VTU-July 2019]

11. Apply Milne's predictor-corrector formulae to compute y(0.3) given, $\frac{dy}{dx} = x + y^2$ with

x	0.0	0.1	0.2	0.3
y	1.0000	1.1000	1.2310	1.4020

(VTU Model 2019)