AJIET

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Mathematics -I for CSE Stream - Lecture Notes

Subject Code : BMATS101

Module 1- Calculus

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Module - 1 Calculus

Syllabus : Introduction to polar coordinates and curvature relating to Computer Science and engineering:

Polar coordinates, Polar curves, angle between the radius vector and tangent, angle between two curves. Pedal equations. Curvature and Radius of curvature - Cartesian, Parametric, Polar and Pedal forms. Problems.

Prerequisites

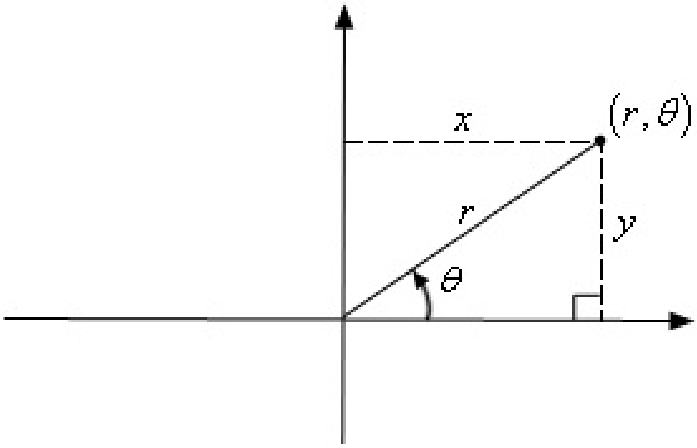
It will be helpful if you can recall the formulas of differentiation, trignometric functions and allied angles.

Notations :

The first order derivative of y $\implies$ y' or y_1 or $\frac{dy}{dx}$ or $D(y)$

The Second order derivative of y $\implies$ y'' or y_2 or $\frac{d^2y}{dx^2}$ or $D^2(y)$

Polar curves :



We are familiar with Cartesian coordinate system for specifying a point in the xy - plane. Another useful system for similar purpose is Polar coordinate system, in which each point P on a plane is determined by a distance r from a fixed point O that is called the **pole** (or origin) and an angle θ measured from x-axis. The point P is represented by the ordered pair (r, θ) where r and θ are called **polar coordinates**.

If the equation of a curve is specified in terms of r and θ, then the curve is referred to as polar curve. The transformation from Cartesian to polar is given by

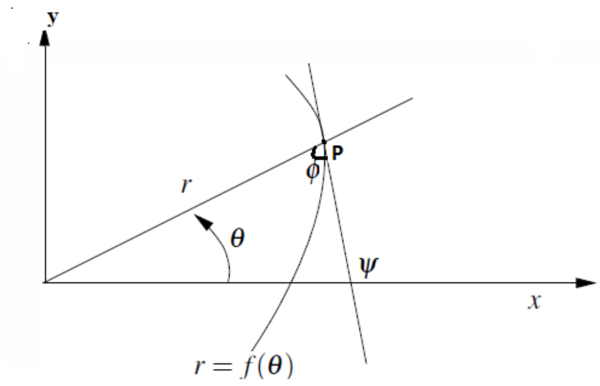
$$r = \sqrt{x^2 + y^2} \qquad \theta = tan^{-1}\left(\frac{y}{x}\right)$$

The transformation from Cartesian to polar is given by

$$x = rcos\theta \qquad y = rsin\theta$$

1.1 Angle between the radius vector and the tangent

Let Φ be the angle between the radius vector OP and the tangent AB at the point 'P' on the polar curve $r = f(\theta)$.



Let Ψ be the angle made by the tangent with positive direction of $x - axis$.

From the figure,

$$\Psi = \theta + \Phi$$

$$tan\Psi = tan(\theta + \Phi) = \frac{tan\theta + tan\Phi}{1 - tan\theta tan\Phi}$$

$$i.e. \frac{dy}{dx} = \frac{tan\theta + tan\Phi}{1 - tan\theta tan\Phi} \cdots\cdots (1)$$

On the other hand, we have $x = rcos\theta$; $y = rsin\theta$ as the relation between polar and Cartesian Coordinates.

differentiating these, w.r.t θ , we obtain

$$\frac{dx}{d\theta} = -rsin\theta + cos\theta\frac{dr}{d\theta}$$

$$\text{and} \frac{dy}{d\theta} = rcos\theta + sin\theta\frac{dr}{d\theta}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{rcos\theta + sin\theta\frac{dr}{d\theta}}{-rsin\theta + cos\theta\frac{dr}{d\theta}}$$

dividing the Numerator and Denominator by $cos\theta\frac{dr}{d\theta}$

$$\frac{dy}{dx} = \frac{\frac{r}{\frac{dr}{d\theta}} + tan\theta}{-\frac{r}{\frac{dr}{d\theta}tan\theta + 1}}$$

$$= \frac{tan\theta + \frac{r}{\frac{dr}{d\theta}}}{1 - \frac{r}{\frac{dr}{d\theta}}tan\theta} \cdots\cdots (2)$$

Comparing equations (1) and (2), we get

$$\boxed{tan\Phi = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{r_1} = r\frac{d\theta}{dr}}$$

Note : We can also find angle Φ by using

$$\boxed{cot\Phi = \frac{1}{r}\frac{dr}{d\theta}}$$

Problem 1.1.1. Find the angle between the radius vector and the tangent to the curve $r = a(1 + cos\theta)$. Also find slope of the tangent at $\theta = \frac{\pi}{3}$

Solution :

$$r = a(1 + cos\theta) \cdots\cdots (1)$$

Applying log on both sides

$$log(r) = log(a) + log(1 + cos\theta)$$

differentiate w.r.to θ,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-sin\theta}{1 + cos\theta}$$

$$cot\Phi = \frac{-2sin\frac{\theta}{2}cos\frac{\theta}{2}}{2cos^2\frac{\theta}{2}}$$

$$i.e.cot\Phi = -tan\frac{\theta}{2} = cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

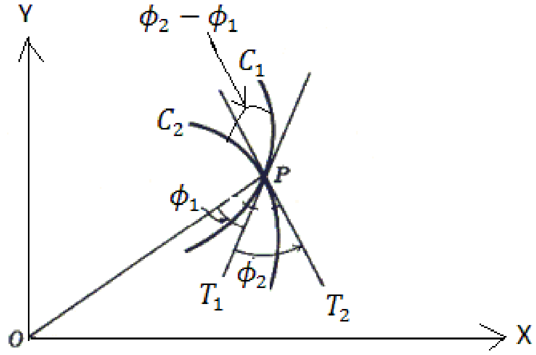
$$\therefore \Phi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\text{At}\theta = \frac{\pi}{3}, \Phi = \frac{\pi}{2} + \frac{\pi}{6}$$

$$\text{Slope} = tan\Psi = tan\,(\theta + \Phi) = tan\pi = 0$$

1.2 Angle of intersection of two polar curves

If Φ_1 and Φ_2 are the angles between the common radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$, then the angle of intersection of the curves is given by $|\phi_1 - \Phi_2|$.



Note 1. Two curves intersect orthogonally if any one of the following conditions are satisfied.

a) $$|\Phi_1 - \Phi_2| = \frac{\pi}{2}$$
or

b) $$tan\Phi_1 tan\Phi_2 = -1$$
or

c) $$cot\Phi_1 cot\Phi_2 = -1$$

Note 2. If the angles Φ_1 and Φ_2 can not be obtained explicitly, then angle of intersection can be found by using the formula

$$|tan(\Phi_1 - \Phi_2)| = \frac{tan\Phi_1 - tan\Phi_2}{1 + tan\Phi_1 tan\Phi_2}$$

Problem 1.2.1. Show that the following curves intersect each other orthogonally. $r = a(1 + cos\theta)$ and $r = b(1 - cos\theta)$ **(VTU 2015, July 2003)**

Solution : Given

$$r = a(1 + cos\theta)$$

Applying log on both sides

$$log(r) = log(a) + log(1 + cos\theta)$$

differentiate w.r.toθ,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-sin\theta}{1 + cos\theta}$$

$$cot\Phi_1 = \frac{-2sin\frac{\theta}{2}cos\frac{\theta}{2}}{2cos^2\frac{\theta}{2}}$$

$$i.e. cot\Phi_1 = -tan\frac{\theta}{2} = cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \Phi_1 = \frac{\pi}{2} + \frac{\theta}{2}$$

Now consider the curve, $r = b(1 - cos\theta)$

Applying log on both sides

$$log(r) = log(b) + log(1 - cos\theta)$$

differentiate w.r.toθ,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{sin\theta}{1 - cos\theta}$$

$$cot\Phi_1 = \frac{2sin\frac{\theta}{2}cos\frac{\theta}{2}}{2sin^2\frac{\theta}{2}}$$
$$i.e.cot\Phi_2 = cot\frac{\theta}{2}$$
$$\therefore \Phi_2 = \frac{\theta}{2}$$

Consider $|\Phi_1 - \Phi_2| = \left|\frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}\right| = \frac{\pi}{2}$

This shows that the given curves intersect each other orthogonally.

Problem 1.2.2. Show that the following curves intersect each other orthogonally. $r = a\theta$ and $r = \frac{a}{\theta}$

Solution :

$$\text{Consider} r = a\theta$$
Applying log on both sides
$$logr = log(a) + log\theta$$
differentiate w.r.toθ
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta}$$
$$i.e.cot\Phi_1 = \frac{1}{\theta}$$
$$\implies tan\Phi_1 = \theta$$
$$\text{Now consider} r = \frac{a}{\theta}$$
Applying log on both sides
$$logr = log(a) - log\theta$$
differentiate w.r.toθ
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-1}{\theta}$$
$$cot\Phi_2 = \frac{-1}{\theta}$$
$$i.e.\ tan\Phi_2 = -\theta$$
$$\text{Using} r = a\theta \text{and} r = \frac{a}{\theta}, \text{ we can write}$$
$$a\theta = \frac{a}{\theta}$$
$$i.e\theta^2 = 1$$
$$\implies \theta = \pm 1$$

Case 1 : If $\theta = 1$, then $tan\Phi_1 = 1$ and $tan\Phi_2 = -1$

$\therefore tan\Phi_1 tan\Phi_2 = -1$

Case 2 : If $\theta = -1$, then $tan\Phi_1 = -1$ and $tan\Phi_2 = 1$

$\therefore tan\Phi_1 tan\Phi_2 = -1$

This shows that the given curves intersect orthogonally.

Problem 1.2.3. Show that the following curves intersect each other orthogonally. $r = a(1 - \sin\theta)$, $r = b(1 + \sin\theta)$

$$r = a(1 - \sin\theta)$$

Solution : Diff. w.r.to θ we get, $r_1 = a(-\cos\theta)$

$$\therefore \tan\phi_1 = \frac{r}{r_1} = -\frac{(1 - \sin\theta)}{\cos\theta}$$

For the curve, $r = b(1 + \sin\theta)$

Diff. w.r.to θ we get, $r_1 = b(\cos\theta)$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{(1+\sin\theta)}{\cos\theta}$

$\tan\phi_1 \cdot \tan\phi_2 = -\frac{(1-\sin^2\theta)}{\cos^2\theta} = -1$

This shows that the given curves intersect orthogonally.

Problem 1.2.4. Show that the following curves intersect each other orthogonally. Show that the curves $r^m = a^m \cos m\theta$ and $r^m = a^m \sin m\theta$

Solution: Equations of curves are $r^m = a^m \cos m\theta$ (1) $r^m = a^m \sin m\theta$ (2) Take natural logarithm of both sides of (1)

$$m \log r = m \log a + \log(\cos m\theta)$$

Differentiate both sides w. r. t θ

$$\frac{m}{r} r_1 = -\frac{m \sin m\theta}{\cos m\theta}$$

$$\therefore \frac{r}{r_1} = -\cot m\theta$$

$$\tan\phi_1 = \frac{r}{r_1} = -\cot m\theta$$

Take natural logarithm of both sides of (2) $m\log r = m\log a + \log(\sin m\theta)$ Differentiate both sides w. r. t θ

$$\frac{m}{r}r_1 = \frac{m\cos m\theta}{\sin m\theta}$$
$$\therefore \frac{r}{r_1} = \tan m\theta$$
$$\tan\phi_2 = \frac{r}{r_1} = \tan m\theta$$

$$\therefore \tan\phi_1 \cdot \tan\phi_2 = -\cot m\theta \times \tan m\theta = -1$$

$\therefore$ curves (1) and (2) cut orthogonally.

Problem 1.2.5. Find the angle between the following curves $r = \sin\theta + \cos\theta \quad , r = 2\sin\theta$

Sol :

$$r = \sin\theta + \cos\theta$$

Diff. w.r.to θ we get,

$$r_1 = \cos\theta - \sin\theta$$
$$\therefore \tan\phi_1 = \frac{r}{r_1} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$
$$= \frac{1+\tan\theta}{1-\tan\theta} \quad \text{(By dividing each term in Numerator and Denominator b y } \cos\theta)$$
$$= \tan\left(\theta + \frac{\pi}{4}\right)$$
$$\Rightarrow \quad \phi_1 = \theta + \frac{\pi}{4}$$

$$r = 2\sin\theta$$

Diff. W.r.to θ we get,

$$r_1 = 2\cos\theta$$
$$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$
$$\Rightarrow \quad \phi_2 = \theta$$

Angle of intersection is

$$|\phi_1 - \phi_2| = |\theta + \frac{\pi}{4} - \theta| = \frac{\pi}{4}$$

Problem 1.2.6. Find the angle between the following curves : $r = alog\theta$ and $r = \frac{a}{log\theta}$ **(VTU Jan**

2015, July 2005, Aug 2001)

Consider the curve $r = alog\theta$ Applying log on both sides

$logr = log(a) + log(log\theta)$

differentiate w.r.to θ, $\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta log\theta}$

$cot\Phi_1 = \frac{1}{\theta log\theta}$

$\therefore tan\Phi_1 = \theta log\theta$

Consider the curve $r = \frac{a}{log\theta}$

Applying log on both sides

$logr = log(a) - log(log\theta)$

differentiate w.r.to θ, $\frac{1}{r}\frac{dr}{d\theta} = \frac{-1}{\theta log\theta}$

$cot\Phi_2 = \frac{-1}{\theta log\theta}$

$\therefore tan\Phi_2 = -\theta log\theta$

Using the given curves $r = alog\theta$ and $r = \frac{a}{log\theta}$, we can write

$alog\theta = \frac{a}{log\theta}$

$(log\theta)^2 = 1$

i.e. $\theta = e$ (assuming positive value)

Consider

$$|tan(\Phi_1 - \Phi_2)| = \frac{tan\Phi_1 - tan\Phi_2}{1 + tan\Phi_1 tan\Phi_2} = \frac{2e}{1 - e^2}$$

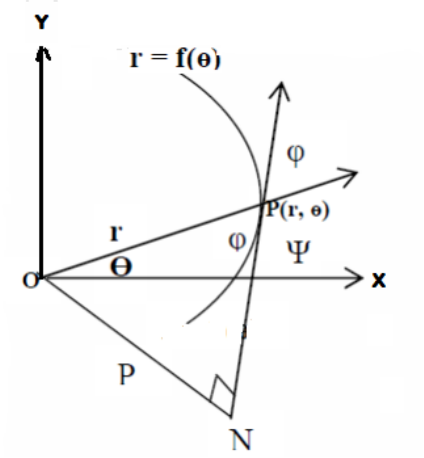
$\therefore |\Phi_1 - \Phi_2| = tan^{-1}\left(\frac{2e}{1-e^2}\right) = 2tan^{-1}e$

The length p of perpendicular from pole to the tangent in a polar curve

The length p of perpendicular from pole to the tangent at a point (r, θ) in a polar curve is given by

$$p = rsin\Phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Proof :



From the right angled triangle OPN,

$$\begin{aligned}
sin\phi &= \frac{ON}{OP}\\
i.e..sin\phi &= \frac{p}{r} \implies \boxed{p = rsin\phi}\\
\text{Consider } \frac{1}{p} &= \frac{1}{rsin\phi} = \frac{1}{r}cosec\phi\\
\therefore \frac{1}{p^2} &= \frac{1}{r^2}cosec^2\phi = \frac{1}{r^2}(1 + cot^2\phi)\\
&= \frac{1}{r^2}\left[1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right]
\end{aligned}$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

1.3 Pedal equations ($p - r$ equations)

For a plane curve $r = f(\theta)$ and a given point $P(r, \theta)$ on this curve, the pedal equation of the curve is a relation between r and p where r is the distance from O(origin) to a point $P(r, \theta)$, and p is the perpendicular distance from O to the tangent to the curve at the point P.

Working rules to find pedal equations :

Let the polar equation of any curve be,

$$f(r, \theta) = 0 \tag{1}$$

Let Φ be the angle between the radius vector and the tangent, then we know that,

$$tan(\Phi) = r.\frac{d\theta}{dr} \quad (2)$$

and perpendicular distance from O to the tangent is given by using the formula

$$p = rsin\Phi \quad (3)$$

or

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad (4)$$

Now , if we eliminate θ between the equations (1) and (3) or (1) and (4) , then we shall get an equation in terms of p and r and thus will be required an equation of the curve .

Problem 1.3.1. Find the pedal equation for the curve $\frac{2a}{r} = 1 - cos\theta$

Solution : Given equation can be written as

$$r = \frac{2a}{1 - cos\theta} = \frac{2a}{2sin^2\frac{\theta}{2}} = acosec^2\frac{\theta}{2} \quad (1)$$

$$\frac{dr}{d\theta} = a2cosec^2\frac{\theta}{2}\left(-cosec\frac{\theta}{2}\right)cot\frac{\theta}{2}.\frac{1}{2}$$

$$= -acosec^2\frac{\theta}{2}cot\frac{\theta}{2}$$

$$tan\Phi = r\frac{d\theta}{dr} = acosec^2\frac{\theta}{2} \times \frac{1}{-acosec^2\frac{\theta}{2}cot\frac{\theta}{2}}$$

$$= tan\frac{\theta}{2} = tan\left(\pi - \frac{\theta}{2}\right)$$

$$\implies \Phi = \left(\pi - \frac{\theta}{2}\right)$$

Now $p = rsin\Phi = rsin\left(\pi - \frac{\theta}{2}\right) = rsin\frac{\theta}{2}$

Squarring $p^2 = r^2sin^2\frac{\theta}{2} \implies sin^2\frac{\theta}{2} = \frac{p^2}{r^2}$

Now (1) can be written as

$r = \frac{2a}{2sin^2\frac{\theta}{2}} = \frac{a}{\frac{p^2}{r^2}} = \frac{ar^2}{p^2}$

$\implies p^2 = ar$ This is the required pedal equation.

Problem 1.3.2. Find the pedal equation of the curve $r^n = a^n \sin n\theta$

Solution : Equation of the curve is $r^n = a^n \sin n\theta$

Take the natural logarithm of both sides

$$n \log r = n \log a + \log(\sin n\theta)$$

Differentiate both sides w.r.t. θ

$$n\,\frac{1}{r} r_1 = \frac{n \cos n\theta}{\sin n\theta}$$
$$\therefore \quad \tan\phi = \frac{r}{r_1} = \frac{\sin n\theta}{\cos n\theta} = \tan n\theta$$

$\therefore \quad \phi = n\theta$

$\therefore \quad p = r\sin\phi = r\sin n\theta = r\frac{r^n}{a^n}$ from (1)

$\therefore \quad pa^n = r^{n+1}$

which is the pedal equation of the curve.

Problem 1.3.3. Find the pedal equation for the polar curve $r^m = a^m(\cos(m\theta) + \sin(m\theta))$.

For this curve in the usual way, taking log on both sides,

$m \log r = \log(a^m) + \log(\cos(m\theta) + \sin(m\theta))$

Diff. w.r.to θ,

$$m\,\frac{1}{r} r_1 = 0 + \frac{1}{\cos(m\theta) + \sin(m\theta)} \cdot (-m\sin m\theta + m\cos m\theta)$$
$$\cot\phi = \frac{(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$
$$\tan\phi = \frac{(\cos m\theta + \sin m\theta)}{(\cos m\theta - \sin m\theta)}$$
$$= \frac{1 + \tan m\theta}{1 - \tan m\theta} \quad \text{(By dividing each term in Numerator and Denominator b y } \cos m\theta\text{)}$$
$$= \tan\left(\frac{\pi}{4} + m\theta\right)$$
$$\Rightarrow \phi = \frac{\pi}{4} + m\theta$$

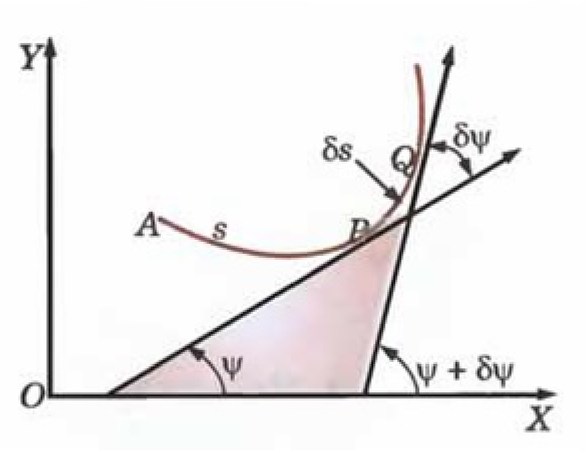
Pedal equation is

$$\begin{aligned} p = r\sin\phi &= r\sin\left(\frac{\pi}{4}+m\theta\right) \\ &= r\left(\sin\left(\frac{\pi}{4}\right)\cos\theta+\cos\left(\frac{\pi}{4}\right)\sin\theta\right) \\ &= \frac{r}{\sqrt{2}}(\cos(m\theta)+\sin(m\theta)) \\ &= \frac{r}{\sqrt{2}}\left(\frac{r^m}{a^m}\right) \quad \text{(from the given equation)} \\ &= \frac{r^{m+1}}{a^m\sqrt{2}}. \end{aligned}$$

Curvature.

The curvature is the concept in geometry that indicates the change in direction of the curve at a certain point. While the radius of curvature gives the radius of the approximate circle that matches the curve at a particular point.

Consider a smooth curve C in XY-plane and let P, Q be any two neighbouring points on it. Let $arc\ AP = s$ and $arc\ PQ = \Delta s$. Let the tangents drawn to the curve at P, Q respectively make angles Ψ and $\Psi + \Delta\Psi$ with X-axis i.e., the angle between the tangents at P and Q is $\Delta\Psi$. While moving from P to Q through a distance Δs, the tangent has turned through the angle $\Delta\Psi$. This is called the bending of the arc PQ. Geometrically, a change in Ψ represents the bending of the curve C and the ratio $\frac{\Delta Psi}{\Delta s}$ represents the ratio of bending of C between the point P & Q and the arc length between them.



Rate of bending of Curve at P is

$$\frac{d\Psi}{ds} = lim_{Q\to P}\frac{\Delta\Psi}{\Delta s}$$

Curvature is a numerical measure of bending of the curve. Or, more simply, it measures the rate of change of direction of the curve, and it is denoted by κ (kappa). Thus

$$\boxed{\kappa = \frac{d\Psi}{ds}}$$

Note : Intuitively the curvature of a plane curve at a point P can be thought of as the curvature of that circle which approximates the curve most closely near that point. The curvature of a circle is directly defined by the reciprocal of the length of its radius. The shorter the radius, the greater the curvature of the arc in the vicinity of any point P on it. The longer the radius, the bigger the circle, and the less the curvature of the arc in the vicinity of any point P on it. For a very large circle the curvature of an arc at some point P approaches that of a straight line and the curvature of a straight line is zero since there exist no bending.

1.4 Radius of Curvature

If $\kappa \neq 0$, then $\frac{1}{\kappa}$ is called the radius of curvature and is denoted by ρ.

ie.

$$\boxed{\rho = \frac{1}{\kappa} = \frac{ds}{d\Psi}}$$

The sign of $\frac{d\Psi}{ds}$ indicates the convexity and concavity of the curve in the neighbourhood of the point.

Radius of curvature in Cartesian form

Suppose $y = f(x)$ is the Cartesian equation of a curve, then we know that

$$\tan\psi = dy/dx = y_1$$

or

$$\psi = \tan^{-1}(y_1)$$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\frac{d(y_1)}{dx}$$
$$= \frac{y_2}{1+y_1^2}$$

$$\text{Curvature } K = \frac{d\psi}{ds}$$
$$= \frac{d\psi}{dx} \times \frac{dx}{ds}$$

From the derivative of arc length, we have

$$\frac{dS}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$
$$i.e.\ \frac{dx}{ds} = \frac{1}{\sqrt{(1+y_1^2)}}$$
$$\therefore K = \frac{d\psi}{dx} \times \frac{dx}{ds}$$
$$= \frac{y_2}{1+y_1^2} \times \frac{1}{\sqrt{(1+y_1^2)}}$$
$$= \frac{y_2}{\left(1+y_1^2\right)^{3/2}}$$
$$\therefore \text{ Radius of curvature } \rho = \frac{1}{K}$$
$$= \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

the expression for radius of curvature is,

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$i.e.\ \boxed{\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}}$$

Alternate formula for radius of curvature :

An alternate formula for radius of curvature is

$$\boxed{\rho = \frac{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}}$$

This formula can be used at a point where $\frac{dy}{dx}$ doesn't exist such as a point on a curve where the tangent line is parallel to the y-axis. (i.e. when $\frac{dy}{dx} = \infty$)

Radius of curvature in Parametric form:

For the curve $x = x(t)$ and $y = y(t)$, where t is the parameter,

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Where $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, \ddot{y} = \frac{d^2y}{dt^2}$.

Note : When parametric form $x = x(t)$ and $y = y(t)$ is given, we can also find the radius of curvature using

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$$

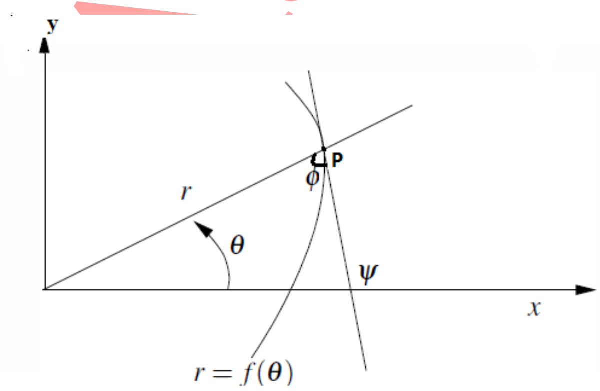
where

$$y_1 = \frac{dy/dt}{dx/dt}$$

and

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}(y_1) \times \frac{dt}{dx}$$

Radius of curvature in polar form



For the curve $r = f(\theta)$ i.e., the curve in polar coordinates

With the usual notations, we have

$$\psi = \theta + \phi$$

Differentiating w.r.t. s,

$$\begin{aligned}\kappa &= \frac{d\psi}{ds} \\ &= \frac{d\theta}{ds} + \frac{d\phi}{ds} \\ &= \frac{d\theta}{ds} + \frac{d\phi}{d\theta}, \frac{d\theta}{ds} \\ &= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \qquad (1)\end{aligned}$$

Also we know that

$$\begin{aligned}\tan\phi &= r\frac{d\theta}{dr} = \frac{r}{r_1} \\ \text{or } \phi &= \tan^{-1}\left(\frac{r}{r_1}\right)\end{aligned}$$

Differentiating w.r.t. θ,

$$\begin{aligned}\frac{d\phi}{d\theta} &= \frac{1}{1+(r/r_1)^2} \times \frac{r_1 \cdot r_1 - rr_2}{r_1^2} \\ &= \frac{r_1^2 - rr_2}{r^2 + r_1^2}\end{aligned} \qquad (2)$$

Also,

$$\frac{ds}{d\theta} = \sqrt{(r^2 + r_1^2)} \qquad (3)$$

Substituting the value from (2) and (3) in (1),

$$\begin{aligned}\kappa &= \frac{1}{\sqrt{r^2 + r_1^2}} + \left(1 + \frac{r_1^2 - r_2}{r^2 + r_1^2}\right) \\ &= \frac{r^2 + r_1^2 - rr_2}{(r^2 + r_1^2)^{3/2}}\end{aligned}$$

$$\therefore \boxed{\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + r_1^2 - rr_2}}$$

where $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$

Radius of Curvature in Pedal form

For the curve in terms of p and r (i.e., the curve in pedal form):

$$\boxed{\rho = r\frac{dr}{dp}}$$

Problem 1.4.1. Find the radius of curvature for $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point where it meets the line $y=x$.

Solution : On the line $y=x$, we have $\sqrt{x}+\sqrt{x}=\sqrt{a}$, i.e. $2\sqrt{x}=\sqrt{a}$ or $x=\frac{a}{4}$

Let

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

differentiating w.r.to$x$,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y_1=0$$

$$i.e. y_1=-\frac{\sqrt{y}}{\sqrt{x}}\cdots\cdots(1)$$

$$\therefore y_1|_{\frac{a}{4},\frac{a}{4}}=-1$$

differentiating(1) $w.r.tox,$

$$y_2=\frac{\sqrt{x}\frac{1}{2\sqrt{y}}y_1-\sqrt{y}\frac{1}{2\sqrt{x}}}{x}$$

$$y_2|_{\frac{a}{4},\frac{a}{4}}=\frac{4}{a}$$

$$\therefore \rho=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}=\frac{(1+(-1)^2)^{\frac{3}{2}}}{\frac{4}{a}}=\frac{a}{\sqrt{2}}$$

Problem 1.4.2. Find ρ at the point $\left(\frac{3a}{2},\frac{3a}{2}\right)$ of the Folium $x^3+y^3=3axy$ (VTU July 2017, July 2016, Model 2014, 2015, 2008)

Solution : Given

$$x^3+y^3=3axy \tag{1}$$

Differentiating (1) w.r.to x, we get

$$x^2+yy_1=a(xy_1+y)=(y^2-ax)y_1=ay-x^2$$

$$i.e. y_1=\frac{ay-x^2}{y^2-ax}\cdots\cdots(2)$$

$$y_1|_{\left(\frac{3a}{2},\frac{3a}{2}\right)}=-1$$

Differentiating (2) w.r.to x, we get

$$y_2=\frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

$$y_2|_{\left(\frac{3a}{2},\frac{3a}{2}\right)}=\frac{-32}{3a} \text{ on simplification}$$

$$\therefore \text{ Radius of Curvature, } \rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+1)^{\frac{3}{2}}}{\frac{-32}{3a}} = \frac{3a}{8\sqrt{2}}$$

Problem 1.4.3. Find the radius of curvature for the curve $x = alog(sect + tant), y = asect$ **(VTU July 2015)**

Solution:

$$\text{Given } x = alog(sect + tant) - - - - - - - - -(1)$$

$$\text{and } y = asect - - - - - - - - -(2)$$

differentiating (1)w.r.to t, we get

$$\frac{dx}{dt} = a\frac{secttant + sec^2t}{sect + tant} = asect$$

differentiating (2) w.r.to t, we get

$$\frac{dy}{dt} = asecttant$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dy}{dt}} = tant - - - - - - - - -(3)$$

differentiating (3) w.r.to x, we get

$$y_2 = sec^2t\frac{dt}{dx} = \frac{1}{a}sect$$

$$\therefore \text{ Radius of Curvature, } \rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+tan^2t)^{\frac{3}{2}}}{\frac{1}{a}sect} = asec^2t$$

Problem 1.4.4. Show that the R.O.C at any point of the cardioid $r = a(1 - cos\theta)$ varies as $\sqrt{r}$. **(VTU 2003)**

Solution : Given that

$$r = a(1 - cos\theta) \qquad (1)$$

differentiating (1) w.r.to θ, we get

$r_1 = asin\theta$ and $r_2 = acos\theta$

$\therefore$ Radius of Curvature ,

$$\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(a^2(1-cos\theta)^2 + a^2sin^2\theta)^{\frac{3}{2}}}{a^2(1-cos\theta)^2 + 2a^2sin^2\theta - a^2(1-cos\theta)cos\theta}$$

$$= \frac{2\sqrt{2a}}{3}\sqrt{r} \text{ on simplification}$$

$\therefore \rho \propto \sqrt{r}$

Problem 1.4.5. Find the radius of the curvature of $r^n = a^n \cos n\theta$

Solution : Given curve is $r^n = a^n \cos n\theta$ Diff. w.r.to θ we get,

$$\begin{aligned}
nr^{n-1}r_1 &= -na^n \sin n\theta \\
n\frac{r^n}{r}\, r_1 &= -na^n \sin n\theta \\
r_1 &= \frac{-na^n \sin n\theta\, r}{a^n \cos n\theta} \quad \text{(from the given equation)} \\
\therefore \quad r_1 &= -r \tan n\theta \\
r_2 &= -r_1 \tan n\theta - nr \sec^2 n\theta \\
&= r \tan^2 n\theta - nr \sec^2 n\theta \\
\rho &= \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \\
&= \frac{r^3 \left(1 + \tan^2 n\theta\right)^{\frac{3}{2}}}{r^2 + 2r^2 \tan^2 n\theta - r\left(r \tan^2 n\theta - nr \sec^2 n\theta\right)} \\
&= \frac{r \sec^3 n\theta}{1 + \tan^2 n\theta + n \sec^2 n\theta} \\
&= \frac{r \sec n\theta}{n+1} \\
&= \frac{ra^n}{(n+1)r^n} = \frac{a^n}{(n+1)r^{n-1}}.
\end{aligned}$$

Problem 1.4.6. Find the radius of the curvature of $r^3 = 2ap^2$

Solution : Here $r^3 = 2ap^2$

Differentiating w.r.t. p, we get

$$3r^2 \cdot \frac{dr}{dp} = 4ap$$
$$\Rightarrow \frac{dr}{dp} = \frac{4ap}{3r^2}$$

Hence,

$$\rho = r \cdot \frac{dr}{dp} = r \cdot \frac{4ap}{3r^2} = \frac{4ap}{3r}$$

$$\text{where } p=\left(\frac{r^3}{2a}\right)^{\frac{1}{2}}$$

$$\rho=\frac{4a\cdot\left(\frac{r^3}{2a}\right)^{\frac{1}{2}}}{3r}$$

$$=\frac{4ar^{\frac{3}{2}}}{3r\sqrt{2a}}=\frac{2\sqrt{2ar}}{3}$$

Problem 1.4.7. Find the radius of the curvature of $p^2=ar$

Solution : Here $p^2=ar$

Differentiating w.r.t. p, we get

$$2p=a\cdot\frac{dr}{dp}$$

$$\Rightarrow\quad\frac{dr}{dp}=\frac{2p}{a}$$

where $p=\sqrt{ar}$. $\rho=r\frac{dr}{dp}=r\cdot\frac{2\cdot\sqrt{ar}}{a}=\frac{2r^{\frac{3}{2}}}{\sqrt{a}}$

Problem 1.4.8. Find the radius of curvature of the curve $y^2=\frac{a^2(a-x)}{x}$ at the point $(a,0)$.

Solution :

$$y^2=\frac{a^2(a-x)}{x}$$

$$y^2x=a^3-a^2x$$

Differentiating w.r.t x

$$2xyy_1+y^2=-a^2\quad\Rightarrow y_1=-\frac{a^2+y^2}{2xy}.$$

at $(a,0)$, $y_1=\infty$

therefore $\rho=\frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}=\frac{1}{x_2}$.

$$x_1=\frac{dx}{dy}=-\frac{2xy}{y^2+a^2}$$

then $x_1=0$ at $(a,0)$

$$\left(y^2+a^2\right)x_1=-2xy.$$

differentiating w.r.t y

$$\left(y^2+a^2\right)x_2+2yx_1=-2x-2yx_1$$

$$\Rightarrow\left(y^2+a^2\right)x_2=-2x-2yx_1-2yx_1$$

Then

$$x_2 = -\frac{2}{a} \text{ at } (a, 0)$$
$$\therefore \rho = \frac{1}{x_2} = -\frac{a}{2}$$

$\therefore$ The radius of curvature of the given curve is $\frac{a}{2}$.

Problem 1.4.9. Find ρ at any point on $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

SOl : Here $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$

Differentiating w.r.t. θ

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \frac{dy}{d\theta} = a\sin\theta$$
$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)}$$
$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$
$$y_1 = \tan\frac{\theta}{2}$$

Again differentiating w.r.t. θ

$$y_2 = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right)$$
$$= \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right) \times \frac{d\theta}{dx}$$
$$= \sec^2\left(\frac{\theta}{2}\right) \times \frac{1}{2} \times \frac{1}{a(1+\cos\theta)}$$
$$= \frac{\sec^2\frac{\theta}{2}}{2a \times 2\cos^2\frac{\theta}{2}}$$
$$y_2 = \frac{1}{4a\cos^4\frac{\theta}{2}}$$

$$\rho = \frac{\left\{1 + y_1^2\right\}^{\frac{3}{2}}}{y_2}$$
$$= \frac{\left\{1 + \tan^2\frac{\theta}{2}\right\}^{\frac{3}{2}}}{\left\{\frac{1}{4a\cos^4\frac{\theta}{2}}\right\}}$$
$$= \left\{\sec^2\left(\frac{\theta}{2}\right)\right\}^{\frac{3}{2}} \times 4a\cos^4\left(\frac{\theta}{2}\right)$$
$$= \frac{1}{\cos^3\left(\frac{\theta}{2}\right)} \times 4a\cos^4\left(\frac{\theta}{2}\right)$$
$$= 4a\cos\left(\frac{\theta}{2}\right).$$

Problem 1.4.10. Find the radius of curvature at any point on the curve $y = a\log\sec(x/a)$.

Sol : Radius of curvature $\rho = \frac{\{1+y_1^2\}^{\frac{3}{2}}}{y_2}$

Here, $y = a \log \sec\left(\frac{x}{a}\right)$

$y_1 = a \times \frac{1}{\sec\left(\frac{x}{a}\right)} \cdot \sec\left(\frac{x}{a}\right) \tan\left(\frac{x}{a}\right) \cdot \frac{1}{a}$

$y_1 = \tan\left(\frac{x}{a}\right)$

$y_2 = \sec^2\left(\frac{x}{a}\right) \cdot \frac{1}{a}$

$$\text{Hence } \rho = \frac{\left\{1+\tan^2\left(\frac{x}{a}\right)\right\}^{\frac{3}{2}}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)} = \frac{\left\{\sec^2\left(\frac{x}{a}\right)\right\}^{3/2}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)} = \frac{a\sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)} = a\sec\left(\frac{x}{a}\right)$$

$\therefore$ Radius of curvature $= a\sec(x/a)$

Question Bank-Module 2

Polar curves - Angle between the radius vector and tangent

1. With usual notations, prove that $tan\phi = r\frac{d\theta}{dr}$ (VTU March 2022, Aug 2021, Jan 2020, Jan 2019, July 2019, June 2018, Model 2018)
2. Find the angle between radius vector and tangent for the following curves.
3. $r = a(1+sin\theta)$ **Ans :** $\Phi = \frac{\pi}{4} + \frac{\theta}{4}$
4. $r^2cos2\theta = a^2$ **Ans :** $\Phi = \frac{\pi}{2} - 2\theta$
5. $\frac{2a}{r} = 1 - cos\theta$ **Ans :** $\Phi = \frac{-\theta}{2}$

Angle of intersection of two polar curves

1. Show that the following curves intersect each other orthogonally. $r = a(1+cos\theta)$ and $r = b(1-cos\theta)$ (VTU Jan 2020, Model 2018, Jan 2018, 2015, July 2003)
2. Show that the following curves intersect each other orthogonally. $r^n = a^ncosn\theta$ and

$r^n = b^n sinn\theta$ (VTU Jan 2019, Jan 2018)

3. Show that $r = 4\sec^2\theta/2$ and $r = 9\,\mathrm{cosec}^2\,\theta/2$ the pair of curves cut orthogonally. (VTU March 2022)

4. Find the angle of intersection of the following curves :

   (i) $r = alog\theta$ and $r = \frac{a}{log\theta}$ (VTU March 2022, Aug 2021, Jan 2018, Jan 2015, July 2005, Aug 2001)

   (ii) $r^2 sin2\theta = 4$ and $r^2 = 16sin2\theta$ (VTU Model 2015) **Ans :** $\frac{\pi}{3}$

   (iii) $r = sin\theta + cos\theta$ and $r = 2sin\theta$ (VTU Jan 2020, July 2004, 2002) **Ans :** $\frac{\pi}{4}$

   (iv) $r = 2sin\theta$ and $r = 2(sin\theta + cos\theta)$ (VTU July 2017).

   (v) $r = \frac{a}{1+cos\theta}$ and $r = \frac{b}{1-cos\theta}$ (VTU June 2019, Jan 2017, Jun 2012, June 2009, Jul 2008)

   (vi) $r = a(1 + sin\theta)$ and $r = b(1 - sin\theta)$ (VTU July 2017)

   (vii) Find the angle between the pair of curves $r = 6cos\theta$ and $r = 2(1 + cos\theta)$ (VTU June 2018)

   (viii) $r = a(1 - cos\theta)$ and $r = b(1 + cos\theta)$ (VTU July 2019)

   (ix) $r = 2\sin\theta$ and $r = 2\cos\theta$ (VTU Jan 2021)

   (x) $r^n = a^n(cosn\theta + sinn\theta)$ and $r^n = a^n\, sinn\theta$, **Ans :** $\frac{\pi}{4}$

   (xi) $r^n = a^n cosn\theta$ and $r^n sinn\theta = b^n$, **Ans :** $\frac{\pi}{2}$

   (xii) $r^m = a^m cosm\theta$ and $r^m = b^m\, sinm\theta$, **Ans :** $\frac{\pi}{2}$

Pedal equations ($p - r$ equations)

1. With usual notations prove the pedal equation in the form $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ (VTU Aug 2021, Jan 2018)

2. Find the pedal equations of the following curves.

   (i) $r = ae^{\theta cot\alpha}$ (VTU Jan 2019)

   (ii) $r^n = sechn\theta$ (VTU Jan 2015) **Ans :** $\frac{1}{p^2} = \frac{-1}{r^2}(r^{2n} - 2)$

   (iii) $r^m = a^m sinm\theta + b^m cosm\theta$ (VTU June 2019, July 2015, Jan-2005) **Ans :** $p^2 = \frac{r^{2(m+1)}}{a^{2m}+b^{2m}}$

(iv) $r^n = a^n cosn\theta$ (VTU March 2022, Jan 2020, July 2017, Jan 2016, Model 2014, Jan 2014, June 2014, Jan 2010, May 2001, Aug 2000) **Ans :** $pa^n = r^{n+1}$

(v) $r^m = a^m(sinm\theta + cosm\theta)$ (VTU Aug 2021, June 2018, 2010) **Ans :** $r^{m+1} = \sqrt{2}a^m p$

(vi) $r^m cosm\theta = a^m$ (VTU July 2016) **Ans :** $pr^{m-1} = a^m$

(vii) $\frac{2a}{r} = 1 - cos\theta$ (VTU Jun 2011) **Ans :** $p^2 = ar$

(viii) $r = a + bcos\theta$ (VTU July 2017)

(ix) $\frac{2a}{r} = 1 + cos\theta$ (VTU Jan 2017)

(x) $r = 2(1 + cos\theta)$ (VTU Jan 2018)

(xi) $r = a(1 + cos\theta)$ (VTU Aug 2021, June 2018)

(xii) $r^m = a^m cosm\theta$ (VTU June 2019)

(xiii) $\frac{\ell}{r} = 1 + ecos\theta$ **Ans :** $\frac{1}{p^2} = 1 - \frac{1}{e^2} + \frac{2}{lr}$

(xiv) $r = ae^{\theta cot\alpha}$ **Ans :** $r^2 = p^2 cosec^2\alpha$

(xv) $r = a\theta$ **Ans :** $p = \frac{r^2}{\sqrt{r^2+a^2}}$

(xvi) $r = a(1 + cos\theta)$ **Ans :** $r^3 = 2ap^2$

(xvii) $r^n = a^n sinn\theta$ **Ans :** $pa^n = r^{n+1}$

Radius of Curvature in Cartesian, Polar, Parametric and Pedal forms

1. With usual notations prove that $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$ (VTU March 2022)
2. Find ρ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the Folium $x^3 + y^3 = 3axy$ (VTU Aug 2021, Model 2018, July 2017, July 2016, Model 2014, 2015, 2008)
3. Find the radius of curvature for the curve $r^2 = a^2 cos2\theta$. (VTU Aug 2021, July 2017)
4. Find the radius of curvature for the curve $r^n = a^n cosn\theta$. (VTU Jan 2017)
5. Find the radius of curvature for the curve $r^n = a^n sinn\theta$. (VTU Jan 2018)
6. Show that the radius of curvature at any point of the cardioid $r = a(1 - cos\theta)$ varies as $\sqrt{r}$. (VTU 2003)
7. Find the radius of curvature of the curve $r = a(1 + cos\theta)$ (VTU March 2022, June 2019, Jan 2018)

8. Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ where the curve meets the x-axis. (VTU Jan 2018, July 2018, Jan 2014, 2000) **Ans :** $\frac{a}{2}$

9. Find the radius of curvature at $(a, 0)$ of the curve $y = x^3(x - a)$ (VTU June 2019, June 2010)

10. Find the radius of curvature of $a^2y = x^3 - a^3$ at the point where the curve cuts x-axis. (VTU July 2014) **Ans :** $\frac{5\sqrt{10}a}{3}$

11. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1,1) (VTU Jan 2017, July 2016) **Ans :** $\frac{-\sqrt{2}}{3}$

12. For the curve $y = \frac{ax}{(a+x)}$, show that. $\left(\frac{2\rho}{a}\right)^{\frac{2}{3}} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$(VTU Jan 2021, July 2017, 2008)

13. Find the radius of curvature for the curve $y = a\ logsec(\frac{x}{a})\ at\ (x, y)$ (VTU Jan 2020, Jan 2019)

14. show that. $\left(\frac{2\rho}{a}\right)^2 = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$, For the curve $y = \frac{ax}{(a+x)}$ (VTU June 2018)

15. Prove that the radius of curvature ρ at any point (x,y) on the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ is given by $\frac{2(ax+by)^{\frac{3}{2}}}{ab}$ (VTU Jan 2014)

16. Show that the radius of curvature at $x = \frac{\pi}{2}$ of the curve $y = 4sinx - sin2x$ is $\frac{5\sqrt{5}}{4}$ (VTU 2009)

17. Show that the radius of curvature of the curve $x^3 + y^3 = 3xy$ at $\left(\frac{3}{2}, \frac{3}{2}\right)$ is $\frac{-3}{8\sqrt{2}}$ (VTU June 2019, Jan 2015)

18. Find ρ at $(-2a, 2a)$ for the curve $x^2y = a(x^2 + y^2)$ (VTU Jan 2018)

19. Find the radius of curvature for the curve $x = alog(sect + tant), y = asect$ (VTU July 2015)

20. Show that for the curve $r(1 - cos\theta) = 2a,\ \rho^2$ varies as r^3. (VTU July 2019)

21. Find the radius of curvature for the curve $x = a(t - sint), y = a(1 - cost)$ (VTU July 2017)

22. Show that the radius of curvature at any point of the cycloid, $x = a(\theta + sin\theta), y = a(1 - cos\theta)$ is $4acos\frac{\theta}{2}$ (VTU Model 2018, Jan 2016)

23. Find the radius of curvature for the curve $x = a(cost + logtan\frac{t}{2}), y = asint$ (VTU July 2014) **Ans :** $acott$

24. Find the radius of the curvature of each of the following curves:
(i) $r^3 = 2ap^2$(Cardiod)
(ii) $p^2 = ar$
(iii) $pr = a^2$
(iv) $r^3 = a^2p$
(v) $p^2(a^2 + b^2 - r^2) = a^2b^2$
(vi) $1/p^2 = 1/a^2 + 1/b^2 - r^2/a^2b^2$

25. For the curve $p = \frac{r^{n+1}}{a^n}$, Show that ρ varies inversely as $(n-1)$th power of the radius vector.

26. For the curve $r = a(1 + cos\theta)$, show that $\frac{\rho^2}{r}$ is a constant.

27. For the curve $r(1 - cos\theta) = 2a$, show that ρ^2 varies as r^3.

28. For the curve $r^m = a^m cosm\theta$, show that $\rho = \frac{a^m}{(m+1)r^{m-1}}$

29. If ρ_1 & ρ_2 are radii of curvature at the extremities of any chord of the cardioid $r = a(1 + cos\theta)$ which passes through the pole, then show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$

30. Find the radius of curvature for the curve $r^n = a^n cosn\theta$. (VTU Jan 2017)

31. Show that the R.O.C at any point of the lemniscate $r^2 = a^2cos2\theta$ is inversely proportional to r

32. Find ρ for the curve $\frac{2a}{r} = (1 - cos\theta)$

33. Show that for the equiangular spiral $r = ae^{\theta cot\alpha}$, $\frac{\rho}{r}$ is a constant.

34. Find the radius of curvature of the polar curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - cos^{-1}\left(\frac{a}{r}\right)$

35. Find the radius of curvature for the curve $r^n = a^n sinn\theta$

36. Show that the R.O.C at any point of the cardioid $r^2sec2\theta = a^2$ is $\rho = \frac{a^2}{3r}$

37. Find the radius of curvature of $y = ccosh\left(\frac{x}{c}\right)$ at the point where it crosses the y-axis. Ans : c

38. Find the radius of curvature of the curve $xy = c^2$ at (c, c) Ans : $\sqrt{2}c$

39. Find the radius of curvature of the curve $y^2 = x^2\frac{(a+x)}{(a-x)}$ at the point $(-a, 0)$ Ans : $\frac{a}{4}$

40. Find the radius of curvature at 't' on $x = e^t cost, y = e^t sint$ Ans : $\sqrt{2}t$

41. Find the radius of curvature of $r = ae^{\theta cot\alpha}$ Ans : $\rho = rcosec\alpha$

42. Find the radius of curvature at any point of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ Ans : $3asin\theta cos\theta$

43. P.T the R.O.C at any point of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is 3 times the length of the perpendicular from the origin to the tangent at that point.

44. Show that the R.O.C at the end of the major axis $(a, 0)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the semi - latus rectum.

45. Find the radius of curvature for the curve $x = a(cost + tsint), y = a(sint - tcost)$ **Ans : at**

46. Find the radius of curvature for the curve $x = a(\theta - sin\theta), y = a(1 - cos\theta)$ **(VTU 2003)** **Ans :** $4asin\left(\frac{\theta}{2}\right)$

47. Show that for the rectangular hyperbola $xy = c^2$, $\rho = \frac{(x^2+y^2)^{\frac{2}{3}}}{2xy}$

48. Show that the radius of curvature of the curve $y = 4sinx - sin2x$ at $x = \frac{\pi}{2}$ is $\frac{5\sqrt{5}}{4}$

49. Find the radius of curvature of $xy^2 = a^3 - x^3$ at $(a, 0)$.
Hint : Here $y_1 = \infty$ Hence find $x_1 = \frac{dx}{dy} = 0$ and $x_2 = \frac{d^2x}{dy^2} = \frac{-2}{3a}$ and use the formula $\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}$ **Ans :** $\frac{3a}{2}$

50. If ρ is the R.O.C at the point P on the parabola $y^2 = 4ax$ & S be its focus, then show that ρ^2 varies with $(SP)^3$

51. Find ρ at $(-2a, 2a)$ for the curve $x^2y = a(x^2 + y^2)$ **Ans :** $2a$

Module 2 - Series Expansion and Multivariable Calculus

Syllabus

Taylor's and Maclaurin's series expansion for one variable (Statement only) – problems. Indeterminate forms - L'Hospital's rule. Problems. Partial differentiation, total derivative - differentiation of composite functions. Jacobian and problems. Maxima and minima for a function of two variables. Problems.

2.5 Taylor's series expansion of a function of single variable:

If $y = f(x)$ be a function,then the Taylor's series expansion for $y = f(x)$ about $x = a$ is given by

$$\boxed{f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + - - - - - - - -} \quad (1)$$

where $f'(a), f''(a), f'''(a)$ —— stands for successive derivatives of $f(x)$ at $x = a$.

2.6 Maclaurin's series :

If $y = f(x)$ be a function,then the Maclaurin's series expansion for $y = f(x)$ about is given by

$$\boxed{f(x) = f(0) + \frac{x\,f'(0)}{1!} + \frac{x^2}{2!}\,f''(0) + \frac{x^3}{3!}\,f'''(0) + \cdots}$$

Problem 2.6.1. Expand $y = cosx$ by Maclaurin's series.

Solution: By Maclaurin's series expansion

$$f(x) = f(0) + \frac{xf'(0)}{1!} + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \quad (1)$$

$$f(x) = cosx, \quad f(0) = cos0 = 1$$
$$f'(x) = -sinx, \quad f'(0) = 0$$
$$f''(x) = -cosx, \quad f''(0) = 1$$
$$f'''(x) = sinx, \quad f'''(0) = 0$$
$$f''''(x) = cosx, \quad f''''(0) = 1$$

Substituting in (1), we get

$$\therefore cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Problem 2.6.2. Expand $\sqrt{(1 + sin2x)}$ by Maclaurin's series. (VTU July 2021, June 2019, June 2018,Model 2018, Jan 2016, July 2014)

Solution: By Maclaurin's series expansion

$$f(x) = f(0) + \frac{xf'(0)}{1!} + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \quad (1)$$

$$f(x) = \sqrt{(1 + sin2x)} = \sqrt{(sinx + cosx)^2} = sinx + cosx$$
$$f(0) = sin0 + cos0 = 1$$
$$f'(x) = cosx - sinx \qquad f'(0) = 1$$
$$f''(x) = -sinx - cosx \qquad f''(0) = -1$$
$$f'''(x) = -cosx + sinx \qquad f'''(0) = -1$$
$$f''''(x) = sinx + cosx \qquad f''''(0) = 1$$

$\therefore \sqrt{(1 + sin2x)} = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Problem 2.6.3. Obtain the Maclaurin series of $f(x) = \log \sec x$ upto 6th degree term. (VTU Jan 2018, July 2017, 2009)

Solution : $y = f(x) = \log \sec x$

$\therefore f(0) = \log 1 = 0$

$$y_1 = f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x \qquad \therefore f'(0) = 0$$

$$y_2 = f''(x) = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2 \qquad \therefore f''(0) = 1$$

$$y_3 = 2y_1y_2 \qquad \therefore f'''(0) = 2(0)(1) = 0$$

$$y_4 = 2y_2^2 + 2y_1y_3 \qquad \therefore f^{iv}(0) = 2(1)^2 + 2(0)(0) = 2$$

$$y_5 = 4y_2y_3 + 2y_2y_3 + 2y_1y_4 = 6y_2y_3 + 2y_1y_4 \qquad \therefore f^{v}(0) = 6(1)(0) + 2(0)(2) = 0$$

$$y_6 = 6y_3^2 + 8y_2y_4 + 2y_1y_5 \qquad \therefore f^{v'}(0) = 6(0) + 8(1)(2) + 2(0) = 16$$

$\therefore$ By Maclaurin series expansion

$$\begin{aligned} \ln \sec x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \\ &= \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \cdots \end{aligned}$$

Problem 2.6.4. Obtain the Maclaurin series of $log(1 + cosx)$ upto 6th degree term. (VTU June 2019, July 2017)

Solution:

$$y(x) = \log(1 + \cos x)$$

$$y(0) = \log(1 + 1) = \log 2$$

Diff y with respect to x.

$$y_1(x) = \frac{(0 - \sin x)}{(1 + \cos x)}$$

$$(1 + \cos x)y_1(x) = -\sin x.$$

$$\text{Put } x = 0$$

$$y_1(0) = 0$$

Diff again w.r.to x,

$(1 + \cos x)(y_2) + y_1(-\sin x) = -\cos x$

Put $x = 0$, we get $y_2(0) = \frac{-1}{2}$

Diff again w.r.to x,

$(1 + \cos x)y_3 - \sin x(y_2) - \sin x(y_2) - \cos x(y_1) = \sin x$

Put $x = 0$, we get $y_3(0) = 0$

Diff again w.r.to x,

$(1+\cos x)y_4 - \sin x(y_3) - 2\left[\sin x(y_3) + \cos x(y_2)\right]$

$-\left[\cos x\ (y_2) - \sin x\ (y_1)\right] = \cos x$

Put $x = 0$, we get $y_4(0) = \frac{-1}{4}$

$\therefore$ By Maclaurin series expansion

$$\begin{aligned}\ln \sec x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots \\ &= \log 2 - \frac{x^2}{4} - \frac{x^4}{96} + \cdots\end{aligned}$$

Problem 2.6.5. Using Maclaurin's series Prove that $\sqrt{1+\cos 2x} = \sqrt{2}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right]$ (VTU Jan 2021)

$$y = \sqrt{1+\cos 2x} = \sqrt{2\cos^2 x}$$

$$\Rightarrow y = \sqrt{2}\cos x, \qquad \Rightarrow y(0) = \sqrt{2}\cos(0) = \sqrt{2}$$

$$y'(x) = \sqrt{2}\times(-\sin x) \qquad \Rightarrow y'(0) = -\sqrt{2}\sin(0) = -\sqrt{2}(0) = 0$$

$$\therefore y''(x) = -\sqrt{2}\times\cos x = -\sqrt{2}\cos x, \qquad \Rightarrow y''(0) = -\sqrt{2}\cos 0 = -\sqrt{2}$$

$$y'''(x) = -\sqrt{2}(-\sin x) = \sqrt{2}\sin x$$

$$\Rightarrow y'''(0) = \sqrt{2}\sin(0) = 0$$

Maclaurin's series expansion is given by

$$\begin{aligned} y(x) &= y(0) + \frac{x}{2!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots \\ \Rightarrow \sqrt{1+\cos 2x} &= \sqrt{2} + \frac{x}{1!}(0) + \frac{x^2}{2!}(-\sqrt{2}) + \frac{x^3}{3!}(0) + \ldots\ldots \\ &= \sqrt{1+\cos 2x} = \sqrt{2} + \frac{x^2}{2!}(-\sqrt{2}) - 0 + \ldots. \\ &= \sqrt{1+\cos 2x} = \sqrt{2} - \frac{\sqrt{2}x^2}{2} + \ldots \end{aligned}$$

2.7 Indeterminate Forms:

If $f(x)$ and $g(x)$ be two functions such that $lim_{x\to a}\quad f(x)$ and $lim_{x\to a}\quad g(x)$ both exists, then

$lim_{x\to a}\frac{f(x)}{g(x)} = \frac{lim_{x\to a}f(x)}{lim_{x\to a}g(x)}$.

If $lim_{x\to a}\quad f(x) = 0$ and $lim_{x\to a}\quad g(x) = 0$ then,

$$lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$$

Which do not have any definite value. Such an expression is called indeterminate form.

The other indeterminate forms are $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 1^{∞}, 0^0, ∞^0,

L'-Hospital's Rule:

If two functions $f(x)$ and $g(x)$ are such that

(i) $lim_{x \to a} \quad f(x) = 0$ and $lim_{x \to a} \quad g(x) = 0$

(ii) $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$

Then,

$$lim_{x \to a} \frac{f(x)}{g(x)} = lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note : If $lim_{x \to a} \frac{f'(x)}{g'(x)}$ is in $\frac{0}{0}$ form, we have to apply L'Hospital's rule again.i.e.

$$lim_{x \to a} \frac{f'(x)}{g'(x)} = lim_{x \to a} \frac{f''(x)}{g''(x)}$$

and so on.

Note:

(i) L'Hospital's rule is applicable only when the expression has the form $\frac{0}{0}$, or $\frac{\infty}{\infty}$ in the limit.

(ii) Note that we do not take the derivative of the ratio using the quotient rule, but rather separately find the derivatives of the numerator and denominator functions, then find the limit of their ratio.(Do not confuse L'Hopital's Rule with the quotient rule for derivatives.)

(iii) Sometimes we need to repeat L'Hospital's Rule more than once till we get definite value of the limit.

(iv) **Important Limits :**

(a) $lim_{x \to 0} \quad \frac{sinx}{x} = 1$

(b) $lim_{x \to 0} \quad \frac{x}{sinx} = 1$

(c) $lim_{x \to 0} \quad \frac{tanx}{x} = 1$

(d) $lim_{x\to 0} \quad \frac{x}{tanx} = 1$

(e) $lim_{x\to 0}\,(1+nx)^{\frac{1}{x}} = e^n$. In Particular, $lim_{x\to 0}\,(1+x)^{\frac{1}{x}} = e$

(f) $lim_{x\to \infty}\,\left(1+\frac{n}{x}\right)^x = e^n$. In Particular, $lim_{x\to \infty}\,\left(1+\frac{1}{x}\right)^x = e$

(g) $lim_{x\to\infty}e^x = \infty$ and $lim_{x\to\infty}e^{-x} = 0$

(h) $lim_{x\to 0}ln(x) = -\infty$

(i) $lim_{x\to 1}logx = 0$

(j) $lim_{x\to\infty}logx = \infty$

(k) $log(e) = 1$

The indeterminate form $0.\infty$:

If $lim_{x\to a}f(x)g(x)$ assumes the indeterminate form $0.\infty$ then the product may be converted to one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing $f(x)g(x)$ as

$$f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}} \text{ or } \frac{g(x)}{\frac{1}{f(x)}}$$

and then we can apply L'Hospitals rule to get the limit.

Problem 2.7.1. Evaluate the following limits. $lim_{x\to 0}log_{sinx}sin2x$

Solution:

$$k = lim_{x\to 0}log_{sinx}sin2x$$

$$= lim_{x\to 0}\frac{logsin2x}{logsinx} \quad \left(\frac{\infty}{\infty}\ form\right)$$

By applying L'Hospital's rule,

$$k = lim_{x\to 0}\frac{2cot2x}{cotx} \quad \left(\frac{\infty}{\infty}\ form\right)$$

$$= 2 \quad lim_{x\to 0}\frac{tanx}{tan2x} \quad \left(\frac{0}{0}\ form\right)$$

Again by applying L'Hospital's rule,

$$= 2 \quad lim_{x\to 0}\frac{sec^2x}{2sec^2 2x}$$

$$= 1$$

Problem 2.7.2. Evaluate $\lim_{x\to 0}\log_{\tan x}\tan 2x$

Solution :

$$\lim_{x\to 0} \log_{\tan x} \tan 2x = \lim_{x\to 0} \frac{\log \tan 2x}{\log \tan x} \quad : \frac{\infty}{\infty}$$

Applying L'Hospital's rule

$$= \lim_{x\to 0} \frac{1}{\tan 2x} \cdot 2 \cdot \sec^2 2x \cdot \frac{\tan x}{\sec^2 x} \quad : \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{\sin x \cdot \cos x \cdot 2}{\sin 2x + \cos 2x}$$

$$= \lim_{x\to 0} \frac{2 \cdot \frac{1}{2} \sin 2x}{\frac{1}{2} \sin 4x}$$

$$= \lim_{x\to 0} \cdot \frac{\sin 2x}{2x} \cdot \frac{4x}{\sin 4x}$$

$$= 1 \cdot 1$$

$$= 1$$

Problem 2.7.3. Evaluate $lim_{x\to 0} \frac{a^x - b^x}{x}$

Solution: Let

$$k = lim_{x\to 0} \frac{a^x - b^x}{x} \qquad (\frac{0}{0}\text{form})$$

By applying L'Hospital's rule ,

$$= lim_{x\to 0} \frac{a^x loga - b^x logb}{1}$$

$$= loga - logb$$

$$= log\frac{(a)}{(b)}$$

The indeterminate form $\infty - \infty$:

If $lim_{x\to a} f(x) - g(x)$ assumes the indeterminate form $\infty - \infty$, can be transformed into a type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by rewriting the expression as one function.

Problem 2.7.4. Evaluate $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right)$

Solution:

$$K = \lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right)$$

$$= \lim_{x\to 1} \frac{x \cdot \ln(x) - x + 1}{(x-1) \cdot \ln(x)}$$

$$= \lim_{x\to 1} \frac{\ln(x)}{\frac{x-1}{x} + \ln(x)}$$
$$= \lim_{x\to 1} \frac{x\cdot\ln(x)}{x-1+x\cdot\ln(x)}$$
$$= \lim_{x\to 1} \frac{1+\ln(x)}{1+1+\ln(x)}$$
$$= \lim_{x\to 1} \frac{1+\ln(x)}{2+\ln(x)}$$
$$= \frac{1}{2}$$

Problem 2.7.5. Evaluate $lim_{x\to 2} \left[\frac{1}{(x-2)} - \frac{1}{log(x-1)}\right]$

Solution :

$$k = lim_{x\to 2} \left[\frac{1}{(x-2)} - \frac{1}{log(x-1)}\right] \quad (\infty - \infty\text{form})$$
$$k = lim_{x\to 2} \left[\frac{log(x-1)-(x-2)}{(x-2)log(x-1)}\right] \quad (\frac{0}{0} form)$$

By applying L'Hospital's rule,

$$k = lim_{x\to 2} \left[\frac{2-x}{(x-2)+(x-1)log(x-1)}\right] \quad (\frac{0}{0} form)$$

Again applying L'Hospital's rule,

$$k = lim_{x\to 2} \left[\frac{-1}{2+log(x-1)}\right]$$
$$= \frac{-1}{2}$$

The indeterminate forms $1^\infty, 0^0, \infty^0$:

If $\lim_{x\to a} f(x)^{g(x)}$ assumes the indeterminate form $1^\infty, 0^0$ or ∞^0, then let $y = \lim_{x\to a} f(x)^{g(x)}$ and taking logarithm on both sides, we get $logy = \lim_{x\to a} g(x)\,(log(f(x))$. This limit will be in any one of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ or $0.\infty$ and can be evaluated easily. If this limit is K then the required limit is obtained as $y = \lim_{x\to a} f(x)^{g(x)} = e^K$

Problem 2.7.6. Evaluate $lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ (VTU Model 2022, Jan 2021, Jan 2020,June 2019, Jan 2018, July 2015, Model 2014)

Solution: Let

$$k = lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} \quad (1^\infty \quad form)$$

Apply log on both sides

$$logk = log \quad lim_{x\to 0} \quad \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$$
$$= lim_{x\to 0} \quad log \quad \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$$
$$= lim_{x\to 0} \quad \frac{1}{x} \quad log \quad \left(\frac{a^x+b^x+c^x}{3}\right)$$
$$= lim_{x\to 0} \quad \frac{log \quad (a^x+b^x+c^x)-log3}{x} \quad (\frac{0}{0} form)$$

By applying L'Hospital's rule,

$$logk = lim_{x\to 0} \quad \frac{a^x loga + b^x logb + c^x logc}{a^x+b^x+c^x}$$
$$= \frac{(loga+logb+logc)}{3}$$
$$= \frac{log(abc)}{3}$$
$$i.e. logk = log(abc)^{\frac{1}{3}}$$
$$\therefore k = (abc)^{\frac{1}{3}}$$

Problem 2.7.7. Evaluate $lim_{x\to \frac{\pi}{2}} \quad (tanx^{cosx})$

Solution : Let

$$k = lim_{x\to \frac{\pi}{2}} \quad (tanx^{cosx}) \quad (\infty^0 \quad form)$$

Apply log on both sides

$$logk = log \quad lim_{x\to \frac{\pi}{2}} \quad (tanx)^{cosx}$$
$$logk = lim_{x\to \frac{\pi}{2}} \quad log \quad (tanx)^{cosx}$$
$$= lim_{x\to \frac{\pi}{2}} \quad cosx \quad log \quad (tanx) \quad (0*\infty \quad form)$$

By applying L'Hospital's rule,

$$logk = lim_{x\to \frac{\pi}{2}} \quad \frac{log \quad tanx}{secx} \quad (\frac{\infty}{\infty} form)$$

By simplifying we get,

$$logk = lim_{x\to \frac{\pi}{2}} \quad cotx \quad cosecx$$
$$i.e logk = 0 \implies k = e^0 = 1$$

Problem 2.7.8. Prove that $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$.

Solution : Let

$$L = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$$

$$\log L = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to a}\frac{1}{\left(2-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)\frac{1}{\left(-\operatorname{cosec}^2\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} \text{ [Applying L'Hospital's rule]}$$

$$= \frac{2}{\pi}$$

$$\log L = \frac{2}{\pi}$$

$$L = e^{\frac{2}{\pi}}$$

Problem 2.7.9. Prove that $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$.

Solution: Let

$$L = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \qquad [1^\infty \text{ form}] \quad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

$$\log L = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)\cdot\frac{1}{2x} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^3} \quad \left[\frac{0}{0}\right] \quad \left[\because \lim_{x\to 0}\frac{x}{\tan x} = 1\right]$$

$$= \lim_{x\to 0}\frac{\sec^2 x + x\cdot 2\sec^2 x\tan x - \sec^2 x}{6x^2} \text{ [Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{\sec^2 x}{3}\cdot\frac{\tan x}{x} = \frac{1}{3}$$

$$\log L = \frac{1}{3}$$

$$L = e^{\frac{1}{3}}$$

Problem 2.7.10. Evaluate $\lim_{x\to\pi/2}(\sin x)^{\tan x}$

Solution :

$$K = \lim_{x\to\pi/2} (\sin x)^{\tan x} \quad (= 1^{\infty})$$

$$\Rightarrow \log k = \lim_{x\to\pi/2} \log\left[(\sin x)^{\tan x}\right]$$

$$= \lim_{x\to\pi/2} \tan x \cdot \log(\tan x)$$

$$= \lim_{x\to\pi/2} \frac{\log(\sin x)}{(1/\tan x)}$$

$$\Rightarrow \log k = \lim_{x\to\pi/2} \frac{\log(\sin x)}{\cot x} \quad \left(= \frac{0}{0}\right)$$

Applying L-Hospital's rule,

$$\log k = \lim_{x\to\pi/2} \frac{\frac{1}{\sin x}\cdot(\cos x)}{(-\operatorname{cosec}^2 x)}$$

$$= \lim_{x\to\pi/2} -\frac{\cot x}{\operatorname{cosec}^2 x}$$

$$= -\frac{\cot(\pi/2)}{\operatorname{cosec}(\pi/2)}$$

$$= \frac{0}{2} = 0$$

$$\Rightarrow k = e^0 = 1$$

2.8 Partial Differentiation :

Let $z = f(x, y)$ be a function of two variables x and y. The first order partial derivative of z w.r.to x, denoted by $\frac{\partial z}{\partial x}$ or z_x is defined as

$$\frac{\partial z}{\partial x} = lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

From the above definition, we understand that $\frac{\partial z}{\partial x}$ is obtained by differentiating z only w.r.to x, treating y as a constant.

Similarly, the first order partial derivative of z w.r.to y, denoted by $\frac{\partial z}{\partial y}$ or z_y is defined as

$$\frac{\partial z}{\partial y} = lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

From the above definition, we understand that $\frac{\partial z}{\partial y}$ is obtained by differentiating z only w.r.to y, treating x as a constant.

The second order partial derivatives are

$$\frac{\partial^2 z}{\partial x^2} \text{ or } z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial y^2} \text{ or } z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$
$$\frac{\partial^2 z}{\partial x \partial y} \text{ or } z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$
$$\frac{\partial^2 z}{\partial y \partial x} \text{ or } z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Problem 2.8.1. If $u = sinxy$ then find $u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{yx}$. Also verify that $u_{xy} = u_{yx}$.

Solution :

$$u = sinxy \tag{1}$$

differentiate (1) partially w.r.to x,

$$u_x = \frac{\partial}{\partial x}[sin(xy)] = ycos(xy)$$

differentiate (1) partially w.r.to y,

$$u_y = \frac{\partial}{\partial y}[sin(xy)] = xcos(xy)$$

differentiate u_x partially w.r.to x,

$$u_{xx} = \frac{\partial}{\partial x}[ycos(xy)] = -y^2 sin(xy)$$

differentiate u_y partially w.r.to y,

$$u_{yy} = \frac{\partial}{\partial y}[xcos(xy)] = -x^2 sin(xy)$$

differentiate u_y partially w.r.to x,

$$u_{xy} = \frac{\partial}{\partial x}[xcos(xy)] = -xysin(xy) + cos(xy)$$

differentiate u_x partially w.r.to y,

$$u_{yx} = \frac{\partial}{\partial y}[ycos(xy)] = -xysin(xy) + cos(xy)$$

From last 2 results ,it is clear that $u_{xy} = u_{yx}$.

Problem 2.8.2. If $u = e^{(ax+by)} f(ax - by)$,then prove that $bu_x + au_y = 2abu$. (VTU Model 2022)

Solution :

$$u = e^{(ax+by)}f(ax-by) \tag{1}$$

differentiate (1) partially w.r.to x,

$$u_x = \frac{\partial}{\partial x}[e^{(ax+by)}f(ax-by)]$$

$$u_x = a[e^{(ax+by)}f'(ax-by)] + au$$

differentiate (1) partially w.r.to y,

$$u_y = \frac{\partial}{\partial y}[e^{(ax+by)}f(ax-by)]$$

$$u_y = -b[e^{ax+by}f'ax - by] + bu$$

Consider,

$$LHS = bu_x + au_y = b\left[ae^{ax+by}f'(ax-by) + au\right] + a\left[-be^{ax+by}f'(ax-by) + bu\right]$$
$$= 2abu$$
$$= RHS$$

Problem 2.8.3. If $u = f(x+ct) + g(x-ct)$,then prove that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$ (VTU Jan 2018)

Solution :

$$u = f(x+ct) + g(x-ct) \tag{1}$$

differentiate (1) partially w.r.to x,

$$u_x = f^{'}(x+ct) + g^{'}(x-ct)$$

differentiate again partially w.r.to x,

$$u_{xx} = f^{''}(x+ct) + g^{''}(x-ct) \tag{2}$$

differentiate (1) partially w.r.to t,

$$u_t = c[f^{'}(x+ct) - g^{'}(x-ct)]$$

differentiate again partially w.r.to t,

$$u_{tt} = c^2[f^{''}(x+ct) + g^{''}(x-ct)] \tag{3}$$

Using (2) in (3), we get $u_{tt} = c^2u_{xx}$

Problem 2.8.4. If $f = tan^{-1}(\frac{y}{x})$ then find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$. Also verify that $f_{xy} = f_{yx}$.

Solution: We have

$$f = \tan^{-1}\left(\frac{y}{x}\right) \tag{1}$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \left(\frac{-y}{x^2+y^2}\right) \tag{2}$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2} \tag{3}$$

Differentiating (2) partially with respect to y, we get

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \tag{4}$$

Differentiating (3) partially with respect to x, we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)(1) - x(2x)}{(x^3+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \tag{5}$$

$\therefore$ From eqns. (4) and (5), we get $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Problem 2.8.5. If $u = log(x^3+y^3+z^3-3xyz)$, then prove that $u_x + u_y + u_z = \frac{3}{x+y+z}$ and hence show that

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$ (VTU Jan 2014)

Solution : Given $u = \log\left(x^3+y^3+z^3-3xyz\right)$

$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$ $\ldots(i)$

Similarly, $\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$ $\ldots(ii)$

and $\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$. $\ldots(iii)$

Adding eqns. (i), (ii) and (iii), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$\text{or } =\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$$

As $a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$

or $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$ (iv)

$$\begin{aligned}\text{Now, } \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u \\ &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \\ &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \text{ from } (iv) \\ &= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right] \\ &= 3\left[-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right] \\ &= \frac{-9}{(x+y+z)^2}\end{aligned}$$

Problem 2.8.6. If $u=\left(1-2xy+y^2\right)^{-\frac{1}{2}}$ and $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=y^2u^k$, then find the value of K. (VTU Jan 2016)

Solution : $u=\left(1-2xy+y^2\right)^{-\frac{1}{2}}$

Differentiate partially w. r. t. x and y

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{1}{2}\left(1-2xy+y^2\right)^{-\frac{3}{2}}(-2y) \\ &= u^3y \\ \frac{\partial u}{\partial y} &= -\frac{1}{2}\left(1-2xy+y^2\right)^{-\frac{3}{2}}(-2x+2y) \\ &= (x-y)u^3\end{aligned}$$

$$\therefore x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=u^3xy+\left(-xy+y^2\right)u^3=u^3y^2$$

Problem 2.8.7. If $z(x+y)=x^2+y^2$, show that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$.

Solution:

$$z(x+y) = x^2 + y^2$$
$$\Rightarrow z = \frac{x^2+y^2}{x+y} \tag{1}$$

Differentiate (1) partially w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2} = \frac{x^2 - y^2 + 2xy}{(x+y)^2} \tag{2}$$

By Symmetry

$$\frac{\partial z}{\partial y} = \frac{y^2 - x^2 + 2xy}{(x+y)^2}$$

$\therefore$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2\left(x^2-y^2\right)}{(x+y)^2}\right]^2 = \frac{4(x+y)^2(x-y)^2}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2} \tag{3}$$

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2} = 1 - \frac{4xy}{(x+y)^2} = \frac{(x+y)^2 - 4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2} \tag{4}$$

From (3) and (4)

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

2.9 Total Derivative

If $u = f(x,y)$ is a function of two variables, Then its total differential is denoted by du and is defined as

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}.$$

Similarly if $u = f(x,y,z)$ is a function of three variables then

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz}$$

Problem 2.9.1. Find the total derivative for the following function : $u = x^3 + y^3 + x^2y + xy^2$

Solution :

$$du = \frac{\partial(x^3+y^3+x^2y+xy^2)}{\partial x}dx + \frac{\partial(x^3+y^3+x^2y+xy^2)}{\partial y}dy.$$
$$= (3x^2+2xy+y^2)dx + (3y^2+x^2+2xy)dy$$

Problem 2.9.2. Find the total derivative of $u = xy^2z^3$

Solution :

$$du = \frac{\partial(xy^2z^3)}{\partial x}dx + \frac{\partial(xy^2z^3)}{\partial y}dy + \frac{\partial(xy^2z^3)}{\partial z}dz$$
$$= (y^2z^3)dx + (2xyz^3)dy + (3xy^2z^2)dz$$

Problem 2.9.3. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2+y^2+z^2$ and $x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t.$

Solution : Here u is a function of x, y, z and $x, y.z$ are in turn functions of t. Thus u is a function ' t ' via the intermediate variables x, y, z. Then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$
$$= 2x \cdot 2e^{2t} + 2y \cdot \left(2e^{2t}\cos 3t - 3e^{2t}\sin 3t\right)$$
$$+ 2z\left(2e^{2t}\sin 3t + 3e^{2t}\cos 3t\right)$$

Rewriting in terms of $x, y =$

$$= 2x \cdot 2 \cdot x + 2 \cdot y(2y - 3 \cdot z) + 2z(2z + 3y)$$
$$= 4\left(x^2+y^2+z^2\right)$$

or in terms of t

$$\frac{du}{dt} = 4\left(e^{4t} + e^{4t}\left(\cos^2 3t + \sin^2 3t\right)\right) = 8e^{4t}$$

verification by direct substitution:

$$u = x^2+y^2+z^2 = e^{4t} + e^{4t}\cos^2 3t + e^{4t}\sin^2 3t = 2e^{4t}$$

$\frac{du}{dt} = 8e^{4t}$

Problem 2.9.4. Find the total differential coefficient of x^2y w.r.t. x when $x.y$ are connected by $x^2+xy+y^2 = 1$

Solution : Let $u = x^2y$. then the total differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus the total differential coefficient of u w.r.t. x is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$
$$\frac{du}{dx} = 2xy + x^2\frac{dy}{dx}$$

From the implicit relation $f = x^2 + xy + y = 1$, we calculate

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x+y}{x+2y}$$

so

$$\frac{du}{dx} = 2xy + x^2 \cdot \frac{dy}{dx} = 2xy + x^2\left(-\frac{(2x+y)}{(x+2y)}\right)$$
$$\frac{du}{dx} = 2xy - \frac{x^2(2x+y)}{(x+2y)}$$

2.10 Differentiation of Composite functions :

Note 1:

If $u = f(x,y)$ and x and y are functions of single variable t, then

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

Note 2 : (Chain Rule)

If $u = f(x,y)$ and x and y are functions of r, s, then u is a function of r, s. In this case,

$$\boxed{\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}}$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}}$$

Note : The above formulae can be extended to functions of three or more variables.

Problem 2.10.1. If $u = sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ find $\frac{du}{dt}$ as a function of t.

Solution : We have $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

i.e $\frac{du}{dt} = \frac{1}{y}cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}cos\left(\frac{x}{y}\right)2t$

$= \frac{e^t}{t^2}cos\left(\frac{e^t}{t^2}\right) - 2\frac{e^t}{t^3}cos\left(\frac{e^t}{t^2}\right)$

$= e^t cos\left(\frac{e^t}{t^2}\right)\left[\frac{1}{t^2} - \frac{2}{t^3}\right]$

Problem 2.10.2. If z is a function of x and y and $x = e^u + e^{-v}, y = e^{-u} - e^v$ Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution : Here z is a function of x and y, where x and y are functions of u and v.

$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \cdots$ (i) and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \cdots$ (ii)

Also given that

$x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

$\therefore \frac{\partial x}{\partial u} = e^u, \ \frac{\partial x}{\partial v} = -e^{-v}, \ \frac{\partial y}{\partial u} = -e^{-u}, \ \frac{\partial y}{\partial v} = -e^y$

$\therefore$ From (i) we get

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u})$ (iii)

and from (ii) we get

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{-v}) \ldots$ (iv)

Subtracting (iv) from (iii) we get

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$$
$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Problem 2.10.3. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then Prove that $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$ (VTU June 2019, July 2017)

Solution : Let $r = 2x - 3y, s = 3y - 4z, t = 4z - 2x$, then

$$u \to (r, s, t) \to (x, y, z)$$
$$\therefore u \to (x, y, z)$$
$$\frac{\partial r}{\partial x} = 2, \frac{\partial r}{\partial y} = -3, \frac{\partial r}{\partial z} = 0,$$
$$\frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial y} = 4, \frac{\partial s}{\partial z} = -4,$$
$$\frac{\partial t}{\partial x} = -2, \frac{\partial t}{\partial y} = 0, \frac{\partial t}{\partial z} = 4$$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$
$$= 2\left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right)$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$
$$= 3\left(-\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}\right)$$
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$
$$= 4\left(-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right)$$
$$\therefore \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = \left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right) + \left(-\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}\right) + \left(-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right)$$

$$= 0$$

Problem 2.10.4. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then Prove that $xu_x + yu_y + zu_z = 0$ (VTU Model 2022, June 2019, June 2018, July 2017, July 2016)

Solution : Let $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$, then $u \to (r, s, t) \to (x, y, z)$

$$\therefore u \to (x, y, z)$$

$$\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \left(\frac{\partial u}{\partial r}\right)\frac{1}{y} + 0 - \left(\frac{\partial u}{\partial t}\right)\left(\frac{-z}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \left(\frac{\partial u}{\partial r}\right)\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + 0$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= 0 - \left(\frac{\partial u}{\partial s}\right)\left(\frac{1}{z^2}\right) + \left(\frac{\partial u}{\partial t}\right)\left(\frac{1}{x}\right)$$

$$\therefore \; xu_x + yu_y + zu_z$$

$$= \left(\frac{\partial u}{\partial r}\right)\frac{x}{y} - \frac{\partial u}{\partial t}\left(\frac{-z}{x}\right) + \left(\frac{\partial u}{\partial r}\right)\left(\frac{-x}{y}\right) + \left(\frac{\partial u}{\partial s}\right)\left(\frac{y}{z}\right) - \left(\frac{\partial u}{\partial s}\right)\left(\frac{1}{z}\right) + \left(\frac{\partial u}{\partial t}\right)\left(\frac{z}{x}\right)$$

$$= 0$$

Problem 2.10.5. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then prove that $x^2u_x + y^2u_y + z^2u_z = 0$ (VTU July 2014)

Solution : Let

$$v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}, \quad w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

Differentiate V and W partially w.r.to x, y, and z

$$\frac{\partial v}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial v}{\partial z} = 0,$$

$$\frac{\partial w}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = \frac{1}{z^2}$$

Now

$$u = u(v, w)$$

$\therefore$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = -\frac{1}{x^2}\frac{\partial u}{\partial v} - \frac{1}{x^2}\frac{\partial u}{\partial w}$$
$$= -\frac{1}{x^2}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} = \frac{1}{y^2}\frac{\partial u}{\partial v} \tag{2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial z} = \frac{1}{z^2}\frac{\partial u}{\partial w} \tag{3}$$

from (1), (2), and (3),

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} = 0$$

2.11 Jacobians (J) :

If u and v are any 2 functions of 2 independent variables x and y ,then the Jacobian of (u,v) w.r. to (x,y) is denoted by $J = \frac{\partial(u,v)}{\partial(x,y)}$ and is defined as $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ Similarly , If (u,v,w) are functions of 3 independent variables (x,y,z)then the Jacobian of (u,v,w) w.r. to (x,y,z) is denoted by $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ and is defined as $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Problem 2.11.1. If $u = x + y + z \quad v = y + z \quad w = z$, then find $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution :

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= 1$$

Problem 2.11.2. If $u = x^2 + y^2 + z^2, \quad v = xy + yz + zx, \quad w = x + y + z$,then Find

$J\frac{(u,v,w)}{(x,y,z)}$

Solution :

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2x[(x+z)-(y+z)] - 2y[(y+z)-(y+x)] + 2z[(y+z)-(x+z)]$$

$$== 2(xz - xy - yz + xy + yz - xz)$$

$$= 0$$

Problem 2.11.3. If $u = \frac{yz}{x}, \quad v = \frac{zx}{y}, \quad w = \frac{xy}{z}$,then prove that $J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$ (VTU July 2015, July 2014).

Solution : We have

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz} \begin{vmatrix} \frac{-yz}{x} & z & y \\ z & \frac{-xz}{y} & x \\ y & x & \frac{-xy}{z} \end{vmatrix}$$

$$= \frac{1}{xyz}[4xyz]$$

$$= 4$$

Problem 2.11.4. If $u + v = e^x cosy \quad u - v = e^x siny$,then find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution : We have $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

By adding given equations, we get $u = \frac{e^x}{2}(cosy + siny)$

By subtracting equations ,$v = \frac{e^x}{2}(cosy - siny)$

$$\therefore J = \begin{vmatrix} \frac{e^x}{2}(cosy + siny) & \frac{e^x}{2}(cosy - siny) \\ \frac{e^x}{2}(cosy - siny) & \frac{-e^x}{2}(cosy + siny) \end{vmatrix}$$

$$J = \frac{e^{2x}}{4}\begin{vmatrix} (cosy + siny) & (cosy - siny) \\ (cosy - siny) & (cosy + siny) \end{vmatrix}$$

$$= \frac{-e^{2x}}{4}(2) = \frac{-e^{2x}}{2}$$

Problem 2.11.5. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Solution :

Given $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin\theta\sin^2\theta$$

$$= r^2\sin\theta$$

Problem 2.11.6. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - x - y$, then find $J\frac{(u,v,w)}{(x,y,z)}$ at $(1, -1, 0)$ (VTU June 2019, June 2018)

Solution :

$$u = x + 3y^2 - z^3 \quad \cdots (1)$$

$$v = 4x^2yz \quad \cdots (2)$$

$$w = 2z^2 - x - y \quad \cdots (3)$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -1 & -1 & 4z \end{vmatrix}$$

At the point $(1, -1, 0)$,

$$J = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ -1 & -1 & 0 \end{vmatrix}$$

$$= 1(0-4) + 6(0-4) = -4 - 24 = -28$$

2.12 Maxima and Minima for a function of two variables

A function $f(x, y)$ is said to be a maximum at (a, b) when $f(a, b)$ is greater than $f(x, y)$ for all values of x and y in the neighborhood of (a, b).

Similarly, $f(a, b)$ is said to be a minimum at (a, b) when $f(a, b)$ is less than $f(x, y)$ for all values of x and y in the neighborhood of (a, b).

These definitions may be stated in analytical form as follows:

If, for all values of h and k numerically less than some small positive quantity,

- If $f(a+h, b+k) - f(a, b) =$ a negative number, then $f(a, b)$ is a maximum value of $f(x, y)$.
- If $f(a+h, b+k) - f(a, b) =$ a positive number, then $f(a, b)$ is a minimum value of $f(x, y)f(x, y)$.

Necessary and sufficient Conditions for extreme values

We know that a necessary condition that a function of one variable to have a maximum or a minimum for a given value of the variable is that its first derivative should be zero for the given value of the variable. Similarly, for a function $f(x, y)$ of two independent variables, a necessary condition that $f(a, b)$ should be a maximum or a minimum (i.e. a turning value) is that for $x = a, y = b$,
$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0.$$
Hence the **following steps are used** for finding maximum and minimum values of a function $f(x, y)$

Step 1. Solve the simultaneous equations $\dfrac{\partial f}{\partial x} = 0, \quad \dfrac{\partial f}{\partial y} = 0.$ and find the values of (x, y) in the form $(a, b), (c, d), (e, f)$...These points are called as stationary points.

Step 2. For each stationary point, calculate the value of $\Delta = rt - s^2$ where $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ and $t = \frac{\partial^2 f}{\partial y^2}$

Step 3. The function will have a

maximum if $\Delta > 0$ and $r < 0$

minimum if $\Delta > 0$ and $r > 0$

neither a maximum nor a minimum if $\Delta < 0$

The question is undecided if $\Delta = 0$

Problem 2.12.1. Examine $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ for its extreme values

Solution :

$$f_x = 3x^2 - 12, \ f_y = 3y^2 - 12$$

$$r = f_{xx} = 6x, \ s = f_{xy} = 0, \ t = f_{yy} = 6y$$

Solving $f_x = 0$ and $f_y = 0$, we get

$$\Rightarrow 3x^2 - 12 = 0, \quad \text{and} \quad 3y^2 - 3 = 0$$

$$\Rightarrow x^2 - 4 = 0 \quad \text{and} \quad y^2 - 1 = 0$$

$$\Rightarrow x^2 = 4 \quad \text{and} \quad y^2 = 1$$

$$\Rightarrow x = \pm 2 \quad \text{and} \quad y = \pm 1$$

Hence stationary points are $(2, 1), (2, -1), (-2, 1), (-2, -1)$

$\Delta = rt - s^2 = (6x)(6y) - 0 = 36xy$

At (2,1) $\Delta = 72 > 0$ and $r = 12 > 0$.

Therefore function takes minimum value at (2,1)

and the Minimum Value = $f(2, 1) = 2$

At (2,-1) and (-2,1), $\Delta = -72 < 0$ Hence (-2,1) and (2,-1) are saddle points.

At (-2,-1) $\Delta = 72 > 0$ and $r = -12 < 0$. Therefore function takes maximum value at (2,1)

Maximum Value= $f(-2, -1) = 38$

Problem 2.12.2. Find the extreme points of the function $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$.

Solution : $f(x, y) = 2(x^2 - y^2) - x^4 + y^4 \qquad \cdots (1)$

diff (1) w,r.t x & y partially

$\frac{\partial f}{\partial x} = 4x - 4x^3 \qquad \cdots (2)$

$\frac{\partial f}{\partial y} = -4y + 4y^3 \qquad \cdots (3)$

Solving $f_x = 0$ and $f_y = 0$, we get

$$\Rightarrow 4x - 4x^3 = 0 \quad \text{and} \quad -4y + 4y^3 = 0$$

$$\Rightarrow x - x^3 = 0 \quad \text{and} \quad -y + y^3 = 0$$

$$\Rightarrow x(1 - x^2) = 0 \quad \text{and} \quad y(-1 + y^2) = 0$$

$$\Rightarrow x = 0, \; x^2 = 1 \quad \text{and} \quad y = 0, \; y^2 - 1 = 0$$

$$\Rightarrow x = 0, \; x = 1, \; x = -1 \quad \text{and} \quad y = 0, \; y = 1, \; y = -1$$

Hence stationary points are

$(0,0),\ (0,1),\ (0,-1),\ (1,0),\ (1,1),\ (1,-1), (-1,0),\ (-1,1),\ (-1,-1)$

$$r = f_{xx} = 4 - 12x^2, \quad s = f_{xy} = 0, \quad t = f_{yy} = -4 + 12y^2$$

Points	r	s	t	$rt - s^2$	conclusion
$(0,0)$	$4 > 0$	0	-4	$-16 < 0$	Saddle point
$(0,1)$	$4 > 0$	0	8	$32 > 0$	minimum
$(1,0)$	$-8 < 0$	0	-4	$32 > 0$	maximum
$(1,1)$	$-8 < 0$	0	8	$-64 < 0$	saddle point
$(0,-1)$	4	0	8	$32 > 0$	minimum
$(1,-1)$	-8	0	8	$-64 < 0$	saddle point
$(-1,0)$	$-8 < 0$	0	-4	$32 > 0$	maximum
$(-1,1)$	-8	0	8	$-64 < 0$	saddle point
$(-1,-1)$	-8	0	8	$-64 < 0$	saddle point

$$\max f = f(1,0) = f(-1,0) = 2(1-0) - 1 + 0 = 2 - 1 = 1$$

$$\min f = 2(0-1) - 0 + 1 = -2 + 1 = -2$$

Problem 2.12.3. Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3axy$

Solution : We have, $\quad f(x,y) = x^3 + y^3 - 3axy$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3ay, \quad q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = -3a, \quad t = \frac{\partial^2 f}{\partial y^2} = 6y$$

For maxima and minima

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$3x^2 - 3ay = 03y^2 - 3ax = 0$

$\Rightarrow \quad x^2 = ay \Rightarrow y = \frac{x^2}{a} \quad \ldots$ (1)

$\Rightarrow y^2 = ax \quad \ldots$ (2)

Putting the value of y from (1) in (2), we get

$$x^4 = a^3x \Rightarrow x\left(x^3 - a^3\right) = 0$$

$$\Rightarrow \quad x(x-a)\left(x^2 + ax + a^2\right) = 0$$

$$\Rightarrow \quad x = 0, a$$

Putting $x = 0$ in (1), we get $y = 0$,

Putting $x = a$ in (1), we get $y = a$,

Stationary pairs	$(0,0)$	(a,a)
r	0	$6a$
s	$-3a$	$-3a$
t	0	$6a$
$rt - s^2$	$-9a^2 < 0$	$27a^2 > 0$

At $(0,0)$ there is no extremum value, since $rt - s^2 < 0$.

At $(a,a), rt - s^2 > 0, r > 0$

Therefore (a,a) is a point of minimum value.

The minimum value of $f(a,a) = a^3 + a^3 - 3a^3 = -a^3$

Problem 2.12.4. Find the extreme values of the function $f(x,y) = x^4 + y^4 – 2x^2 + 4xy – 2y^2$

Solution: $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For stationary point of $f(x,y)$, we solve $f_x = 4x^3 - 4x + 4y = 0$ or $x^3 - x + y = 0$ (1)

$f_y = 4y^3 + 4x - 4y = 0$ or $y^3 + x - y = 0$ (2)

Adding (1) and (2), we get $x^3 + y^3 = 0$

$\therefore y = -x$

Substituting this value in (1), we get $x^3 - x - x = 0$

$x^3 - 2x = 0 \rightarrow x(x^2 - 2) = 0 \rightarrow x = 0, x = \sqrt{2}, -\sqrt{2}$

for $x = 0, y = 0$; for $x = \sqrt{2}, y = -\sqrt{2}$ and for $x = -\sqrt{2}, \quad y = -\sqrt{2}$

$\therefore$ Stationary points of f are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

$$r = f_{xx} = 12x^2 - 4, \; s = f_{xy} = 4, \; t = f_{yy} = 12y^2 - 4$$

Point	r	s	t	$rt - s^2$
$(0, 0)$	-4	4	-4	0
$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	4	20	$384 > 0$
$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	4	20	$384 > 0$

$\therefore$ Minima at both the points $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ and minimum value at both these points $= 4 + 4 - 4 - 8 - 4 = -8$.

Problem 2.12.5. Examine the function $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (VTU Model ME 2022)

Solution :

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 \quad \cdots (1)$$
$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \quad \cdots (2)$$
$$\frac{\partial f}{\partial y} = 6xy - 30y \quad \cdots (3)$$

For stationary point of $f(x, y)$, we solve

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$
$$3x^2 + 3y^2 - 30x + 72 = 0 \quad \cdots (4)$$
$$\text{and } 6xy - 30y = 0 \quad \cdots (5)$$
$$\Rightarrow xy - 5y = 0$$
$$\Rightarrow y(x - 5) = 0$$
$$\Rightarrow y = 0, \quad x - 5 = 0$$
$$\Rightarrow y = 0 \quad x = 5$$

When $y = 0$, from (4) we get

$$x^2 - 10x + 24 = 0$$
$$\Rightarrow x^2 - 4x - 6x + 24 = 0$$
$$\Rightarrow x(x - 4) - 6(x - 4) = 0$$
$$\Rightarrow (x - 4)(x - 6) = 0$$
$$\Rightarrow x = 4, 6$$

$\therefore$ When $y = 0$ stationary points are $(4, 0), (6, 0)$, when $x = 5$ from (4) we get

$$\Rightarrow 25 + y^2 - 50 + 24 = 0$$

$$y^2 - 1 = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

$\therefore$ when $x = 5$ the stationary point are $(5, 1), (5, -1)$

$r = \dfrac{\partial^2 f}{\partial x^2} = 6x - 30$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 6y$

$t = \dfrac{\partial^2 f}{\partial x \partial y} = 6x - 30$

Points	r	s	t	$rt - s^2$	conclusion
$(4, 0)$	$-6 < 0$	0	-6	$36 > 0$	max
$(6, 0)$	$6 > 0$	0	6	$36 > 0$	min
$(5, 1)$	0	6	0	$-36 < 0$	saddle point
$(5, -1)$	0	-6	0	$-36 < 0$	saddle point

Maximum value is $f(4, 0) = 4^3 + 0 - 15(4^2) - 0 + 72(4) = 112$

Minimum value is $f(6, 0) = 6^3 + 0 - 15(6^2) - 0 + 72(6) = 108$

Question Bank- Module 2

Maclaurin's series :

1. Determine the Maclaurin series expansion of the function, $y = \sqrt{1 + sin2x}$ by Maclaurin's series. (VTU July 2021, June 2019, June 2018,Model 2018, Jan 2016, July 2014)

   Ans : $1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

2. Determine the Maclaurin series expansion of the function, $f(x) = log(1 + cosx)$ upto the term containing x^4. (VTU Model CV 2022, June 2019, July 2017) **Ans :** $\log 2 - \frac{x^2}{4} - \frac{x^4}{96} + \cdots$

3. Expand $log(secx)$ upto 6^{th} degree term by Maclaurin's series (VTU Jan 2018, July 2017, 2009) **Ans :** $\frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \cdots$

4. Expand $tanx$ by Maclaurin's series (VTU July 2015)

5. Expand $sin(e^x - 1)$ by Maclaurin's series (VTU June 2014) **Ans :** $x + \frac{x^2}{2!} - 5\frac{x^4}{4!} + \ldots$

6. Expand $f(x) = e^{sinx}$ by Maclaurin's series (VTU Model EE 2022, June 2018, 2011) **Ans :** $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

7. Expand $y = log(1 + e^x)$ by Maclaurin's series (VTU Model CV 2022, July 2016) **Ans :** $log2 + \frac{x}{2} + \frac{x^2}{8} + \cdots$

8. Expand $y = \frac{x}{(e^{x-1})}$ by Maclaurin's series (VTU Jan 2015) **Ans :** $e(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots$

9. Expand $\frac{e^x}{(1+e^x)}$ using Maclaurin's series upto 3rd degree terms. (VTU Jan 2017)

10. Expand $y = secx$ using Maclaurin's series up to x^4 term. (VTU Jan 2016)

11. Using Maclaurin's series Prove that $\sqrt{1 + \cos 2x} = \sqrt{2}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right]$ (VTU Jan 2021)

12. Expand $log(1 + sinx)$ up to the term containing x^4 using Maclaurin's series (VTU Model 2022) **Ans :** $x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

13. Expand $log \cos x$ by Maclaurin's series up to the term containing x^6 (VTU Model ME 2022)

14. Expand $f(x) = e^{cosx}$ by Maclaurin's series up to the term containing x^4 (VTU Model ME 2022)

15. Determine the Maclaurin series expansion of the function, $tanx$ (VTU July 2015) **Ans :** $x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \ldots$ **Ans :** $log2 - \frac{1}{2} - \frac{x^2}{2!} - \frac{1}{4} + \frac{x^4}{4!} + \ldots$

16. Determine the Maclaurin series expansion of the function, $f(x) = e^x sinx$ **Ans :** $1 + x + \frac{x^2}{2!} - \frac{3}{4!}x^4 - \frac{8}{5!}x^5 + \ldots$

17. Determine the Maclaurin series expansion of the function, $log(1 + x)$ **Ans :** $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

18. Determine the Maclaurin series expansion of the function, $e^{tan^{-1}x}$ **Ans :** $1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - 7\frac{x^4}{4!} + \ldots$

19. Determine the Maclaurin series expansion of the function, e^{xcosx} **Ans :** $1 + x + \frac{x^2}{2!} - 2\frac{x^3}{3!} - 11\frac{x^4}{4!} + \ldots$

20. Determine the Maclaurin series expansion of $y = e^{tanx}$ **Ans :** $1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \cdots$

Indeterminate Forms:

1. Evaluate $\overset{lim}{x \to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ (VTU Model 2022, Jan 2021, Jan 2020,June 2019, Jan 2018, July 2015, Model 2014) **Ans :** $(abc)^{\frac{1}{3}}$
2. Evaluate $lim_{x\to a} \left[2 - \frac{x}{a}\right]^{tan\left(\frac{\pi x}{2a}\right)}$ (VTU May 2010) **Ans :** $e^{\frac{2}{\pi}}$
3. Evaluate $lim_{x\to 0} (cosx)^{cot^2 x}$ (VTU July 2017)
4. Evaluate $lim_{x\to 0} \left[\frac{tanx}{x}\right]^{\frac{1}{x^2}}$ (VTU Model EE 2022, Jan 2017, July 2016) **Ans :** $e^{\frac{1}{3}}$
5. Evaluate $lim_{x\to 0} \left[\frac{sinx}{x}\right]^{\frac{1}{x^2}}$ (VTU Model ME 2022, July 2017)
6. Evaluate $lim_{x\to 0} \left[\frac{sinx}{x}\right]^{\frac{1}{x}}$ (VTU Jan 2015)
7. Evaluate $lim_{x\to 0} \left(\frac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}}$ (VTU Jan 2018, Jan 2014)
8. Evaluate $lim_{x\to 0} \left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}}$ (VTU Model ME 2022, July 2021, June 2019, June 2018, Model 2018, July 2017, Jan 2016, July 2014, Dec 2011)
9. Evaluate $lim_{x\to 0} \left[\frac{1}{x} - \frac{log(1+x)}{x^2}\right]$ (VTU Jan 2018)
10. Evaluate $lim_{x\to 0} \left[\frac{1}{x}\right]^{2sinx}$ (VTU Model 2018)
11. Evaluate $lim_{x\to 0} \left(\frac{1}{x}\right)^{2\sin x}$ (VTU Jan 2021)
12. Evaluate $lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$ (VTU Model EE 2022, July 2021)

    $lim_{x\to 0} \quad [a^x + x]^{\frac{1}{x}}$ (VTU Model 2022) **Ans :** ae
13. $lim_{x\to 0} \left[\frac{tanx}{x}\right]^{\frac{1}{x}}$ (VTU Model 2022)
14. Evaluate $lim_{x\to \frac{\pi}{2}} \quad (tanx^{tan2x})$ (VTU Model EE 2022)
15. Evaluate $lim_{x\to \frac{\pi}{2}} \quad (tanx^{cosx})$ **Ans :** 1
16. Evaluate $\lim_{x\to\pi/2}(\sin x)^{\tan x}$ (VTU Model CV 2022, June 2018)

    Ans : 1
17. Evaluate $\lim_{x\to 0}(cotx)^{tanx}$ (VTU Model CV 2022)

Partial Differentiation :

1. If $u = e^{(ax+by)} f(ax - by)$,then prove that $bu_x + au_y = 2abu$. (VTU Model 2022)
2. If $z = sin(ax + y) + cos(ax - y)$, prove that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$ (VTU July 2017)
3. If $u(x + y) = x^2 + y^2$, then prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$ (VTU Model ME 2022, July 2017)
4. If $x = rcos\theta, y = rsin\theta$ then show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ (VTU July 2017)
5. If $u = log(x^3 + y^3 + z^3 - 3xyz)$, then prove that $u_x + u_y + u_z = \frac{3}{x+y+z}$ and hence show that
$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$ (VTU Jan 2014)
6. If $V = (1 - 2xy + y^2)^{-\frac{1}{2}}$ and $x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} = y^2 V^k$, then find the value of K. (VTU Jan 2016)
Ans : 3
7. If $u = f(x + ct) + g(x - ct)$,then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ (VTU Jan 2018)
8. If $u = \log(\tan x + \tan y + \tan z)$ show that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$. (VTU Model 2022)
9. If $u = \frac{1}{\sqrt{(x^2+y^2+z^2)}}$, then prove that $v_{xx} + v_{yy} + v_{zz} = 0$
10. If $u = \log[\frac{x^2+y^2}{x+y}]$ then show that $xu_x + yu_y = 1$
11. If $u = log(x^2 + y^2 + z^2)$, then show that $(x^2 + y^2 + z^2)(u_{xx} + u_{yy} + u_{zz}) = 2$
12. If $u = tan^{-1}(\frac{y}{x})$ then find the value of $u_{xx} + u_{yy}$ (VTU Model EE 2022)
Ans : 0

Differentiation of Composite functions :

1. Find total derivative of u with respect to t where $u = tan^{-1}\left(\frac{y}{x}\right), x = e^t - e^{-t}, y = e^t + e^{-t}$ (VTU Model EE 2022, July 2017)
2. If $u = x^3y^2 + x^2y^3, x = at^2, y = 2at$, then find $\frac{du}{dt}$
3. If $u = xy^2 + x^2y, x = at^2, y = 2at$, then find $\frac{du}{dt}$ (VTU Model ME 2022)

4. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then find the value of $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$ (VTU June 2019, July 2017)

5. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then Prove that $xu_x + yu_y + zu_z = 0$ (VTU Model 2022, June 2019, June 2018, July 2017, July 2016)

6. If $u = f\,(x - y, y - z, z - x)$ then Prove that $u_x + u_y + u_z = 0$ (VTU Model ME 2022, July 2021, Jan 2020, Model 2018, Model 2014, July 2003)

7. If $u = f\,(y - z, z - x, x - y)$ then Prove that $u_x + u_y + u_z = 0$ (VTU July 2019)

8. If $u = f\,(xz, \frac{y}{z})$ then Prove that $xu_x - yu_y - zu_z = 0$ (VTU July 2004)

9. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then prove that $x^2u_x + y^2u_y + z^2u_z = 0$ (VTU Model EE 2022 , July 2014)

10. If $z = f(x, y)$, $x = rcos\theta, y = rsin\theta$, then prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$ (VTU Jan 2018, July 2018, Jan 2015)

11. If $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, then prove that $xz_x - yz_y = z_u - z_v$. (VTU Jan 2016)

12. If $u = \log(\tan x + \tan y + \tan z)$ show that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$. (VTU Model 2022) If $u = e^{(ax+by)}f(ax - by)$,then prove that $bu_x + au_y = 2abu$ by using the concept of composite functions. (VTU Model 2022)

13. If $u = sin^{-1}(x - y), x = 3t, y = 4t^3$, then show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

14. If $z = f(x, y)$ where $x = u - v$, $y = uv$ then show that

    $(i) uz_u - vz_v = (u + v)z_x$

    $(ii) z_u + z_v = (u + v)z_y$

Jacobians (J) :

1. If $u = \frac{yz}{x}, \quad v = \frac{zx}{y}, \quad w = \frac{xy}{z}$,then prove that $J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$ (VTU Model EE 2022, Model 2018, June 2018, July 2015, July 2014).

2. If $u = x^2 + y^2 + z^2, \quad v = xy + yz + zx, \quad w = x + y + z$,then Find $J\frac{(u,v,w)}{(x,y,z)}$ (VTU Model ME 2022, Jan 2020, Jan 2018, July 2017, Jan 2015) Ans : 0

3. If $u = x + y + z \quad vu = y + z \quad uvw = z$, then find $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ (VTU Model 2022, Jan 2016) **Ans :** u^2v

4. If $x = rsin\theta cos\phi, \quad y = rsin\theta sin\phi, \quad z = rcos\theta$,then find the Jacobian of x, y, z w.r.to r, θ, ϕ. (VTU Model 2022, July 2016, July 2014) Ans : $r^2 \sin\theta$

5. If $u = \frac{x}{y-z}, v = \frac{y}{z-x}, w = \frac{z}{x-y}$, find the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. Determine whether u, v, w are functionally dependent. (VTU July 2017)

6. If $u = \frac{x_2x_3}{x_1}, \quad v = \frac{x_1x_3}{x_2}, \quad w = \frac{x_1x_2}{x_3}$,then find the value of $J = \frac{\partial(u,v,w)}{\partial(x_1,x_2,x_3)}$ (VTU Jan 2017)

7. If $u = x + 3y^2, \quad v = 4x^2yz, \quad w = 2z^2 - xy$, then Find $J\frac{(u,v,w)}{(x,y,z)}$ at $(1, -1, 0)$ (VTU Jan 2018)

8. If $u = x + 3y^2 - z^3, \quad v = 4x^2yz, \quad w = 2z^2 - x - y$, then find $J\frac{(u,v,w)}{(x,y,z)}$ at $(1, -1, 0)$ (VTU June 2019, June 2018) Ans : -28

9. If $u = x + 3y^2 - z^3, \quad v = 4x^2yz, \quad w = 2z^2 - xy$, then Find $J\frac{(u,v,w)}{(x,y,z)}$ at $(1, -1, 0)$ (VTU Model CV 2022)

10. If $u = 3x + 2y - z, v = x - 2yz, w = x^2 + 2xy - xz$, then Show that $J\left(\frac{u,v,w}{x,y,z}\right) = 0$ (VTU 2021)

11. If $u = x + y + z \quad v = y + z \quad w = z$, then find $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ Ans : 1

12. If $u + v = e^x cosy \quad u - v = e^x siny$,then find $\frac{\partial(u,v)}{\partial(x,y)}$. Ans : $\frac{-e^{2x}}{2}$

Maxima and minima for a function of two variables

Find the extreme values of the following functions

1. $f(x, y) = x^3 + 3xy^2 - 3y^2 - 3x + 4$ (VTU Model 2022)
2. $f(x, y) = x^2 + y^2 + 6x - 12$ (VTU Model 2022)
3. $f(x, y) = x^3 + 3x^2 + 4xy + y^2$ (VTU Model ME 2022)

4. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (VTU Model ME 2022) Ans :

   Maximum value is $f(4, 0) = 112$ and Minimum value is $f(6, 0) = 108$

5. $f(x, y) = x^3 + y^3 - 3axy$ (VTU Model EE 2022)

   Ans :

   Extreme value=$-a^3$

6. $f(x, y) = x^4 + y^4 – 2x^2 + 4xy – 2y^2$ Ans :

   Minimum value attwo points $= -8$.

7. $f(x, y) = x^3 y^2 (1 – x – y)$

8. $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

9. $f(x, y) = x^3 + y^3 – 3x – 12y + 20$ (VTU Jan 2020, June 2019)

10. $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ Ans :

    Minimum Value = $f(2, 1) = 2$, Maximum Value= $f(-2, -1) = 38$

11. $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ (VTU Jan 2021)

12. S.T. $f(x, y) = xy(a - x - y)$ is max. at $\left(\frac{a}{3}, \frac{a}{3}\right)$. Find the maximum value. (VTU Model CV 2022, July 2021)

13. Find the extreme points of the function $f(x, y) = 2\left(x^2 - y^2\right) - x^4 + y^4$ (VTU Model CV 2022) Ans :

    $\max f = 1 \& \min f = -2$

14. Find the extreme values of the function $f(x, y) = SinxSinySin(x + y)$ Where $0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2}$

15. In a plane triangle find the maximum value of $CosACosBCosC$ (VTU June 2019) Ans :

    maximum value is $\frac{1}{8}$)

16. In a plane triangle find the maximum value of $\sin x \sin y \sin z$ (VTU 2021)

17. Examine the function $f(x, y) = Sinx + Siny + Sin(x + y)$ for extremum. **Ans :** Max value is $\frac{3\sqrt{3}}{2}$

18. Examine the function $f(x, y) = 1 + Sin(x^2 + y^2)$ for extremum

    (**Ans :** minimum value is $f(0, 0) = 1$)

19. Find the stationary points of $z = x^2 - xy + y^2 - 2x + y$ and hence find its maximum and minimum values. (**Ans:**Minimum value is $f(1, 0) = -1$)

20. Show that the function $f(x, y) = x^3 + y^3 - 3xy + 1$ is minimum at the point $(1, 1)$ (VTU Model EE 2022)

Module 3-Ordinary Differential Equations (ODE's) of first order

Syllabus

Linear and Bernoulli's differential equations. Exact and reducible to exact differential equations, Orthogonal trajectories, L-R & C-R circuits, Problems

Non-linear differential equations: Introduction to general and singular solutions, solvable for p only, Clairaut's equations, reducible to Clairaut's equations. Problems.

Self-Study: Applications of ODE, Solutions of nonlinear ODEs-Solvable for x and y

Definition of ODE: A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives. A differential equation is called an **ordinary differential equation(ODE)**, if it has ordinary derivatives in it. An ODE involves derivatives of an unknown function with respect to a single independent variable, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t).

e.g. : $\frac{dy}{dx} + 5y = sinx$ is an ordinary differential equation.

Partial Differential Equations:

A differential equation is called a **partial differential equation(PDE)**, if it has partial derivatives of an unknown function of two or more variables.

Order of a Differential Equation: The order of a differential equation is the order of the highest order derivative occurring in it.

Degree of a Differential Equation: The degree of a differential equation is the degree of the highest order derivative occurring in it, when the derivatives are free from radicals and fractions.

Solution of a Differential Equation: A solution or Integral of a differential equation is a relation between the independent and the dependent variables which satisfies the given differential equation.

General Solution or Complete Integral: A solution containing the number of arbitrary constants equal to the order of the equation is called the general solution or complete integral.

Particular Solution : Any solution obtained from the general solution by giving specific values to one or more of the arbitrary constants is called a particular solution.

Initial Value Problem: In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an initial condition $y(x_0) = y_0$ with given values and , that is used to determine a value of the arbitrary constant c. Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane. An ODE, together with an initial condition, is called an initial value problem. Thus, if the ODE is explicit, $y' = f(x, y)$ the initial value problem is of the form $y' = f(x, y),\ y(x_0) = y_0$

3.13 Exact Differential equation:

A differential Equation of the form $Mdx + Ndy = 0$ or $M(x, y)dx + N(x, y)dy = 0$ is said to be exact if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

and its solution is obtained by using the formula,

$$\boxed{\int \underset{(y-constant)}{M} dx + \int (\text{Terms of N not containing } x)dy = c}$$

Problem 3.13.1. Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Solution: Given equation is of the form $Mdx + Ndy = 0$

where $M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$

$\therefore \frac{\partial M}{\partial y} = -4x - 4y, \frac{\partial N}{\partial x} = -4y - 4x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given equation is exact.

$\therefore$ solution is $\int \underset{(y-constant)}{M} dx + \int (\text{Terms of N not containing } x)dy = c$

i.e. $\int (x^2 - 4xy - 2y^2)dx + \int y^2 dy = c$

$\frac{x^3}{3} - 4\frac{x^2}{2}y - 2y^2x + \frac{y^3}{3} = c$

i.e. $x^3 - 6x^2y - 6xy^2 + y^3 = 3c = C$

Problem 3.13.2. Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$. (VTU Jan 2014, Model

2014)

Solution : Here

$$M = y^2 e^{xy^2} + 4x^3, \quad N = 2xye^{xy^2} - 3y^2$$

$$\therefore \frac{\partial M}{\partial y} = y^2 \cdot e^{xy^2} 2xy + e^{xy^2} \cdot 2y = \left(2y + 2xy^3\right) e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2e^{xy^2} y + 2xy \left[e^{xy^2} y^2\right] = \left(2y + 2xy^3\right) e^{xy^2}$$

and since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

$\int\limits_{(y-constant)} M \, dx + \int (\text{Terms of N not containing } x) dy = c$ (1)

Integrating M w.r.t. x keeping y as constant, we get

$$\int \left(y^2 e^{xy^2} + 4x^3\right) dx = y^2 \frac{e^{xy^2}}{y^2} + 4\frac{x^4}{4} = e^{xy^2} + x^4$$

Integrating the terms of N which are free from x, w.r.t. y, we have

$$\int \left(-3y^2\right) dy == -3\frac{y^3}{3} = -y^3$$

Substituting in (1) we get the solution in the form :

$x^4 + e^{xy^2} - y^3 = c$

Problem 3.13.3. Solve $\left(1 + e^{\frac{x}{y}}\right) dx + \left(1 - \frac{x}{y}\right) e^{\frac{x}{y}} dy = 0.$ (VTU June 2014)

Solution : Here

$$M = 1 + e^{\frac{x}{y}}, \quad N = e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) = e^{\frac{x}{y}} - \frac{x}{y} e^{\frac{x}{y}}$$

$\therefore \quad \frac{\partial M}{\partial y} = e^{\frac{x}{y}} \left(-\frac{x}{y^2}\right)$

and $\quad \frac{\partial N}{\partial x} = e^{x/y} \cdot \frac{1}{y} - \frac{1}{y} \left(e^{\frac{x}{y}} + xe^{\frac{x}{y}} \frac{1}{y}\right) = -\frac{x}{y^2} e^{\frac{x}{y}}$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Solution is given by $\int\limits_{(y-constant)} M \, dx + \int (\text{Terms of N not containing } x) dy = c$ (1)

Integrating M w.r.t. x keeping y as constant, we have

$$\int\limits_{(y-constant)} \left(1 + e^{\frac{x}{y}}\right) dx = x + \frac{e^{\frac{x}{y}}}{(1/y)} = x + ye^{xy}$$

since in N there is no term free from x, hence the solution (1) becomes

$$x + ye^{x/y} = c$$

Problem 3.13.4. Solve $(ycosx + siny + y)dx + (sinx + xcosy + x)dy = 0$ (VTU Jan 2016)

(or)

Solve $\frac{dy}{dx} + \frac{(ycosx+siny+y)}{(sinx+xcosy+x)} = 0$ (VTU Jan 2020, Jan 2019, June 2018, Jan 2018)

Solution : Given equation is of the form $Mdx + Ndy = 0$

Here M $= y\cos x + \sin y + y,$ N $= \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1, \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Solution is given by

$\int\limits_{(y-constant)} M dx + \int(\text{Terms of N not containing } x)dy = c$

$\underset{(y-constant)}{(y\cos x + \sin y + y)dx} = c$

$i.e. y\sin x + (\sin y + y)x = c.$ $(\because \int kdx = kx,$ where k is a constant)

Problem 3.13.5. Solve $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x\sin y]\, dy = 0$ (VTU Jan 2021)

Solution : Given equation is of the form $Mdx + Ndy = 0$

Here

$$M = y\left(1 + \frac{1}{x}\right) + \cos y$$

$$N = x + \log x - x\sin y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y = \frac{dN}{dx}$$

Equation is exact.

Solution is given by $\int\limits_{(y-constant)} M dx + \int(\text{Terms of N not containing } x)dy = c$

$$\int \underset{(y-constant)}{\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]} dx + 0 = C$$

i.e. $y(x + \log x) + x\cos y = C$

Problem 3.13.6. Solve $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$ (VTU Jan 2018)

Solution :Given equation is of the form $Mdx + Ndy = 0$

Here M $= ye^{xy},$ N $= xe^{xy} + 2y$

$$\frac{\partial \mathrm{M}}{\partial y} = y\left(e^{xy} \cdot x\right) + e^{xy} = e^{xy}(xy+1)$$
$$\frac{\partial \mathrm{N}}{\partial x} = x\left(e^{xy} \cdot y\right) + e^{xy} = e^{xy}(xy+1)$$
$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}, \text{ the equation is exact.}$$

Solution is given by $\int \underset{(y-constant)}{M} dx + \int (\text{Terms of N not containing } x) dy = c$

$$\int \underset{(y-constant)}{ye^{xy}} dx + \int 2y dy = c$$

$$y \cdot \frac{e^{xy}}{y} + 2 \cdot \frac{y^2}{2} = c \quad \text{or} \quad e^{xy} + y^2 = c.$$

Problem 3.13.7. Solve $(e^y + 1)\cos x dx + e^y \sin x dy = 0$.

Solution : Here $M = (e^y + 1)\cos x, N = e^y \sin x$

$\Rightarrow \quad \frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x.$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Solution is given by

$\int \underset{(y-constant)}{M} dx + \int (\text{Terms of N not containing } x) dy = c$

Integrating M w.r.t. x keeping y as constant, we have

$$\int (e^y + 1)\cos x dx = (e^y + 1)\sin x$$

Now since in N there is no term free from x, the solution is

$(e^y + 1)\sin x = c$

Problem 3.13.8. Solve:

$$\left(5x^4 + 3x^2y^2 - 2xy^3\right) dx + \left(2x^3y - 3x^2y^2 - 5y^4\right) dy = 0$$

Solution : Here, $M = 5x^4 + 3x^2y^2 - 2xy^3, \quad N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now Solution is given by $\int M dx + \int (\text{ terms of } N \text{ is not containing } x) dy = C \quad (\ y \text{ constant})$

$$\int \left(5x^4 + 3x^2y^2 - 2xy^3\right) dx + \int -5y^4 dy = C$$

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

3.14 Equations Reducible to exact form

Sometimes a differential equation which is not exact may become exact on multiplication by a suitable function known as an **Integrating Factor.**(I.F.)

Consider the general differential equation $Mdx + Ndy = 0$

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then compute $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ and check whether it is near to M or N.

Case 1 : If it is near to M then compute $\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M}$ and if this gives a function of y only, say $f(y)$, then

$I.F. = e^{\int f(y)dy}$

Case 2 : If it is near to N then compute $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}$ and if this gives a function of x only, say $g(x)$, then

$I.F. = e^{\int g(x)dx}$

Finally multiply the given differential equation by the integrating factor which reduces to the exact form. and the Solution is given by

$\int_{(y-constant)} M \, dx + \int (\text{Terms of N not containing } x) dy = c$

Problem 3.14.1. Solve $(x^2 + y^2 + x)dx + xydy = 0.$ (VTU July 2017, Aug 2001, Mar 2000)

SOlution : Here $M = x^2 + y^2 + x$ and $N = xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Consider $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 2y - y = y$ which is near to N.

$\therefore$ compute $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = \frac{1}{xy} y = \frac{1}{x} = g(x)$

$I.F. = e^{\int g(x)dx} = e^{\int \frac{1}{x}dx} = e^{logx} = x$

Multiplying the given differential equation by the integrating factor (i.e. x), we get

$(x^3 + xy^2 + x^2)dx + x^2ydy = 0.$

This is in exact form.

$\therefore$ its solution is

$$\int_{(y-constant)} M \, dx + \int (\text{Terms of N not containing } x) dy = c$$

i.e. $\int (x^3 + xy^2 + x^2)dx + 0 = c$

i.e. $\frac{x^4}{4} + \frac{(x^2y^2)}{2} + \frac{x^3}{3} = c$

Problem 3.14.2. Solve $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0.$ (VTU Jan 2015)

Solution : Here $M = xy^3 + y, \quad N = 2\left(x^2y^2 + x + y^4\right).$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1, \quad \frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\text{Now} \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{(4xy^2+2) - (3xy^2+1)}{y\left(1+xy^2\right)}$$

$$= \frac{xy^2+1}{y\left(1+xy^2\right)}$$

$$= \frac{1}{y}$$

$$\text{Hence I.F.} = e^{\int \frac{1}{y} dy}$$

$$= e^{\log y}$$

$$= y$$

Multiplying the equation by y , we have

$$\left(xy^4 + y^2\right)dx + 2\left(x^2y^3 + xy + y^5\right)dy = 0$$

Above equation is exact.

Solution is given by

$$\int_{(y-constant)} M \, dx + \int (\text{Terms of N not containing } x) dy = c$$

$$i.e. \quad \int_{(y-constant)} (xy^4 + y^2) dx + 2\int y^5 dy = cc$$

or

$$\frac{x^2y^4}{2} + y^2x + 2\frac{y^6}{6} = c$$

or

$$3x^2y^4 + 6y^2x + 2y^6 = c_1$$

Problem 3.14.3. Solve $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$

Solution :

$$\text{Here} \quad \text{M} = y^4 + 2y, \ \text{N} = xy^3 + 2y^4 - 4x$$
$$\frac{\partial \text{M}}{\partial y} = 4y^3 + 2,$$
$$\frac{\partial \text{N}}{\partial x} = y^3 - 4$$
$$\frac{\frac{\partial \text{N}}{\partial x} - \frac{\partial \text{M}}{\partial y}}{\text{M}} = \frac{(y^3 - 4) - (4y^3 + 2)}{y^4 + 2y}$$
$$= \frac{-3\left(y^3 + 2\right)}{y\left(y^3 + 2\right)}$$
$$= -\frac{3}{y} = f(y)$$
$$\therefore \quad \text{I.F.} = e^{\int f(y)dy}$$
$$= e^{\int -\frac{3}{y}dy}$$
$$= e^{-3\log y}$$
$$= e^{\log y^{-3}}$$
$$= y^{-3} = \frac{1}{y^3}$$

Multiplying the given equation by $\frac{1}{y^3}$, it becomes

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

Which is exact.

Hence the solution is

$$\int_{(y-constant)} M \, dx + \int (\text{Terms of N not containing } x)dy = c$$

$$i.e. \quad \int_{y-constant} \left(y + \frac{2}{y^2}\right)dx + \int 2ydy = c$$

$$or \quad \left(y + \frac{2}{y^2}\right)x + y^2 = c.$$

Problem 3.14.4. Solve $(x^2 + y^2 + 2x)\,dx + 2ydy = 0$

Solution : Here $\quad M = (x^2 + y^2 + 2x)\,, \quad N = 2y$

$\Rightarrow \quad \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 0$

Now $\quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2y}{2y} = 1$

This can be treated as a function of x only.

Hence, the I.F. $= e^{\int dx} = e^x$

Multiplying the equation by e^x, we have

$$e^x\left(x^2+y^2+2x\right)dx+2ye^x dy=0$$

Above equation is exact.

Solution is given by

$\int\limits_{(y-constant)} M\, dx+\int(\text{Terms of N not containing } x)dy=c$

Hence the solution is $\int e^x\left(x^2+y^2+2x\right)dx+0=c$

($\because$ $2ye^x$ is not free from x)

or $\int\left(x^2+2x\right)e^x dx+y^2\int e^x dx=c$

Integrating by parts

$$\left(x^2+2x\right)e^x-\int(2x+2)e^x dx+y^2\int e^x dx=c$$

$$\left(x^2+2x\right)e^x-\left[(2x+2)e^x-\int 2e^x dx\right]+y^2e^x=c$$

$$\left(x^2+2x\right)e^x-(2x+2)e^x+2e^x=c$$

$$\left[x^2+2x-2x-2+2\right]e^x=c$$

or $\left(x^2+y^2\right)e^x=c$

Problem 3.14.5. Solve $x^2y dx-(x^3+y^3)dy=0$ (VTU Jan 2010, Dec 2009)

Solution : The given equation is $x^2y dx-\left(x^3+y^3\right)dy=0$

Comparing it with $\mathrm{M}dx+\mathrm{N}dy=0$

$$\mathrm{M}=x^2y;\quad \mathrm{N}=-x^3-y^3$$

$$\frac{\partial \mathrm{M}}{\partial y}=x^2;\ \frac{\partial \mathrm{N}}{\partial x}=-3x^2$$

$\frac{\frac{\partial \mathrm{N}}{\partial x}-\frac{\partial \mathrm{M}}{\partial y}}{M}=\frac{-3x^2-x^2}{x^2y}=\frac{-4}{y}$

Hence I.F. is

$$I.F.=e^{\int\frac{-4}{y}dy}=e^{-4logy}=e^{logy^{-4}}=y^{-4}=\frac{1}{y^4}$$

Multiplying the given equation by $\frac{1}{y^4}$ it becomes

$$\frac{x^2}{y^3}dx-\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0$$

Solution is given by

$$\int\limits_{(y-constant)} M\, dx+\int(\text{Terms of N not containing } x)dy=c$$

Hence the solution is

$$\int_{(y-constant)} \frac{x^2}{y^3} dx - \int \frac{1}{y} dy = c$$

or

$$\frac{x^3}{3y^3} - \log y = c.$$

Problem 3.14.6. Solve $(xy^2 - e^{\frac{1}{x^3}})dx - x^2ydy = 0.$ (VTU Jan 2015)

Solution : Here

$$\begin{aligned}
\mathrm{M} &= xy^2 - e^{\frac{1}{x^3}}, \quad \mathrm{N} = -x^2y \\
\frac{\partial \mathrm{M}}{\partial y} &= 2xy, \quad \frac{\partial \mathrm{N}}{\partial x} = -2xy \\
\frac{\frac{\partial \mathrm{M}}{\partial y} - \frac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} &= \frac{2xy - (-2xy)}{-x^2y} \\
&= \frac{4xy}{-x^2y} \\
&= -\frac{4}{x} = f(x) \\
\text{I.F.} &= e^{\int f(x)dx} \\
&= e^{\int -\frac{4}{x}dx} \\
&= e^{-4\log x} \\
&= e^{\log x^{-4}} \\
&= x^{-4} \\
&= \frac{1}{x^4}
\end{aligned}$$

Multiplying the given equation by $\frac{1}{x^4}$, it becomes

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{\frac{1}{x^3}}\right) dx - \frac{y}{x^2} dy = 0$$

which is exact.

Hence the solution is

$$\int_{(y-constant)} M\, dx + \int (\text{Terms of N not containing } x) dy = c$$

$$\int\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx+0=c$$

$$\text{or}\quad y^2\int\frac{1}{x^3}dx-\int\frac{1}{x^4}e^{\frac{1}{x^3}}dx=c$$

$$y^2\cdot\frac{x^{-2}}{-2}-\int\frac{1}{x^4}e^{\frac{1}{x^3}}dx+c \qquad (*)$$

To evaluate $\int\frac{1}{x^4}e^{\frac{1}{x^3}}dx$, put $\frac{1}{x^3}=t,\ \ i.e.\ \ x^{-3}=t$

$$\therefore -\frac{3}{x^4}dx=dt \quad \text{or} \quad \frac{1}{x^4}dx=-\frac{1}{3}dt$$

$$\therefore \quad \int\frac{1}{x^4}e^{\frac{1}{x^3}}dx=\int e^t\left(-\frac{1}{3}\right)dt$$

$$=-\frac{1}{3}e^t$$

$$=-\frac{1}{3}e^{\frac{1}{x^3}}$$

Hence, from (*), the solution is

$$-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c.$$

Problem 3.14.7. Solve $(x^2+y^3+6x)dx+y^2xdy=0$ (VTU Model 2021, July 2016)

Solution : Here $M=x^2+y^3+6x$ and $N=xy^2$

$$\frac{\partial M}{\partial y}=3y^2, \qquad \frac{\partial N}{\partial x}=y^2$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{xy^2}(3y^2-y^2)=\frac{2}{x}$$

which is a function of x alone.

Hence

$$I.F.=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{logx^2}=x^2$$

Multiplying the given equation by x^2 we get

$$(x^4+x^2y^3+6x^3)dx+x^3y^2dy=0$$

This is exact. Hence the solution is

$$\int_{y-constant}(x^4+x^2y^3+6x^3)dx+0=c$$

i.e. $\frac{x^5}{5}+y^3\frac{x^3}{3}+6\frac{x^4}{4}=c$

3.15 Linear Differential Equation

A differential equation is said to be linear if the dependant variable and its differential co-efficients occur only in the first degree, and are not multiplied together

The first order linear differential equation is of the form

$$\boxed{\frac{dy}{dx} + Py = Q}$$

where P and Q are functions of x only or constants. Its solution is given by

$$\boxed{y(I.F.) = \int Q(I.F.)dx + c \quad \text{where} \quad I.F. = e^{\int P dx}}$$

Note : A first-order ODE is said to be linear if it can be brought into the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only or constants (linear in y)

The equation linear in x can be written as

$$\boxed{\frac{dx}{dy} + Px = Q}$$

where P and Q are functions of y only or constants.

In this case solution is given by

$$\boxed{x(I.F.) = \int Q(I.F.)dy + c \quad \text{where} \quad I.F. = e^{\int P dy}}$$

3.16 Equations reducible to linear

General equation reducible to linear form is $\boxed{f'(y)\frac{dy}{dx} + P\,f(y) = Q}$ (1)

where P and Q are functions of x only or constants.

Putting $f(y) = z$ so that $f'(y)\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes $\frac{dz}{dx} + Pz = Q$, which is linear in z.

Its solution is given by

$z(I.F.) = \int Q(I.F.)dx + c$ where $I.F. = e^{\int P dx}$

3.17 Bernoulli's Linear differential Equation :

General Form :

$$\frac{dy}{dx} + Py = Qy^n \tag{1}$$

where P and Q are functions of x only.

To solve this, we use the following method.

Divide by y^n

then $\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}Py = Q$ (2)

$i.e. y^{-n}\frac{dy}{dx} + y^{1-n}P = Q$

Put $y^{1-n} = t$ then $(1-n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$

$\implies y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$

$\therefore$ (2) becomes $\frac{1}{1-n}\frac{dt}{dx} + Pt = Q$

or $\frac{dt}{dx} + (1-n)Pt = Q(1-n)$ which is linear in t.

Its integrating factor is given by

I.F. = $e^{\int (1-n)P dt}$.

And its solution is given by

$t(I.F.) = \int (1-n)Q(I.F.)dt + c$

Note : Another form of Bernoulli's equation is

$$\frac{dx}{dy} + Px = Qx^n$$

where P and Q are functions of y.

Problem 3.17.1. Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Solution : Dividing by $\cos^2 y$, we get $\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{\sin 2y}{\cos^2 y}x = x^3$

i.e. $\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{2\sin y\cos y}{\cos^2 y}x = x^3$

or $\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$

Putting $\tan y = z$, we get $\sec^2 y\frac{dy}{dx} = \frac{dz}{dx}$

Therefore equation reduces to $\frac{dz}{dx} + 2xz = x^3$

This is in the form $\frac{dz}{dx} + Pz = Q$, where $p = 2x$ and $Q = x^3$

$$\text{I.F.} = e^{\left(\int 2x dx\right)} = e^{\frac{2x^2}{2}} = e^{x^2}$$

Hence the solution is

$$z(I.F.) = \int Q(I.F.)dx + c$$
$$ze^{x^2} = \int x^3 e^{x^2} dx + c$$
$$= \int x \cdot x^2 e^{x^2} dx + c \qquad \cdots (*)$$

Put $t = x^2$

$$dt = 2xdx \Rightarrow xdx = \frac{dt}{2}$$

So (*) becomes

$$ze^{x^2} = \frac{1}{2}\int te^t dt + c$$
$$= \frac{1}{2}\left[te^t - \int e^t dt\right] + c \quad \text{(by using integration by parts)}$$
$$= \frac{1}{2}\left[te^t - e^t\right] + c$$
$$= \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2}\right] + c$$

or

$$z = \frac{x^2}{2} - \frac{1}{2} + ce^{-x^2}$$

or

$$\tan y = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$$

Problem 3.17.2. Solve $\frac{dy}{dx} + ytanx = y^3 secx$ (VTU June 2019, Jan 2018, Jan 2015)

Solution : Dividing by y^3 we get

$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \sec x \qquad \ldots(1)$

Let $\frac{1}{y^2} = z$

Then $\left(\frac{-2}{y^3}\right)(dy/dx) = (dz/dx) \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dz}{dx}$

So given equation can be written as

$\frac{-1}{2}\frac{dz}{dx} + \tan x\, z = \sec x$

$\Rightarrow \quad \frac{dz}{dx} - 2\tan x\, z = -2\sec x$

This is linear equation in variable z.

Here $P = -2\tan x$ and $Q = -2\sec x$

$\therefore \quad$ I.F. $= e^{\int Pdx} = e^{\int -2\tan x dx} = e^{-2\log\sec x} = \frac{1}{\sec^2 x}$

Hence solution is

$$z(I.F.) = \int Q\ (I.F.)dx + c$$

$$\Rightarrow (1/y^2) \cdot \frac{1}{\sec^2 x} = \int (-2\sec x)\ \frac{1}{\sec^2 x} dx + c \qquad [\because z = 1/y^2]$$

$$\Rightarrow (1/y^2) \cdot \frac{1}{\sec^2 x} = \int -2cosx + c$$

$$i.e.\ (1/y^2) \cdot \frac{1}{\sec^2 x} = -2sinx + x + c$$

Problem 3.17.3. Solve $x\frac{dy}{dx} + y = x^3y^6$. (VTU Model 2014)

Solution : Let $x\frac{dy}{dx} + y = x^3y^6$. (1)

Divide both sides by xy^6

$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$ (2)

Put $y^{-5} = t$

Then $-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore y^{-6}\frac{dy}{dx} = \frac{1}{-5}\frac{dt}{dx}$ (3)

Using (3) in (2), we get

$-\frac{1}{5}\frac{dt}{dx} + \frac{t}{x} = x^2$

$i.e. \frac{dt}{dx} - 5\frac{t}{x} = -5x^2$

This is in the form $\frac{dt}{dx} + Pt = Q$,

which is linear in t, where $P = \frac{-5}{x}, Q = -5x^2$

$$\begin{aligned}\therefore \text{I.F.} &= e^{\int Pdx}\\ &= e^{-5\int \frac{1}{x}dx}\\ &= e^{-5logx}\\ &= e^{logx^{-5}}\\ &= \frac{1}{x^5}\end{aligned}$$

Solution is given by

$$\begin{aligned}t(I.F.) &= \int Q(I.F.)dx + c\\ t\frac{1}{x^5} &= -5\int x^{-3}dx + c\\ &= -5\frac{x^{-2}}{-2} + c\\ i.e.\ \frac{1}{x^5y^5} &= \frac{5}{2x^2} + c\end{aligned}$$

Problem 3.17.4. Solve: $(1 + y^2)\,dx = \left(\tan^{-1} y - x\right) dy$.

Solution : The given equation is $(1+y^2)\,dx = \left(\tan^{-1} y - x\right) dy$

$$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

$$\left(1+y^2\right)\frac{dx}{dy} + x = \tan^{-1} y$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1} y}{1+y^2}$$

It is of the form $\frac{dx}{dy} + \mathrm{P}x = \mathrm{Q}$

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

The solution is

$$x(I.F.) = \int Q\,(I.F.)dx + c$$

$$x\, e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2}\; e^{\tan^{-1} y} dy + c$$

$$= \int te^t dt + c, \text{ where } t = \tan^{-1} y,\;\; dt = \frac{1}{1+y^2}dy$$

$$= te^t - \int e^t dt + c \;\; \text{(Integrating by parts)}$$

$$= te^t - e^t + c$$

$$= e^t(t-1) + c$$

$$i.e.\;\; x\, e^{\tan^{-1} y} = e^{\tan^{-1} y}\left(\tan^{-1} y - 1\right) c$$

$$\text{or } x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Problem 3.17.5. Solve: $xy\left(1+xy^2\right)\frac{dy}{dx} = 1.$ VTU July 2017

Solution: The given equation can be written as $\frac{dx}{dy} = xy\left(1+xy^2\right)$

$\frac{dx}{dy} = xy - x^2y^3$

$\frac{dx}{dy} - yx = y^3x^2$

Dividing by x^2, we have $x^{-2}\frac{dx}{dy} - yx^{-1} = y^3$ (1)

Putting $x^{-1} = z$ so that $-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$

or $x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$

Equation (1) becomes $-\frac{dz}{dy} - yz = y^3$

or $\frac{dz}{dy} + yz = -y^3,$

This is in the form $\frac{dz}{dy} + Pz = Q$

which is linear in z, with $P = y$ and $Q = -y^3$

$$\text{I.F.} = e^{\int P dy} = e^{\int y dy} = e^{\frac{1}{2}y^2}$$

$\therefore$ The solution is

$$z\,(I.F.) = \int Q\,(I.F.)dy + c$$

$$z \cdot e^{\frac{1}{2}y^2} = \int \left[-y^3 e^{\frac{1}{2}y^2}\right] dy + c$$

$$= -\int y^2 e^{\frac{1}{2}y^2}\; ydy + c \qquad \cdots(*)$$

Put $t = \frac{1}{2}y^2$ then $dt = \frac{1}{2}(2ydy) = ydy$ and $y^2 = 2t$

Now (*) becomes

$$z \cdot e^{\frac{1}{2}y^2} = -\int 2te^t dt + c,$$

$$= -2\left[te^t - \int 1 \cdot e^t dt\right] + c \qquad \text{(integrating by parts)}$$

$$= -2\left(te^t - e^t\right) + c$$

$$= -2e^t(t-1) + c$$

$$i.e.\; z \cdot e^{\frac{1}{2}y^2} = -2e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + c$$

$$\Rightarrow z = -2\left(\frac{1}{2}y^2 - 1\right) + ce^{-\frac{1}{2}y^2}$$

$$\Rightarrow \frac{1}{x} = 2 - y^2 + ce^{-\frac{1}{2}y^2}$$

Problem 3.17.6. Solve $(x + 2y^3)\frac{dy}{dx} = y$ (VTU July 2015)

Sol: The given equation is $(x + 2y^3)\,\frac{dy}{dx} = y$

$y \cdot \frac{dx}{dy} = x + 2y^3$

Divide by y, we get

$\frac{dx}{dy} = \frac{x}{y} + 2y^2$

$\frac{dx}{dy} - \frac{1}{y} \cdot x = 2y^2$

It is of the form $\frac{dx}{dy} + \mathrm{P}x = \mathrm{Q}$

Here $\mathrm{P} = -\frac{1}{y}, \mathrm{Q} = 2y^2$

$\therefore$ I.F. $= e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$

$\therefore$ The solution is

$$x\,(I.F.) = \int Q\,(I.F.)dy + c$$

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} dy + c$$

$$= \int 2y dy + c$$

$$= y^2 + c$$

$$\text{or} \quad x = y^3 + cy$$

Problem 3.17.7. Solve $rsin\theta - cos\theta\frac{dr}{d\theta} = r^2$ (VTU Jan 2020, June 2018, Model 2018)

Solution : The given equation can be written as

$-\frac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$

Dividing by $r^2\cos\theta$, we get

$-r^{-2}\frac{dr}{d\theta} + r^{-1}\tan\theta = \sec\theta$

Putting $r^{-1} = z$ so that $-r^{-2}\frac{dr}{d\theta} = \frac{dz}{d\theta}$ in the above equation, we get

$$\frac{dz}{d\theta} + z\tan\theta = \sec\theta$$

$$\text{I.F.} = e^{\int \tan\theta d\theta} = e^{\log\sec\theta} = \sec\theta$$

solution is given by

$$z\sec\theta = \int \sec\theta, \sec\theta + C \Rightarrow v\sec\theta = \int \sec^2\theta d\theta + C$$

$$i.e. \ \frac{\sec\theta}{r} = \tan\theta + C$$

$$\Rightarrow r^{-1} = (\sin\theta + C\cos\theta)$$

$$\therefore \quad r = \frac{1}{\sin\theta + C\cos\theta}$$

3.18 Nonlinear differential equations:Differential Equations of First order and higher Degree

We are already familiar with differential equations of the first order and first degree , Now we shall study differential equations of first order and degree higher than the first. We are familiar with the solution of differential equations (d.e.) of first order and first degree Now, we shall study the methods of solving differential equations of first order and higher degree. In addition to the general solution and particular solution associated with the D.E., we also introduce singular solution. The differential equations of first order but not of first degree are also branded as $p-y-x$ equations. If $y=f(x)$, we use the notation $\boxed{\dfrac{dy}{dx}=p}$ throughout this section. A differential equation of first order and nth degree is the form

$$A_0p^n + A_1p^{n-1} + A_2p^{n-2} + \cdots + A_n = 0 \tag{1}$$

Where $A_0, A_1, A_2, \cdots, A_n$ are functions of x and y. This being a differential equation of first order, the associated general solution will contain only one arbitrary constant.

3.19 Differential Equations solvable for p

Supposing that the LHS of (1) is expressed as a product of n linear factors, then the equivalent form of (1) is

$$(p - f_1(x,y))(p - f_2(x,y))\cdots(p - f_n(x,y)) = 0$$

Equating each of the factors to Zero, we have
$p = f_1(x,y), p = f_2(x,y), p = f_3(x,y), \cdots p = f_n(x,y)$
All these are differential equations of first order and first degree. They can be solved by the known methods.
Solving each of these equations of the first order of first degree, we get the solutions

$$F_1(x,y,c) = 0, F_2(x,y,c) = 0, \cdots F_n(x,y,c) = 0$$

respectively.
General solution of (1) may be written as the product of all these solutions. i.e.

$$\boxed{F_1(x,y,c).F_2(x,y,c).F_3(x,y,c)\cdots F_n(x,y,c) = 0}$$

Note : We need to present the general solution with the same arbitrary constant in each factor.

Problem 3.19.1. Solve the equation $p^2 + p(x+y) + xy = 0$ (VTU Jan 2013, Dec 2014)

Solution: The equation can be written as $p^2 + px + py + xy = 0$
i.e. $p(p+x) + y(p+x) = 0$
i.e. $(p+y)(p+x) = 0$
i.e. $p = -y$ and $p = -x$
Solving $p = -y \implies \frac{dy}{dx} = -y$

$i.e. \frac{dy}{y} = -dx$

Integrating on both sides

$logy = -x + c$

i.e. $logy + x - c = 0$ (1)

Similarly Consider $p = -x \implies \frac{dy}{dx} = -x$

i.e. $dy = -xdx$

Integrating,

$y = -\frac{x^2}{2} + c$

i.e. $y + \frac{x^2}{2} - c = 0$ (2)

General Solution is obtaied by multiplying the solutions (1) and (2).

Hence the General Solution is , $\boxed{(logy + x - c)(y + \frac{x^2}{2} - c) = 0}$

Problem 3.19.2. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ (VTU Model ME 2022, June 2018, July 2017, Jan 2016, June 2015, Jan 2016)

Solution : $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

$$p - \frac{1}{p} = \frac{x^2 - y^2}{xy}, \text{ where } p = \frac{dy}{dx}$$

$$xyp^2 - \left(x^2 - y^2\right) p - xy = 0$$

$$p = \frac{(x^2 - y^2) \pm \sqrt{\left(x^2 - y^2\right)^2 + 4x^2y^2}}{2xy}$$

$$= \frac{(x^2 - y^2) \pm (x^2 + y^2)}{2xy}$$

or

$$p = \frac{x}{y} \quad ; \quad p = -\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y} \quad ; \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$ydy = xdx \quad ; \quad \frac{1}{y}dy = -\frac{1}{x}dx$$

Integrating both sides

$$\frac{y^2}{2} = \frac{x^2}{2} + c; \qquad \log y = -\log x + c$$

$$\text{or} \quad y^2 - x^2 - 2c = 0 \quad \text{or} \quad \log xy = c \quad \text{or} \quad xy = e^c$$

$\therefore$ The general solution of given equation is $(y^2 - x^2 - 2c)\left(xy - e^c\right) = 0$.

Problem 3.19.3. $xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\frac{dy}{dx} + xy = 0$ (VTU Model 2022, Jan 2019, Jan 2018) or

$xy\left\{\left(\frac{dy}{dx}\right)^2+1\right\}=(x^2+y^2)\frac{dy}{dx}$ (VTU June 2018)

Solution :

$$xy(dy/dx)^2-\left(x^2+y^2\right)dy/dx+xy=0$$

$$\text{Put } dy/dx=p$$

$$xy\cdot p^2-\left(x^2+y^2\right)p+xy=0$$

This is a quadratic equation in powers of p.

$$\therefore p=\frac{(x^2+y^2)\pm\sqrt{(x^2+y^2)^2-4(xy)(xy)}}{2xy}$$

$$p=\frac{(x^2+y^2)\pm\sqrt{x^4+y^4+2x^2y^2-4x^2y^2}}{2xy}$$

$$=\frac{(x^2+y^2)\pm\sqrt{x^4+y^4-2x^2y^2}}{2xy}$$

$$=\frac{(x^2+y^2)\pm(x^2-y^2)}{2xy}$$

$$\Rightarrow p=\frac{x^2+y^2+x^2-y^2}{2xy}=\frac{2x^2}{2xy}=\frac{x}{y}$$

$$\&p=\frac{x^2+y^2-x^2+y^2}{2xy}=\frac{2y^2}{2xy}=\frac{y}{x}$$

i.e. values of p are

$\frac{dy}{dx}=\frac{x}{y}$ (1)

and $\frac{dy}{dx}=\frac{y}{x}$ (2)

Solution of (1) :

$$\frac{dy}{dx}=\frac{x}{y}$$

$$ydy=xdx$$

$$\int ydy=\int xdx$$

$$\frac{y^2}{2}=\frac{x^2}{2}+c_1$$

$$y^2=x^2+2c_1$$

$$y^2-x^2-2c_1=0 \qquad \cdots(3)$$

Solution of (2) :

$$\frac{dy}{dx} = \frac{y}{x}$$
$$\frac{dy}{y} = \frac{dx}{x}$$
$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\log y = \log x + \log c$$
$$= \log(cx)$$
$$y = cx$$
$$y - cx = 0 \qquad \cdots (4)$$

$\therefore$ The general solution of given equation is

$(y^2 - x^2 - 2c_1)(y - cx) = 0$

Problem 3.19.4. Solve $p^2 + 2pycotx = y^2$. (VTU Model 2022, Jan 2020, July 2016, June 2012)

Solution : Given $p^2 + 2pycotx = y^2$

Adding y^2cot^2x on both sides, The above equation can be written as

$(p + y\cot x)^2 = y^2\left(1 + \cot^2 x\right)$

or $\quad p + y\cot x = \pm y \operatorname{cosec} x$

$\therefore$ The component equations are $p = y(-\cot x + \operatorname{cosec} x)$ (1)

and $p = y(-\cot x - \operatorname{cosec} x)$ (2)

From (1),

$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$

or $\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)dx$

Integrating

$$\log y = -\log\sin x + \log\tan\frac{x}{2} + \log c$$
$$= \log\frac{c\tan\frac{x}{2}}{\sin x}$$
$$y = \frac{c\tan\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$= \frac{c}{2\cos^2\frac{x}{2}}$$
$$or\ y\cos^2\frac{x}{2} = c_1, \quad \text{where } c_1 = \frac{c}{2}$$

From (2), $\frac{dy}{dx} = y\left(-\cot x - cosecx\right)$

or $\frac{dy}{y} = (-\cot x - cosecx)dx$

Integrating

$$\log y = -\log\sin x - \log\tan\frac{x}{2} + \log c$$
$$= \log\frac{c}{\sin x\tan\frac{x}{2}}$$

or

$$y = \frac{c}{2\sin^2\frac{x}{2}} \text{ or } y\sin^2\frac{x}{2} = \text{C}$$

$\therefore$ The general solution of the given equation is $\left(y\cos^2\frac{x}{2} - C\right)\left(y\sin^2\frac{x}{2} - C\right) = 0$

Problem 3.19.5. Solve $p(p+y) = x(x+y)$ (VTU Dec 2014, July 2011, Dec 2011)

Solution: Given equation is

$$p^2 + py = x^2 + xy$$

or

$$p^2 + py - \left(x^2 + xy\right) = 0, \text{ which is quadratic in } p$$

$$\therefore \quad p = \frac{-y \pm \sqrt{y^2 + 4\left(x^2 + xy\right)}}{2}$$
$$= \frac{-y \pm \sqrt{(y+2x)^2}}{2}$$
$$\therefore \quad p = \frac{-y + y + 2x}{2} \quad or \quad p = \frac{-y - y - 2x}{2}$$

Thus $p = x$ (1)

or $p = -y - x$ (2)

From (1), $\frac{dy}{dx} = x$ or $dy = xdx$

Integrating both sides,

$$y = \frac{x^2}{2} + c$$
$$\text{or} \quad y - \frac{x^2}{2} - c = 0 \qquad \cdots (3)$$

and From (2)

$$\frac{dy}{dx} = -y - x$$
$$or \ \frac{dy}{dx} + y = -x$$

which is linear equation in y with $P = 1$ and $Q = -x$

$$\text{I.F.} = e^{\int Pdx} = \int e^x dx = e^x$$

$\therefore$ Its solution is

$$y(I.F.) = \int Q\,(I.F.)dx + c$$

$$\Rightarrow ye^x = \int (-x)e^x + c$$

$$= (-x)e^x - \int (-1)e^x dx + c \text{ [Integrating by parts]}$$

$$= -xe^x + e^x + c$$

$$= (-x+1)e^x + c$$

$$y = -x + 1 + ce^{-x}$$

$$\text{or } y + x - 1 - ce^{-x} = 0 \qquad \cdots(4)$$

Combining (3) and (4), general solution is

$$\left(y - \frac{x^2}{2} - c\right)\left(y + x - 1 - ce^{-x}\right) = 0.$$

3.20 Clairaut's equation

The equation of the form

$$\boxed{y = px + f(p)} \qquad (1)$$

is known as Clairaut's equation.

Using $p = c$ in (1) we obtain the general solution of clairaut's equation in the form

$$y = cx + f(c)$$

Differentiating the general solution partially w.r.to c and eliminating the constant c, we get the singular solution.

Problem 3.20.1. Solve $p = \log(px - y)$

Solution :

or $\quad e^p = px - y \quad$ or $y = px - e^p$, which is Clairaut's equation where $f(p) = -e^p$

$\therefore \quad$ Its solution is obtained by putting $p = c$

$\therefore \quad$ solution is $y = cx - e^c$.

Problem 3.20.2. Solve $\sin px \cos y = \cos px \sin y + p$. Also find the singular solution. (Jan 2014)

Solution :

$$\text{Given} \quad \sin px \cos y - \cos px \sin y = p$$
$$\text{or} \quad \sin(px - y) = p$$
$$\text{or} \quad px - y = \sin^{-1} p$$
$$\text{or} \quad y = px - \sin^{-1} p,$$

this is in the form $y = px + f(p)$ which is Clairaut's form

Solution is obtained by replacing p by c.

Hence the General solution is

$$y = cx + sin^{-1}c \qquad (*)$$

Differentiating equation (*) partially w.r.to c

$0 = x + \frac{1}{\sqrt{1-c^2}}$

$\frac{1}{\sqrt{1-c^2}} = -x$

$1 = -x\sqrt{1-c^2}$

$1 = x^2(1-c^2)$

$1 - c^2 = \frac{1}{x^2}$

$c^2 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$

$c = \frac{\sqrt{x^2-1}}{x}$

Substitute for c in (*) we get, $y = \frac{\sqrt{x^2-1}}{x}x + sin^{-1}\frac{\sqrt{x^2-1}}{x}$

Problem 3.20.3. Obtain the general solution and singular solution of the Clairaut's equation $xp^3 - yp^2 + 1 = 0$. (VTU July 2017, Jan 2015,June 2014, Dec 2011)

Solution: The given equation can be written as

$$y = \frac{xp^3 + 1}{p^2}$$
$$\Rightarrow y = px + \frac{1}{p^2} \text{ which is in the Clairaut's form } y = px + f(p)$$

Thus general solution is given by $y = cx + \frac{1}{c^2}$ (*)

Differentiating partially w.r.to c we get

$$0 = x + \frac{-2}{c^3}$$
$$\Rightarrow \frac{2}{c^3} = x$$
$$\Rightarrow c^3 = \frac{2}{x}$$
$$\Rightarrow c = \left(\frac{2}{x}\right)^{1/3}$$

Thus general solution (*) becomes

$$y = \left(\frac{2}{x}\right)^{1/3} x + \left(\frac{x}{2}\right)^{2/3}$$
$$= 2^{1/3} x^{1-1/3} + \frac{x^{2/3}}{2^{2/3}}$$
$$= x^{2/3}\left(2^{1/3} + \frac{1}{2^{2/3}}\right)$$
$$= x^{2/3}\left(\frac{2+1}{2^{2/3}}\right)$$
$$2^{2/3} y = 3x^{2/3}$$
$$4^{1/3} y = 3(x^2)^{1/3}$$

Cubing both sides, we get

$$4y^3 = 27x^2$$

This is the singular solution of the given Clairaut's equation.

Problem 3.20.4. Find the general and singular solution for $xp^2 + xp - yp + 1 - y = 0$ (VTU June 2018)

Solution :

$$xp^2 + px - py + 1 - y = 0$$
$$xp^2 + px + 1 = y(p+1)$$
$$y = \frac{xp^2 + px + 1}{(p+1)}$$
$$\Rightarrow y = \frac{xp(p+1) + 1}{(p+1)}$$

$y = px + \frac{1}{p+1}$

G.S. is given by putting $p = c$

Hence G.S. is $y = cx + \frac{1}{c+1}$ (1)

Diff. partially w.r.t. c,

$0 = x - \frac{1}{(c+1)^2}$

or $\frac{1}{(c+1)^2} = x$

or $(c+1)^2 = \frac{1}{x}$

$(c+1) = \frac{1}{\sqrt{x}}$

or $\quad c = \frac{1}{\sqrt{x}} - 1$

Hence (1) becomes

$y = \left(\frac{1}{\sqrt{x}} - 1\right) x + \sqrt{x}$

$\Rightarrow y = \left(\frac{1-\sqrt{x}}{\sqrt{x}}\right) x + \sqrt{x}$

$$y = (1-\sqrt{x})\sqrt{x} + \sqrt{x}$$

$$\Rightarrow y = \sqrt{x} - x + \sqrt{x}$$

$$y = 2\sqrt{x} - x$$

$$\Rightarrow x + y = 2\sqrt{x}$$

squaring

$$(x+y)^2 = 4x$$

Reducible to Clairaut's equations

Problem 3.20.5. Solve: $(px-y)(py+x) = 2p$ by reducing into Clairaut's form,taking the substitution $X = x^2, Y = y^2$.

Solution : Put $\mathrm{X} = x^2$ and $\mathrm{Y} = y^2$

so that $d\mathrm{X} = 2xdx$ and $d\mathrm{Y} = 2ydy$

$i.e.\ dx = \frac{dX}{2x}, \quad dy = \frac{dY}{2y}$

$\therefore \quad p = \frac{dy}{dx} = \frac{x}{y}\frac{dY}{dX} = \frac{\sqrt{X}}{\sqrt{Y}}\mathrm{P},$

where $\mathrm{P} = \frac{dY}{dX}$

The given equation becomes

$\left(\frac{\sqrt{X}}{\sqrt{Y}}\mathrm{P}\cdot\sqrt{\mathrm{X}} - \sqrt{\mathrm{Y}}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}\mathrm{P}\cdot\sqrt{\mathrm{Y}} + \sqrt{\mathrm{X}}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}\mathrm{P}$

or $\quad (\mathrm{PX} - \mathrm{Y})(\mathrm{P}+1) = 2\mathrm{P}$

or $\mathrm{PX} - \mathrm{Y} = \frac{2\mathrm{P}}{\mathrm{P}+1}$

or $\quad Y = PX - \frac{2P}{P+1}$, which is of Clairaut's form.

$\therefore$

Its solution is $Y = cX - \frac{2c}{c+1}$
and hence $y^2 = cx^2 - \frac{2c}{c+1}$.

Problem 3.20.6. Solve $(px - y)(py + x) = a^2p$, use the substitution $X = x^2, Y = y^2$. (VTU Jan 2018, July 2017, June 2014, June 2015, Dec 2011)

Solution:

Solution: Given $(px - y)(py + x) = a^2p$ (1)

$$X = x^2 \Rightarrow \frac{dX}{dx} = 2x \Rightarrow dX = 2xdx$$
$$Y = y^2 \Rightarrow \frac{dY}{dy} = 2y \Rightarrow dY = 2ydy$$
$$\frac{dY}{dX} = \frac{ydy}{xdx} = \frac{y}{x}\frac{dy}{dx}$$
$$P = \frac{\sqrt{Y}}{\sqrt{X}}p$$
$$\text{or } p = \frac{\sqrt{X}}{\sqrt{Y}}P$$

where $P = \frac{dY}{dX}, \ \ p = \frac{dy}{dx}$

Substitute for P, x and y in the given equation, i.e. $(px - y)(py + x) = a^2p$ we get

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y} + \sqrt{X}\right) = a^2\frac{\sqrt{X}}{\sqrt{Y}}P$$
$$\left(\frac{XP - Y}{\sqrt{Y}}\right)\left(\sqrt{X}P + \sqrt{X}\right) = a^2\frac{\sqrt{X}}{\sqrt{Y}}P$$

Multiplying both sides by $\sqrt{Y}$ we get

$$(XP - Y)\left(\sqrt{X}P + \sqrt{X}\right) = a^2P\sqrt{X}$$
$$(XP - Y)(P + 1)\sqrt{X} = a^2P\sqrt{X}$$
$$(XP - Y)(P + 1) = a^2P$$
$$(XP - Y) = \frac{a^2P}{P + 1}$$
$$\text{or } Y = XP - \frac{a^2P}{P + 1}$$

This is in the Clairaut's form and hence the associated genertal solution is $Y = Xc - \frac{a^2c}{c+1}$

Resubstituting for X and Y we get

$y^2 = cx^2 - \frac{a^2c}{c+1}$

Problem 3.20.7. Find the general and singular solution of the equation $x^2(y - px) = p^2y$ by reducing into Clairaut's equation using the substitution $X = x^2, Y = y^2$ (VTU July 2016)

SOlution :

$$X = x^2 \Rightarrow \frac{dX}{dx} = 2x \Rightarrow dX = 2xdx$$
$$Y = y^2 \Rightarrow \frac{dY}{dy} = 2y \Rightarrow dY = 2ydy$$
$$\frac{dY}{dX} = \frac{ydy}{xdx} = \frac{y}{x}\frac{dy}{dx}$$
$$P = \frac{\sqrt{Y}}{\sqrt{X}}p$$
$$\text{or } p = \frac{\sqrt{X}}{\sqrt{Y}}P$$

where $P = \frac{dY}{dX}, \;\; p = \frac{dy}{dx}$

Substitute for P, x and y in the given equation, i.e. $x^2(y - px) = p^2y$

$$X\left[\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} \cdot P \cdot \sqrt{X}\right] = \frac{X}{Y}P^2\sqrt{Y}$$
$$X\left[\frac{Y - PX}{\sqrt{Y}}\right] = \frac{X}{\sqrt{Y}} \cdot P^2$$

$\Rightarrow Y - PX = P^2$

$\Rightarrow Y = PX + P^2$ Which is Clairaut's form.

Its solution is

$$Y = cX + c^2$$
$$y^2 = cx^2 + c^2 \qquad \cdots (*)$$

Diff. partially w.r.to c

$$0 = x^2 + 2c$$
$$c = -\frac{x^2}{2}$$

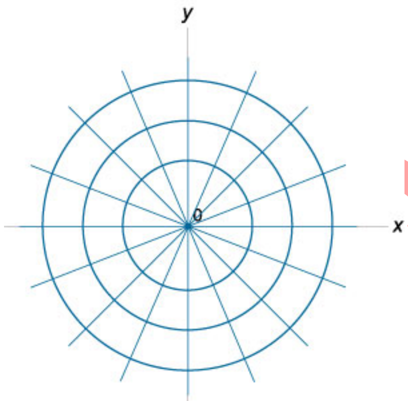
Hence (*) becomes

$$y^2 = -\frac{x^2}{2}x^2 + \left(-\frac{x^2}{2}\right)^2$$
$$y^2 = -\frac{x^4}{2} + \frac{x^4}{4}$$
$$= -\frac{x^4}{4}$$
$$4y^2 = -x^4$$
$$x^4 + 4y^2 = 0$$

3.21 Orthogonal trajectories

Definition: If two family of curves are such that every member of one family intersect every member of the other family at right angles then they are said to be orthogonal trajectories of each other.

Example : the orthogonal trajectory of the family of straight lines defined by the equation $y = kx$, where k is a parameter(the slope of the straight line), is the family of circle having centre at the origin (Figure 1)



Note :

- If $y = f(x)$ be a curve then $\frac{dy}{dx} = m$ = slope of the tangent.
- Two curves intersect each other orthogonally, if the tangents at point of intersection are at right angles.
- Condition for orthogonality of two curves is given by$m_1 m_2 = -1$ where m_1 and m_2 are slopes of tangents.
- If each member of a family cuts every other member of the same family orthogonally, then we say that family of curves is self-orthogonal.

Rules to find the equation of orthogonal trajectories of a family of cartesian curves: Given a relation in the form $f(x, y, c) = 0$ (1)

Step 1. Diff (1) and eliminate c, which gives the differential equation of the given family (1).

Step 2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ which gives the differential equation of orthogonal trajectory(OT).

Step 3. Solve this new equation to get the required orthogonal trajectory.

Rules to find the equation of orthogonal trajectories of a family of Polar curves :

Given a polar family of curves in the form $f(r, \theta, c) = 0$ (1)

(i) Differentiate (1) and eliminate c which gives the differential equation of the given family (1).

(ii) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ which gives the differential equation of orthogonal trajectory(OT).

(iii) Solve this new equation to get the required orthogonal trajectory.

Problem 3.21.1. Find the Orthogonal trajectories of the family of parabola $y^2 = 4ax$ (VTU Jan 2020)

Solution :

$$y^2 = 4ax$$

$$\frac{y^2}{x} = 4a$$

Differentiate w.r.to x

$$\frac{x\ 2y\frac{dy}{dx} - y^2 \cdot 1}{x^2} = 0$$

$$2xy\frac{dy}{dx} - y^2 = 0$$

$$\frac{dy}{dx} = \frac{y}{2x}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to get the D.E. of Orthogonal trajectory in the form

$$\frac{-dx}{dy} = \frac{y}{2x}$$

$$\frac{dy}{dx} = -\frac{2x}{y} \quad \text{(by rearranging the terms)}$$

$$-ydy = 2xdx$$

$$2xdx + ydy = 0$$

integrating both sides

$$\int 2xdx + \int ydy = \int 0$$

$$x^2 + \frac{y^2}{2} = C$$

$$2x^2 + y^2 = 2C$$

$$2x^2 + y^2 = c_1, \quad \text{where } c_1 = 2c$$

Problem 3.21.2. Find the Orthogonal trajectories of the family of confocal ellipses $\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1$, where λ is a parameter . (VTU July 2021, Model 2018, July 2015, Dec 2011, Jule 2011, June 2009, Aug 2000, Mar 2000, Aug 1999)

Solution : Let $\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1$ (1)

Differentiate (1) w.r.to x, we get

$$\frac{2x}{a^2}+\frac{2y}{(b^2+\lambda)}\frac{dy}{dx} = 0$$

$$\implies \frac{x}{a^2}+\frac{y}{(b^2+\lambda)}\frac{dy}{dx} = 0$$

$$\implies \frac{x}{a^2} = \frac{-y\frac{dy}{dx}}{(b^2+\lambda)} \qquad \cdots (2)$$

$$\text{From (1) we have } \frac{x^2}{a^2} - 1 = \frac{-y^2}{(b^2+\lambda)}$$

$$\implies \frac{x^2-a^2}{a^2} = \frac{-y^2}{(b^2+\lambda)} \qquad \cdots (3)$$

Dividing (2) from (3) we obtain,

$$\frac{x}{x^2-a^2} = \frac{1}{y}\frac{dy}{dx}$$

$$\text{Replace}\frac{dy}{dx}\text{by } -\frac{dx}{dy},$$

$$\frac{x}{x^2-a^2} = \frac{-1}{y}\frac{dx}{dy}$$

$$i.e\ \ ydy = -\left(\frac{x^2-a^2}{x}\right)dx$$

$$i.e\ \ ydy = \left(\frac{a^2}{x} - x\right)dx$$

Integrating on both sides

$$\int ydy = \int\frac{a^2}{x} - \int xdx$$

$$\frac{y^2}{2} = a^2 log x - \frac{x^2}{2} + c$$

$$\frac{y^2}{2}+\frac{x^2}{2} = a^2 log x + c$$

Problem 3.21.3. Show that the family of confocal parabolas having x-axis as their axis, is of the form $y^2 = 4a(x+a)$ is self orthogonal (VTU Jan 2021, June 2019, June 2018, Jun 2013)

Solution : The given equation is $y^2 = 4a(x+a)$ (1)

Differentiating w.r. to x,

$y\frac{dy}{dx} = 2a$ (2)

Substituting the value of a from (2) in (1), we get

$y^2 = 2y\frac{dy}{dx}\left(x + \frac{1}{2}y\frac{dy}{dx}\right)$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \quad (3)$$

This is the differential equation of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we obtain

$y\left(\frac{dx}{dy}\right)^2 - 2x\frac{dx}{dy} - y = 0$

Multiplying both sides by $\left(\frac{dy}{dx}\right)^2$, we get

$y - 2x\frac{dy}{dx} - y\left(\frac{dy}{dx}\right)^2 = 0$

Multiplying by -1 and rearranging the terms we get

$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ which is the same as (3).

Thus we see that a system of confocal and coaxial parabolas is self-orthogonal,

i.e., each member of the family cuts every other member of the same family orthogonally.

Problem 3.21.4. Find the orthogonal trajectories of the confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1,\ \lambda$ being the parameter. (VTU July 2017, Jan 2016, Jan 2015, Jan 2013)

Solution: Solution: The given family is $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1.$ (1)

Differentiating (1) with respect to x,

$\frac{2x}{a^2+\lambda} + \frac{2yy_1}{b^2+\lambda} = 0$

$x\left(b^2 + \lambda\right) + yy_1\left(a^2 + \lambda\right) = 0$

$$\lambda\left(x + yy_1\right) = -\left(b^2x + a^2yy_1\right) \Rightarrow \lambda = -\frac{\left(b^2x + a^2yy_1\right)}{\left(x + yy_1\right)} \quad (2)$$

$$\text{Now,} \quad \left(a^2 + \lambda\right) = a^2 - \frac{\left(b^2x + a^2yy_1\right)}{\left(x + yy_1\right)} = \frac{\left(a^2 - b^2\right)x}{\left(x + yy_1\right)} \quad (3)$$

$$\text{Similarly } \left(b^2 + \lambda\right) = b^2 - \frac{\left(b^2x + a^2yy_1\right)}{\left(x + yy_1\right)} = -\frac{\left(a^2 - b^2\right)yy_1}{\left(x + yy_1\right)} \quad (4)$$

On using results (3) and (4) in (1), we get

$$\frac{x\left(x + yy_1\right)}{\left(a^2 - b^2\right)} + \frac{y\left(x + yy_1\right)}{-\left(a^2 - b^2\right)y_1} = 1$$

$$\frac{\left(x + yy_1\right)}{\left(a^2 - b^2\right)}\left[x - \frac{y}{y_1}\right] = 1 \quad (5)$$

Equation (5) is the differential equation of the given family.

In order to find the differential equation of the desired orthogonal trajectory, replace y_1 by $-\frac{1}{y_1}$ in (5) i.e.

$$\frac{\left(x - \frac{y}{y_1}\right)}{\left(a^2 - b^2\right)}\left[x + yy_1\right] = 1 \quad (6)$$

Clearly equations (5) and (6) are the same.

Hence the given system is self orthogonal.

Problem 3.21.5. Find the O.T of the family of astroids $x^{2/3} + y^{2/3} = a^{2/3}$

Solution : Consider $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w.r.to x, we have

$$\frac{2}{3} \cdot x^{-1/3} + \frac{2}{3} \cdot y^{-1/3}\frac{dy}{dx} = 0$$

ie.,

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

. This is the D.E of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0$$

ie., $x^{-1/3}dy = y^{-1/3}dx$

$$y^{1/3}dy = x^{1/3}dx$$

by separating the variables.

$$\Rightarrow \int y^{3/3}dy - \int x^{1/3}dx = c$$

$$\Rightarrow \frac{y^{4/3}}{(4/3)} - \frac{x^{4/3}}{(4/3)} = c$$

or

$$x^{4/3} - y^{4/3} = -\frac{4c}{3} = k\ (say)$$

Thus $x^{4/3} - y^{4/3} = k$ is the required O.T.

Problem 3.21.6. Find the Orthogonal trajectories of the cardioid $r = a(1 - cos\theta)$. (VTU Model 2022, Jan 2015, Jan 2014, Mar 2001, Jul 2007, Jan 2011)

Solution : Let $r = a(1 - cos\theta)$ (1)

Apply log on both sides

$logr = loga + log(1 - cos\theta)$ (2) differentiate (1) w.r.to θ, we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-cos\theta}(sin\theta)$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we get,

$-r\frac{d\theta}{dr} = \frac{1}{1-cos\theta}(sin\theta)$

$i.e.\frac{1-cos\theta}{(sin\theta)}d\theta = \frac{-dr}{r}$

$tan\frac{\theta}{2}d\theta = \frac{-dr}{r}$(This is the differential equation of the orthogonal trajectory)

Integrating on both sides

$\frac{-log\left(cos\frac{\theta}{2}\right)}{\frac{1}{2}} = -\log r - logc$

$i.e.2log\left(cos\frac{\theta}{2}\right) = \log r + logc = log(rc)$

$i.e.cos^2\frac{\theta}{2} = rc$ or $r = \frac{1}{c}cos^2\frac{\theta}{2} = \frac{1}{2c}(1+cos\theta)$

$i.e.r = b(1+cos\theta)$ where $b = \frac{1}{2c}$ and

Problem 3.21.7. Find the Orthogonal trajectories of $r^2 = a^2cos2\theta$ (VTU Jun 2014, Jan 2009, Aug 2003)

Solution: Given $r^2 = a^2\cos 2\theta$ (1)

On differentiating (1), we have

$$r\frac{dr}{d\theta} = -a^2\sin 2\theta \quad (2)$$

Eliminate 'a' from (2), using (1)

$$r\frac{dr}{d\theta} = -r^2\tan 2\theta$$

i.e. $\frac{dr}{d\theta} = -r\tan 2\theta$ (3)

which is the differential equation of the given family.

To obtain the differential of the trajectories, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in equation (3).

i.e. $-r^2\frac{d\theta}{dr} = -r\tan 2\theta$

or $\cot 2\theta d\theta = \frac{dr}{r}$ (4) Integration of (4), gives the equation of the desired system.

$\therefore$ $\frac{\log\sin 2\theta}{2} + \log c = \log r$

$\log(\sin 2\theta) + 2\log(c) = 2\log r$

i.e. $\log(\sin 2\theta) + \log(c^2) = \log r^2$

i.e. $\log(c^2\sin 2\theta) = \log(r^2)$

i.e. $c^2\sin 2\theta = r^2$

Problem 3.21.8. Find the Orthogonal trajectories of $r^n = a^n cosn\theta$ (VTU July 2016, Feb 2003)

Solution :

$$r^n = a^n \cos n\theta$$

Diff. w.r.t. to θ

$$nr^{n-1}\frac{dr}{d\theta} = -a^n \sin n\theta(n)$$
$$\frac{r^n}{r}\frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$
$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left[-r^2\frac{d\theta}{dr}\right] = -\tan n\theta$$
$$\frac{1}{r}dr = \frac{1}{\tan n\theta}d\theta$$

Integrating on both sides

$$\int \frac{1}{r}dr = \int \frac{1}{\tan n\theta}d\theta$$
$$\log r = \int \cot n\theta d\theta$$
$$\log r = \frac{\log(\sin n\theta)}{n} + \log c$$
$$n\log r = \log \sin n\theta + n\log c$$
$$\log r^n = \log(\sin n\theta) + \log c^n$$
$$\log r^n = \log[\sin n\theta \ \ c^n]$$
$$r^n = c^n \sin n\theta$$

Problem 3.21.9. Find the Orthogonal trajectories of $r^n = a^n sinn\theta$ (VTU July 2017)

SOlution :

$$\frac{dr}{d\theta} n r^{n-1} = a^n (\cos n\theta) n$$
$$\frac{r^n}{r}\frac{dr}{d\theta} = a^n \cos n\theta$$
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$
$$\frac{1}{r}\frac{dr}{d\theta} = \text{cotn}\,\theta$$
$$\text{Replace } \left(\frac{dr}{d\theta}\right) = -\frac{r^2 d\theta}{dr}$$
$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot n\theta$$

$\frac{-1}{r\ dr} = \frac{1}{\cot n\theta} d\theta$

Integrating on both sides

$\int \frac{-1}{r} dr = \int \tan n\theta d\theta$

$\Rightarrow -\log r = \frac{\log(\sec n\theta)}{n} - \log c$

$\Rightarrow -n \log r = \log \sec n\theta - n \log c$

$\Rightarrow -\log r^n = \log \sec n\theta - \log c^n$

$\Rightarrow \log c^n - \log r^n = \log \sec n\theta$

$\Rightarrow \log\left(\frac{c^n}{r^n}\right) = \log(\sec n\theta)$

$\Rightarrow r^n = \frac{c^n}{\sec n\theta}$

$\Rightarrow r^n = c^n \cos n\theta$

L-R and C-R circuits

Generally, an electrical circuit is made up of a combination of resistors, inductors and capacitors. If a circuit is made up of resistors and capacitors, then it is known as an **RC circuit**. Similarly, when the circuit is containing capacitors and inductors, then it is known as an LC circuit. An **LR circuit** is a circuit which is made up of resistors and inductors.

Let $i = I(t)$ denote the current in the circuit and $q = q(t)$ denote the charge on the capacitor. Furthermore, let L denote inductance in henrys (H), R denote resistance in ohms (Ω) , and C denote capacitance in farads (F). Last, let E(t) denote electric potential in volts (V).

The formation of differential equation for an electric circuit depends upon the following laws.

(i) $i = \frac{dq}{dt}$,

(ii) Voltage drop across resistance, R is $V_R = Ri$ (Ohm's law)

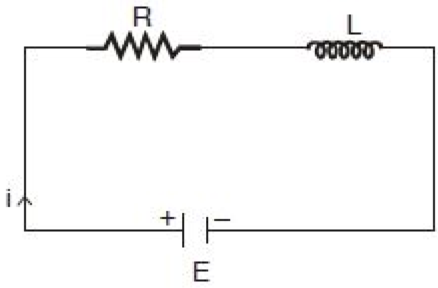
(iii) Voltage drop across inductance L is $V_L = L \cdot \frac{di}{dt}$ (Faraday's law and Lenza's law)

(iv) Voltage drop across capacitance C is, $V_C = \frac{q}{C}$

Kirchhoff's Voltage law :

The algebraic sum of the voltage drop around any closed circuit is equal to the resultant electromotive force in the circuit.

L - R series circuit :



Let i be the current flowing in the circuit containing resistance R and inductance L in series, with voltage source E, at any time t.

By voltage law

$$Ri + L\frac{di}{dt} = \text{E}$$
$$\Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \cdots$$

This is the linear differential equation of the form $\frac{di}{dt} + P\,i = Q$

$$\text{I.F.} = e^{\int P dt} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Its solution is

$$i \times (IF) = \int Q \times (IF)\, dt + c$$

$$\Rightarrow i \times e^{\frac{R}{L}t} = \int \frac{E}{L} \times e^{\frac{R}{L}t}\, dt + c$$

$$= \frac{E}{L} \times \frac{L}{R} e^{\frac{R}{L}t} + c$$

Dividing by $e^{\frac{R}{L}t}$

$$\Rightarrow \quad i = \frac{E}{R} + ce^{-\frac{Rt}{L}}$$

This represents the current in the circuit, at any time t

At $t = 0, \quad i = 0 \Rightarrow c = -\frac{E}{R}$

Thus, above current i becomes

$$i = \frac{E}{R}\left[1 - e^{-\frac{Rt}{L}}\right]$$

Growth of Current

An LR Circuit is analysed in three ways. The first one is the initial state, which is present at the instant of closing the switch or opening the switch in the circuit. The second one is the transient state, which appears at any instant after closing or opening the switch. The third one is **steady-state**, which appears after a long time after closing and opening the switch.

Let's start with the initial and steady states of an LR circuit.

Initial State : Let us assume a circuit of EMF E has the inductance L and the resistance R, as shown in the figure.

The voltage drop across the inductor is V_L and the voltage drop across the resistor is V_R.

At $t = 0$, the inductor offers an infinite opposition to the current flow and hence there is no current flow in the circuit at the time of closing the switch. Due to high opposition to the current flow, the voltage is dropped entirely at the inductor and there is no voltage drop across the resistor.

i.e., at $t = 0$, $V_R = 0$ and $V_L = E$.

Steady State: At a certain point of time, say $t = \infty$, the current in the inductor does not vary with time after closing or opening the switch for a long period of time. We can see that the current has reached its maximum value and therefore the inductor does not offer any position to the current flow. So, the voltage drop across the inductor becomes zero and the entire voltage drops across a resistor.

At $t = \infty$, $V_R = E$ and $V_L = 0$

Transient State : In this state, the voltage is dropped both across the resistor and the inductor. At any instant $t = 0$ and $t = \infty$ is taken for this state. We know that the voltage drop across the inductor is equal to the inductance multiplied by the rate of change in current across the inductor.

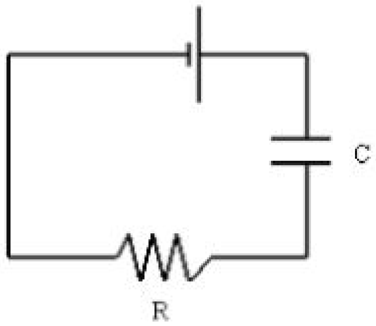
i.e., $V_L = L\frac{dI}{dt}$

And the voltage across the resistor is given by $V_R = IR$.

In the transient state, when the switch is closed gradually, the current starts increasing across the inductor. Due to the increase in the current, there will be a self-induced EMF in the inductor which opposes the change of the current in the circuit.

C $-$ R series circuit :

Let i be current in the circuit containing resistance R, L, and capacitance C in series with voltage source E, at any time t.

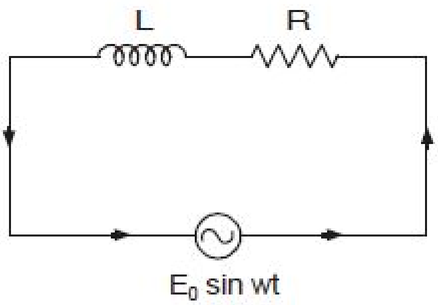


By voltage law

$$Ri + \frac{q}{C} = E$$
$$\Rightarrow \quad R\frac{dq}{dt} + \frac{q}{C} = E \quad \left[\because i = \frac{dq}{dt}\right]$$

Problem 3.21.10. Solve the equation $L\frac{di}{dt} + Ri = E_0 \sin wt$ where L, R and E_0 are constants and discuss the case when t increases indefinitely. (VTU July 2021, Jan 2021)

Solution :



$L\frac{di}{dt} + Ri = E_0 \sin wt$

$$\Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\sin wt$$

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

$$\text{i. } e^{\frac{R}{L}t} = \frac{E_0}{L}\int e^{\frac{R}{L}t}\sin wt dt + c$$

$$\left[\int e^{ax}\sin bx dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right)\right]$$

$$\Rightarrow \quad i\, e^{\frac{R}{L}t} = \frac{E_0}{L}\frac{e^{\frac{R}{L}t}}{\sqrt{\frac{R^2}{L^2}+w^2}}\sin\left(wt - \tan^{-1}\frac{Lw}{R}\right) + c$$

$$i = \frac{E_0}{\sqrt{R^2+L^2w^2}}\sin\left(wt - \tan^{-1}\frac{Lw}{R}\right) + ce^{-\frac{R}{L}t}$$

As t increases indefinitely, then $ce^{-\frac{Rt}{L}}$ tends to zero. SO

$$i = \frac{E_0}{\sqrt{R^2+L^2w^2}}\sin\left(wt - \tan^{-1}\frac{Lw}{R}\right)$$

Problem 3.21.11. If we close a switch in a circuit that contains a battery E , an inductance L, and a resistance R, the current i builds up at a rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long does it will be before the current has reached one- half of its final value if E=6 Volts , R=100 ohms, and L = 0.1 henry? (VTU Model 2022, Jan 2020)

Solution : Let i be the current flowing in the circuit containing resistance R and inductance L in series, with voltage source E, at any time t.

By voltage law

$$Ri + L\frac{di}{dt} = \text{E}$$

$$\Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\ldots$$

This is the linear differential equation of the form $\frac{di}{dt}+P\,i=Q$

$$\text{I.F.} = e^{\int P dt} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

Its solution is

$$i\times\ (IF)\ = \int Q\times\ (IF)\ dt+c$$
$$\Rightarrow i\times e^{\frac{R}{L}t} = \int \frac{E}{L}\times\ e^{\frac{R}{L}t}\ dt+c$$
$$= \frac{E}{L}\times\frac{L}{R}e^{\frac{R}{L}t}+c$$

Dividing by $e^{\frac{R}{L}t}$

$$\Rightarrow \quad i = \frac{E}{R}+ce^{-\frac{Rt}{L}}$$

This represents the current in the circuit, at any time t

Initially, at $t=0$, $\quad i=0 \Rightarrow c=-\frac{E}{R}$

Thus, above current i becomes

$$i=\frac{E}{R}\left[1-e^{-\frac{Rt}{L}}\right]$$

If $E=6$ volts, $R=100$ ohms and $L=0.1$ henry,

Final value of $i=\frac{E}{R}=\frac{3}{50}$

If the current reaches half its final value, $\frac{3}{100}=\frac{3}{50}\left(1-e^{-1000t}\right)$

$$\Rightarrow \frac{1}{2}=1-e^{-1000t} \quad \Rightarrow e^{-1000t}=\frac{1}{2}$$
$$\Rightarrow -1000t=\log\left(\frac{1}{2}\right)$$
$$\Rightarrow t=\frac{\log\left(\frac{1}{2}\right)}{-1000}\approx 0.0007\text{sec.}$$

Problem 3.21.12. Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sinwt satisfies the equation

$iR+L\frac{di}{dt}=Esin\omega t$

Find the value of the current at any time t, if initially there is no current in the circuit. (VTU Model EE 2022)

Solution : By Kirchhoff's first law, we have sum of voltage drops across R and $L=E\sin\omega t$ i.e., $Ri+L\frac{di}{dt}=E\sin\omega t$ This is the required differential equation which can be written as $\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\sin\omega t$

This is a linear equation.

Its I.F. $= e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

$\therefore$ the solution is $i($ I.F. $) = \int \frac{E}{L} \sin \omega t \cdot ($ I.F.$))dt + c$

or $ie^{RttL} = \frac{E}{L} \int e^{Rt/L} \sin \omega t dt + c = \frac{E}{L} \frac{e^{Rt/L}}{\sqrt{[(R/L)^2+\omega^2]}} \sin\left(\omega t - \tan^{-1} \frac{L\omega}{R}\right) + c$

or $\quad i = \frac{E}{\sqrt{(R^2+\omega^2 L^2)}} \sin(\omega t - \phi) + ce^{-Rt/L}$ where $\tan\phi = L\omega/R$

Initially when $t = 0; i = 0$.

$\therefore \quad 0 = \frac{E\sin(-\phi)}{\sqrt{(R^2+\omega^2 L^2)}} + c,$

i.e., $c = \frac{E\sin\phi}{\sqrt{(R^2+\omega^2 L^2)}}$

Thus (i) takes the form $i = \frac{E\sin(\omega t-\phi)}{\sqrt{(R^2+\omega^2 L^2)}} + \frac{E\sin\phi}{\sqrt{(R^2+\omega^2 L^2)}} \cdot e^{-Rt/L}$

or $i = \frac{E}{\sqrt{(R^2+\omega^2 L^2)}} \left[\sin(\omega t - \phi) + \sin\phi \cdot e^{-RttL}\right]$

which gives the current at any time t.

Problem 3.21.13. A resistance of 100 Ω s, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5sec$, if $i = 0$ at $t = 0$

Solution :

$$\mathrm{L}\frac{di}{dt} + Ri = E$$
$$\therefore \frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

Solution is given by,

$$i, e^{\int\left(\frac{R}{L}\right)dt = \int e^{\int\left(\frac{R}{L}\right)dt} \frac{E}{L} dt + c}$$

$$\therefore \text{ i. } e^{\frac{Rt}{L}} = \frac{Ee^{\frac{Rt}{L}}}{R} + c$$

$$\text{At t } = 0, \mathrm{i} = 0 \therefore c = \frac{E}{R}$$

$$\therefore \text{ i. } e^{\frac{Rt}{L}} = \frac{Ee^{\frac{Rt}{L}}}{R} + \frac{-E}{R}$$

$$\therefore \mathrm{i} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

For given condition $R = 100, L = 0.5, E = 20$

$$\therefore i = 0.2\left(1 - e^{-200t}\right)$$

Problem 3.21.14. When a resistance R Ohms is connected in series with an inductance L henries with an emf of E volts, the current i amperes at time t is given by $L\frac{di}{dt} + Ri = E$. If $E = 10sint$ volts, and $i = 0$ when $t = 0$, find i as a function of t. (VTU Model EE 2022)

Solution : We have,

$$L\frac{di}{dt} + Ri = E$$
$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \ldots.$$
$$\therefore I.F. = e^{\int \frac{R}{L}dt}$$
$$= e^{\frac{R}{L}t}$$

Multiplying both sides of (1) by $I.F. = e^{\frac{R}{L}t}$, we get

$$e^{\frac{R}{L}t}\left(\frac{di}{dt} + \frac{R}{L}i\right) = e^{\frac{R}{L}t} \times \frac{E}{L}$$
$$\Rightarrow e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i = e^{\frac{R}{L}t} \times \frac{E}{L}$$

Integrating both sides with respect to t, we get

$$e^{\frac{R}{L}t}i = \frac{E}{L}\int e^{\frac{R}{L}t}dt + C$$
$$\Rightarrow e^{\frac{R}{L}t}i = \frac{E}{L} \times \frac{L}{R}e^{\frac{R}{L}t} + C$$
$$\Rightarrow e^{\frac{R}{L}t}i = \frac{E}{R}e^{\frac{R}{L}t} + C \ldots\ldots$$

Now,

$$i = 0 \text{ at } t = 0$$
$$\therefore e^0 \times 0 = \frac{E}{R}e^0 + C$$
$$\Rightarrow C = -\frac{E}{R}$$

Putting the value of C in (2), we get

$$e^{\frac{R}{L}t}i = \frac{E}{R}e^{\frac{R}{L}t} - \frac{E}{R}$$
$$\Rightarrow i = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t}$$
$$\Rightarrow i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Question Bank-Module 3

Exact, Reducible to Exact Differential Equations

1. Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0.$ (VTU July 2017, Jan 2014, Model 2014) Ans :

   $x^4 + e^{xy^2} - y^3 = c$

2. Solve $\left(1+e^{\frac{x}{y}}\right)dx+\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}dy=0.$ (VTU June 2019, June 2014) Ans : $x+ye^{x/y}=c$

3. Solve $(ycosx+siny+y)dx+(sinx+xcosy+x)dy=0$ (VTU Jan 2016)

   (or)

   Solve $\frac{dy}{dx}+\frac{(ycosx+siny+y)}{(sinx+xcosy+x)}=0$ (VTU Model ME 2022, Jan 2020, Jan 2019, June 2018, Jan 2018, 2016)

   Ans :

   $y\sin x+(\sin y+y)x=c$

4. Solve $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$ (VTU June 2013) Ans : $x^3-6x^2y-6xy^2+y^3=3c=C$

5. Solve $(x^2+y^2+x)dx+xydy=0.$ (VTU Model 2022, July 2021, June 2018, July 2017, Aug 2001, Mar 2000) **Ans :** $\frac{x^4}{4}+\frac{(x^2y^2)}{2}+\frac{x^3}{3}=c$

6. Solve $x^2ydx-(x^3+y^3)dy=0$ (VTU Jan 2010, Dec 2009) **Ans :** $\frac{x^3}{3y^3}-\log y=c$

7. Solve $(x^2+y^3+6x)dx+y^2xdy=0$ (VTU Model ME 2022, July 2016) **Ans :** $\frac{x^5}{5}+y^3\frac{x^3}{3}+6\frac{x^4}{4}=c$

8. Solve $(xy^2-e^{\frac{1}{x^3}})dx-x^2ydy=0.$ (VTU Jan 2015) **Ans :** $-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c.$

9. Solve $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0.$ (VTU Jan 2015) **Ans :** $3x^2y^4+6y^2x+2y^6=c_1$

10. Solve $(4xy+3y^2-x)dx+x(x+2y)dy=0$ (VTU Jan 2020)

11. Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+[x+\log x-x\sin y]\,dy=0$ (VTU Jan 2021) Ans :

    $y(x+\log x)+x\cos y=C$

12. Solve $ye^{xy}dx+(xe^{xy}+2y)dy=0$ (VTU Jan 2018)

    Ans :

    $e^{xy}+y^2=c$

13. $ylogydx+(x-logy)dy=0$ (VTU Model ME 2022) **Ans :** $xlogy-\frac{1}{2}(logy)^2=c$

14. Solve $(e^y+1)\cos x dx+e^y\sin x dy=0.$ Ans : $(e^y+1)\sin x=c$

15. $(1 + 2xycosx^2 - 2xy)dx + (sinx^2 - x^2)dy = 0$ (July 2015) **Ans** $:x + ysinx^2 - x^2y = c$

16. Solve $\frac{2x}{y^3}dx + \frac{(y^2-3x^2)}{y^4}dy = 0.$ **Ans :** $x^2 - y^2 = cy^3$

17. Solve $(5x^4 + 3x2y2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0.$
$x^5 + x^3y^2 - x^2y^3 - y^5 = c$

18. Solve $ysin2xdx - (1 + y^2 + cos^2x)dy = 0.$ **Ans :** $3ycos2x + 6y + 2y^3 = c$

19. Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ **Ans :** $\left(y + \frac{2}{y^2}\right)x + y^2 = c$

20. Solve $(x^2 + y^2 + 2x)\,dx + 2ydy = 0$ **Ans :** $(x^2 + y^2)\,e^x = c$

21. $(x - ylogy + ylogx)dx + x(logy - logx)dy = 0$ **Ans :** $logx + \frac{y}{x}logy - \frac{y}{x}(logx + 1) = c$

22. $y(x + y + 1)dx + x(x + 3y + 2)dy = 0$ **Ans :** $\frac{x^2y^2}{2} + xy^3 + xy^2 = c$

23. $(xy + y^2)dx + (x + 2y - 1)dy = 0$ (VTU Feb 2004) **Ans :** $xye^x - ye^x + y^2e^x = c$

24. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$ (VTU Feb 2002) **Ans :** $x^3y^2 + \frac{x^2}{y} = c$

25. $y(x + y)dx + (x + 2y - 1)dy = 0$ **Ans :** $y(xe^x - e^x) + e^xy^2 = c$

26. $(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy = 0$ (VTU Model CV 2022) **Ans :** $x^4y + x^3y^2 - x^4 = c$

27. $y(2x - y + 1)dx + x(3x - 4y + 3)dy = 0$ **Ans :** $x^2y^3 - xy^4 + xy^3 = c$

28. $y(2xy + 1)dx - xdy = 0$ **Ans :** $x^2 + \frac{x}{y} = c$

Linear and Bernoulli Differential Equations:

1. Solve $x\frac{dy}{dx} + y = x^3y^6.$ (VTU Model 2022, Model 2014) **Ans :** $\frac{1}{x^5y^5} = \frac{5}{2x^2} + c$

2. Solve $xy(1 + xy^2)\frac{dy}{dx} = 1$ VTU June 2018, July 2017 **Ans :** $\frac{1}{x} = 2 - y^2 + ce^{-\frac{1}{2}y^2}$

3. Solve $\frac{dy}{dx} + ytanx = y^3secx$ (VTU June 2019, Jan 2018, Jan 2015) **Ans :** $(1/y^2) \cdot \frac{1}{\sec^2 x} = -2sinx + x + c$

4. Solve $(x + 2y^3)\frac{dy}{dx} = y$ (VTU July 2015) **Ans :** $x = y^3 + cy$

5. Solve $(1+y^2)dx+(x-e^{tan^{-1}y})dy=0$ (VTU Jan 2014, June 2013) **Ans :** $x=\tan^{-1}y-1+ce^{-\tan^{-1}y}$

6. Solve $\frac{dy}{dx}+xsin2y=x^3cos^2y$ (VTU July 2017, Jan 2015, June 2013, July 2014, July 2011) **Ans :** $tany=\frac{1}{2}x^2-1+ce^(-x^2)$

7. Solve $\frac{dy}{dx}=xy^3-xy$ (VTU Jan 2017) **Ans :** $\frac{e^{-x^2}}{y^2}=e^{-x^2}+c$

8. Solve $\frac{dy}{dx}-\frac{2}{x}y=\frac{y^2}{x^3}$ (VTU Jan 2018)

9. Solve $rsin\theta-cos\theta\frac{dr}{d\theta}=r^2$ (VTU Jan 2020, June 2018, Model 2018)

10. Solve $y(2xy+1)dx-xdy=0$ (VTU Model 2022, Model 2018)

11. Solve $\frac{dy}{dx}+ytanx=y^2secx$ (VTU Model ME 2022, Jan 2019) Ans : $secx=(tanx+c)y$

12. Solve $(1+y^2)dx+(x-e^{tan^{-1}y})dy=0$ (VTU Jan 2014, June 2013) **Ans :** $xe^{tan^{-1}y}=tan^{-1}y+c$

13. Solve $e^y\frac{dy}{dx}+1=e^x$. (VTU Jan 2016) **Ans :** $\frac{1}{y}=x^2-2+ce^{\frac{-x^2}{2}}$

14. Solve $(2xlogx-xy)dy+2ydx=0$ (VTU Jan 2017) Ans : $2y\log x-\frac{1}{2}y^2=c$

15. Solve $\frac{dy}{dx}+\frac{y}{x}=y^2x.$ (VTU Model 2022)

Differential Equations solvable for p and Clairaut's equations

1) $xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)\frac{dy}{dx}+xy=0$ (VTU Model 2022, Jan 2019, Jan 2018)

or

$xy\left\{\left(\frac{dy}{dx}\right)^2+1\right\}=(x^2+y^2)\frac{dy}{dx}$ (VTU June 2018) **Ans :** $(y^2-x^2-2c_1)(y-cx)=0$

2) $x^2p^2+3xyp+2y^2=0$ (VTU Jan 2017) Ans:$(x^2y-c)(xy-c)=0$

3) $x^2p^2+xyp-6y^2=0$ (VTU July 2017)

4) Solve the equation $p^2+p(x+y)+xy=0$ (VTU Jan 2013, Dec 2014) **Ans :** $(logy+x-c)(y+\frac{x^2}{2}-c)=0$

5) Solve $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$ (VTU July 2017, July 2013) Ans :$(y - cx^2)(y^2 + 3x^2 - c) = 0$

6) Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ (VTU Model ME 2022, June 2018, July 2017, Jan 2016, June 2015, Jan 2016) **Ans :** $(y^2 - x^2 - 2c)(xy - e^c) = 0$.

7) Solve $p^2 + 2pycotx = y^2$. (VTU Model 2022, Jan 2020, July 2016, June 2012) **Ans :** $\left(y \cos^2 \frac{x}{2} - C\right)\left(y \sin^2 \frac{x}{2} - C\right) = 0$ Ans:$y(1 \pm Cosx) = c$

8) $xp^2 - (2x + 3y)p + 6y = 0$ (VTU Jan 2015) Ans:$(y - 2x - c)(y - cx^3) = 0$

9) Solve $p(p + y) = x(x + y)$ (VTU Dec 2014, July 2011, Dec 2011) **Ans :** $\left(y - \frac{x^2}{2} - c\right)(y + x - 1 - ce^{-x}) = 0$ or $(2y - x^2 - c)(e^x(y + x + 1) - c) = 0$

10) Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$ (VTU Model EE 2022)

11) Solve $p^2 + 2pcoshx + 1 = 0$ (VTU Jan 2014)

12) Solve $p^2 - 2psinhx - 1 = 0$ (VTU Jan 2015)
Ans:$(y - Coshx - Sinhx - c)(y - Coshx + Sinhx - c) = 0$

13) Solve $p^2 + p(x + y) + xy = 0$ (VTU Jan 2013)

14) Solve $y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ (VTU July 2016) Ans:$(y - x - c)(y^2 + x^2 - c) = 0$

15) Solve $\frac{dy}{dx}^2 - 5dy/(dx) + 6 = 0$ Ans:$(y - 3x - c)(y - 2x - c) = 0$

16) Solve $x^2\left(\frac{dy}{dx}\right)^2 - xy\frac{dy}{dx} - 6y^2 = 0$ Ans:$(y - cx^2)(x^3y - c) = 0$

17) Obtain the general solution and singular solution of the equation, $y = px + p^3$ (VTU Jan 2016) Ans :
$y = cx + c^3$

18) Obtain the general solution and singular solution of the Clairaut's equation
$xp^3 - yp^2 + 1 = 0.$
(VTU July 2017, Jan 2015, June 2014, Dec 2011)
Ans : $cx + \frac{1}{c^2}$ and $4y^3 = 27x^2$

19) Obtain the general solution and singular solution of the Clairaut's equation
$(y - px)(p - 1) = p$ (VTU Jan 2015)

20) Solve the equation $y^2(y - xp) = x^4p^2$ using the substitution, $X = \frac{1}{x}$ and $Y = \frac{1}{y}$ (VTU June 2014)

21) Solve $(px - y)(py + x) = 2p$, by reducing into Clairaut's form,taking the substitution $X = x^2, Y = y^2$. (VTU Model 2022, June 2018, July 2017, July 2016, June 2012, Dec 2011).

22) Solve $(px - y)(py + x) = a^2p$, use the substitution $X = x^2, Y = y^2$. (VTU Model 2022, June 2019, Jan 2018, July 2017, June 2014, June 2015, Dec 2011) **Ans :** $y^2 = cx^2 - \frac{a^2c}{c+1}$

23) Solve $x^2(y - px) = yp^2$ using the substitution $X = x^2, Y = y^2$ (VTU Model ME 2022, July 2016) **Ans :** $y^2 = cx^2 + c^2, x^4 + 4y^2 = 0$

24) Find the general and singular solution of the equation $p = log(px - y)$ (VTU Model 2015). Ans: $y = xlogx - x$

25) Solve $y = 2px - y^2p^3$ Take $X = 2x, Y = y^2$

26) Obtain the general solution and singular solution of the equation $sinpxcosy = cospxsiny + p$ (VTU Jan 2013)

Ans : $y = \frac{\sqrt{x^2-1}}{x}x + sin^{-1}\frac{\sqrt{x^2-1}}{x}$

27) Solve $p + CosySinpx = SinyCospx$ Ans: $y = cx + sin^{-1}c$

28) Find the general and singular solution of the equation $xp^2 - py = a = 0$ (VTU Model 2018)

29) Find the general and singular solution for $xp^2 + xp - yp + 1 - y = 0$ (VTU Model ME 2022, June 2018)

Ans : $y = cx + \frac{1}{c+1}$ and $(x + y)^2 = 4x$

Orthogonal trajectories

1. Find the Orthogonal trajectories of the family of parabola $y^2 = 4ax$ (VTU Jan 2020) **Ans :** $2x^2 + y^2 = c_1$

2. Find the Orthogonal trajectories of the family of confocal ellipses $\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1$, where λ is a parameter . (VTU July 2021, Model 2018, July 2015, Dec 2011, Jule 2011, June 2009, Aug 2000, Mar 2000, Aug 1999) **Ans :** $\frac{y^2}{2} + \frac{x^2}{2} = a^2logx + c$

3. Show that the orthogonal trajectories of a family of circles passing through the origin having centres on x-axis is a family of circles passing through the origin having their cetres on y-axis. (VTU Model 2014)

4. P.T. the system of confocal and coaxial parabolas $y^2 = 4a(x + a)$ is self orthogonal. (VTU Model EE 2022, Jan 2021, June 2019, June 2018, Jun 2013)

5. Prove that the family of curves $\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1$, where λ is the parameter is self - orthogonal. (VTU Model 2022, July 2017, Jan 2016, Jan 2015, Jan 2013)

6. Given $y = ke^{-2x} + 3x$ find member of its orthogonal trajectory passing through the point (0, 3) (VTU Feb 2006)

7. Find the Orthogonal trajectories of the family of curves $y^2 = cx^3$ (VTU Jan 2018)

8. Find the O.T of the family of astroids $x^{2/3} + y^{2/3} = a^{2/3}$ **Ans :** $x^{4/3} - y^{4/3} = k$

9. Find the Orthogonal trajectories of the family $y^2 = cx^3$ **Ans :** $3y^2 + 2x^2 = k$

10. Find the Orthogonal trajectories of the family $y = ax^2$ **Ans :**$y^2 + \frac{x^2}{2} = c$

11. Find the Orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\lambda x + c = 0$, λ is a parameter . (VTU Feb 2002) **Ans :** $x^2 + y^2 - ky = c$

12. Show that the orthogonal trajectories of a family of circles passing through the origin having centers on x-axis is a family of circles passing through the origin having their centers on y-axis. (VTU Model 2014)

13. Find the Orthogonal trajectories of the family of curves $x^3 - 3xy^2 = c$ **Ans :** $y^3 - 3x^2y = c$

14. Find the Orthogonal trajectories of the cardioid $r = a(1 + cos\theta)$ (VTU Model 2022, Aug 2001) **Ans :** $r = b(1 - cos\theta)$

15. Find the Orthogonal trajectories of $r^n = a^n sinn\theta$ **Ans :** $r^n = b^n cosn\theta$

16. Find the Orthogonal trajectories of $r = 2acos\theta$, where 'a' is a parameter. VTU July 2017

17. Find the Orthogonal trajectories of $r^2 = a^2cos2\theta$ (VTU Jun 2014, Jan 2009, Aug 2003) **Ans :** $r^2 = c^2sin2\theta$

18. Find the Orthogonal trajectories of $(r+\frac{k^2}{r})cos\theta = a$ where a is the parameter. (VTU Feb 2005) **Ans :** $(r-\frac{k^2}{r})sin\theta = c$

19. Find the Orthogonal trajectories of $\frac{2a}{r} = 1 - cos\theta$ **Ans :** $r(1+cos\theta) = 2b$

20. Find the Orthogonal trajectories of $r = \frac{2a}{1+cos\theta}$ **Ans :**$r = \frac{2b}{1-cos\theta}$

21. Find the Orthogonal trajectories of $r = 4a(sec\theta tan\theta)$. **Ans :** $r(1+sin^2\theta)^{\frac{1}{2}} = c$

22. Find the Orthogonal trajectories of $r^n sinn\theta = a^n$ (VTU Model 2022) **Ans :** $r^n cosn\theta = b^n$

23. Find the Orthogonal trajectories of $r^n cosn\theta = a^n$ (VTU Jan 2017, June 2013, Jun 2011, Jan 2010,, Dec 2009) **Ans :** $r^n sinn\theta = b^n$

24. Find the Orthogonal trajectories of $r^n = asinn\theta$ (VTU Jun 2012,July 2006,Aug 2002 , Mar 1999) **Ans :** $r^n = bcosn\theta$

25. Find the Orthogonal trajectories of $r^n = a^n cosn\theta$ (VTU July 2016, Feb 2003) **Ans :**$r^n = b^n sinn\theta$

26. Find the Orthogonal trajectories of $r^n = a^n sinn\theta$ (VTU July 2017) **Ans :**$r^n = c^n \cos n\theta$

27. $r = csin^2\theta$ **Ans :** $r^2 = bcos\theta$

28. Show that orthogonal trajectories of the family of cardioids $r = acos^2\left(\frac{\theta}{2}\right)$ is another family of cardioids $r = bsin^2\left(\frac{\theta}{2}\right)$ (VTU Jan 2015)

29. Find the orthogonal trajectories of the family curve $r = 2a(cos\theta + sin\theta)$ (VTU Model ME 2022)

L-R and C-R circuits

1. An RL circuit has an emf of 5 V, a resistance of 50ω, an inductance of $1H$, and no initial current. Find the current in the circuit at any time t. Distinguish between the transient and steady-state current.

2. If we close a switch in a circuit that contains a battery E , an inductance L, and a resistance R, the current i builds up at a rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long does it will be before the current has reached one- half of its final value if E=6 Volts , R=100 ohms, and L = 0.1 henry? (VTU Model 2022, Jan 2020) ANS : 0.0006931 sec

3. When a resistance R Ohms is connected in series with an inductance L henries with an emf of E volts, the current i amperes at time t is given by $L\frac{di}{dt} + Ri = E$. If $E = 10sint$ volts, and $i = 0$ when $t = 0$, find i as a function of t. (VTU Model EE 2022)

   ANS : $\frac{10}{L^2+R^2}(RSint - Lcost + Le^{-\frac{Rt}{L}}$

4. A resistance of 100 Ω s, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5sec$, if $i = 0$ at $t = 0$

5. Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sinwt satisfies the equation

   $iR + L\frac{di}{dt} = Esin\omega t$

   Find the value of the current at any time t, if initially there is no current in the circuit. (VTU Model EE 2022)

6. Solve $L\frac{di}{dt} + Ri = E_0 \sin \omega t$, where $L, R \& E_0$ are constants and disuss the case when t increases indefinitely. (VTU July 2021, Jan 2021)

Module 4-Modular Arithmetic

Syllabus

Introduction to Congruences, Linear Congruences, The Remainder theorem (statement only), Solving Polynomials, Linear Diophantine Equation, System of Linear Congruences, Euler's Theorem(statement only), Wilson's Theorem(statement only) and Fermat's little theorem(statement only). Applications of Congruences-RSA algorithm

Integer Divisibility :

If a and b are integers such that $a \neq 0$, then we say a **divides** b and write $a \mid b$ if there exists an integer k such that $b = ka$.

That is, given $a, b \in \mathbb{Z}$ such that $a \neq 0$, we write $a \mid b$ if $\exists k \in \mathbb{Z}$ such that $b = ka$. The notation $\mathbb{Z}$ is used for the set of integers. i.e. $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

If a divides b, we also say a is a **factor** [or **divisor**] of b, and b is a **multiple** of a. If a does not divide b, we write $a \nmid b$.

For example, $2 \mid 4$ and $7 \mid 63$, while $5 \nmid 26$.

Note :

1. $\forall a \in \mathbb{Z}, a \mid 0$
2. If $a \mid b$ and $b \mid c$ then $a \mid c$
3. $\forall a, b, c, m, n \in \mathbb{Z}$, if $c \mid a$ and $c \mid b$ then $c \mid (ma + nb)$

Example : 1) $6 \mid 18$ and $18 \mid 36$ then $6 \mid 36$

2) $5 \mid 10, 5 \mid 25$ then $5 \mid (3 \times 10 + 2 \times 25)$

i.e. $5 \mid 80$

The Division Algorithm :

Given $a, b \in \mathbb{Z}$ such that $b > 0$, there exist unique $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < b$. This q is called the **quotient** and r is called the **remainder** when a is divided by b.

Example 1: If $a = 71$ and $b = 6$, if we divide 71 by 6 we get $71 = 6 \times 11 + 5$. Here the quotient, $q = 11$ and the remainder $r = 5$.

Example 2: If $a = 2589$ and $b = 17$ then we can write $2589 = 152 \times 17 + 5$

Note : In all divisions q may be positive or negative or zero, but r is always positive (or zero) and less than the divisor.

4.22 Common Divisors :

An integer d is called common divisor of the integers a and b if $d \mid a$ and $d \mid b$. If d is a common divisor of a and b, so is $-d$.

Greatest Common Divisor(GCD) :

Given $a, b \in \mathbb{Z}$, not both zero, the **greatest common divisor** is the largest integer that divides both a and b, and is written $\gcd(a, b)$ (or sometimes just (a, b)).

Note : we will write $\gcd(0, 0) = 0$.

A simple way to find GCD is to factorize both numbers and multiply common prime factors.

Example 1 : Find $\gcd(42, 56)$

Divisors of 42 are $1, 2, 3, 7$ and 14,

Divisors of 56 are 2, 7 and 14.

common divisors of both 42 and 56 are $2, 7$, and 14

Since 14 is the largest, $\gcd(42, 56) = 14$.

Example 2 : $\gcd(81, 153) = 9$

Example 3 : The greatest common divisor of 24 and 18 is 6 . In other words, $\gcd(24, 18) = 6$.

Lemma :

If $a, b, q, r \in \mathbb{Z}$ and $a = bq + r$, then $\gcd(a, b) = \gcd(r, b)$.

Euclidian Algorithm:

Euclidian Algorithm is a method of finding the greatest common divisor of two numbers by dividing the larger by the smaller, the smaller by the remainder, the first remainder by the second remainder, and so on until exact division is obtained.

Steps to find gcd using Euclidian Algorithm For any two integers a and b with $a > b$

Step 1: Let a, b be the two numbers with $a > b$.

Step 2: Write a in quotient remainder form $a = bq + r$

Step 3: Let $a = b$ and $b = r$.

Step 4: Repeat Steps 2 and 3 until r is greater than 0. The last non-zero remainder is the required GCD.

Problem 4.22.1. Find the gcd of 621 and 483.

Solution: We run the Euclidean algorithm:

$621 = 1 \times 483 + 138$

$483 = 3 \times 138 + 69$

$138 = 2 \times 69 + 0$

The last non-zero remainder is 69

So $\gcd(621, 483) = 69.$

Problem 4.22.2. find the greatest common divisor of 10672 and 4147

Solution :

$$\begin{aligned}
10672 &= 4147 \times 2 + 2378, \\
4147 &= 2378 \times 1 + 1769, \\
2378 &= 1769 \times 1 + 609, \\
1769 &= 609 \times 2 + 551, \\
609 &= 551 \times 1 + 58, \\
551 &= 58 \times 9 + 29, \\
58 &= 29 \times 2 + 0
\end{aligned}$$

$$\therefore \gcd(4147, 10672) = 29$$

Problem 4.22.3. Find the gcd of 25520 and 19314

Solution :

$$25520 = 1 \cdot 19314 + 6206$$
$$19314 = 3 \cdot 6206 + 696$$
$$6206 = 8 \cdot 696 + 638$$
$$696 = 1 \cdot 638 + 58$$
$$638 = 11 \cdot 58 + 0$$

Thus, $\gcd(25520, 19314) = 58$

Problem 4.22.4. Find $d = \gcd(1820, 231)$ and hence write it in the form $d = 1820x + 231y$

Solution : Here $a = 1820, b = 231$

$$\begin{aligned} 1820 &= 7(231) + 203 \qquad \cdots (1) \\ 231 &= 1(203) + 28 \qquad \cdots (2) \\ 203 &= 7(28) + 7 \qquad \cdots (3) \\ 28 &= 4(7) + 0 \\ d &= \gcd(1820, 231) = 7 \end{aligned} \tag{4.1}$$

$$\begin{aligned} \text{From (3) } 203 &= 7(28) + 7 \\ \Rightarrow 7 &= 203 - 7(28) \\ &= 203 - 7(231 - 203) \quad \text{From (2)} \\ &= 203 - 7(231) + 7(203) \\ &= -7(231) + 8(203) \\ &= -7(231) + 8(1820 - 7(231)) \quad \text{From (1)} \\ &= -7(231) + 8(1820) - 56(231) \\ &= 8(1820) + (-63)(231) \\ &= xa + yb \end{aligned}$$

Then $x = 8$, and $y = -63$

Relatively Prime :

$a, b \in \mathbb{Z}$ are said to be **relatively prime** if $\gcd(a, b) = 1$.

Example 1: The greatest common divisor of 9 and 16 is 1, thus they are relatively prime.

Example 2: 42 and 75 are relatively prime as $\gcd(42, 75) = 1$

Example 3: 59 and 97 are relatively prime as $\gcd(59, 97) = 1$

Example 4: 15 and 18 are not relatively prime, because $(15, 18) = 3 \neq 1$.

Example 5: 4 and 14 are not relatively prime, because $(4, 14) = 2 \neq 1$.

4.23 Congruences :

Let n be a fixed positive integer. Two integers a and b are said to be congruent modulo n (or a is congruent to b modulo n), symbolized by

$$a \equiv b(\text{mod}n)$$

if n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Examples : consider $n = 7$.

then we can write $3 \equiv 24(\text{mod}7)$ because $7 \mid (3 - 24 = -21)$,

$-31 \equiv 11(\text{mod}7)$, because $7 \mid (-31 - 11 = -42)$

$-15 \equiv -64(\text{mod}7)$ because $7 \mid (-15 - (-64) = 49)$

Note : When $n \nmid (a - b)$, we say that a is **incongruent** to b modulo n, and in this case we write $a \not\equiv b(\text{mod}n)$.

Example : $25 \not\equiv 12(\text{mod}7)$, because 7 fails to divide $25 - 12 = 13$.

Note 1: $amodn$ is the smallest nonnegative integer b such that $a \equiv b(\text{mod}n)$.

For example, $100 \text{ mod } 7 = 2$ because $100 \equiv 2(\text{mod}7)$

Note 2 : The relation $a \equiv b(\text{mod}n)$ is equivalent to the relation $a \text{ mod } n = b \text{ mod } n$.

Note 3 : Let a, b, and n be integers with $n > 0$. Then the following statements are equivalent.

$n \mid (a - b)$

$a \equiv b(\text{mod}n)$

$a = b + kn$ for some integer k

$a \text{ mod } n = b \text{ mod } n$

b is the remainder when a is divided by n

Theorem : Let $n > 1$ be fixed and a, b, c, d be arbitrary integers. Then the following properties hold:

(a) $a \equiv a(modn)$.

(b) If $a \equiv b(\text{mod}n)$, then $b \equiv a(\text{mod}n)$.

(c) If $a \equiv b(\text{mod}n)$ and $b \equiv c(\text{mod}n)$, then $a \equiv c(\text{mod}n)$.

(d) If $a \equiv b(\text{mod}n)$ and $c \equiv d(\text{mod}n)$, then $a + c \equiv b + d(\text{mod}n)$, $a - c \equiv b - d(\text{mod}n)$.and $ac \equiv bd(\text{mod}n)$.

(e) If $a \equiv b(\text{mod}n)$, then $a + c \equiv b + c(\text{mod}n)$ and $ac \equiv bc(\text{mod}n)$.

(f) If $a \equiv b(\text{mod}n)$, then $a^k \equiv b^k(\text{mod}n)$ for any positive integer k.

Example : we have $100 \equiv 2(\text{mod}7)$ and $80 \equiv 3(\text{mod}7)$.

Then we can write

$100 + 80 = 180 \equiv 5(\text{mod}7)$

$100 - 80 = 20 \equiv -1 \equiv 6(\text{mod}7)$

$100 \cdot 80 = 8000 \equiv 6(\text{mod}7)$

Problem 4.23.1. Using modulo arithmetic, find the remainder when (246×176) is divided by 9.

Solution: When 246 is divided by 9, the remainder is 3.

i.e. $246 \equiv 3(\text{mod}9)$

When 176 is divided by 9, the remainder is 5.

i.e. $176 \equiv 5(\text{mod}9)$

Hence,

$$(246 \times 176) = (3 \times 5)mod9$$
$$= 15(\text{mod}9)$$
$$= 6(\text{mod}9)$$

Problem 4.23.2. Find the remainder when $(349 \times 74 \times 36)$ is divided by 3 (VTU Model 2022)

Solution: $349 \equiv 1(\text{mod}3) \qquad (1)$

$74 \equiv 2(\text{mod}3) \qquad (2)$

$36 \equiv 0(\text{mod}3) \qquad (3)$

From (1), (2) and (3)

$349 \times 74 \times 36 \equiv 1 \times 2 \times 0 = 0(\text{mod}3)$ Hence the remainder is 0.

Problem 4.23.3. Find the last digit of 7^{105}

Solution: Last digit of 7^{105} is the remainder when 7^105 is divided by 10.

7^{105} can be written as $[(7^4)^26 \times 7]$

so we can say that units place of 7^105 is the units place of $[(7^4)^26 \times 7]$

Now we know that $7^4 = 2401$

i.e. units place 1,

i.e. $7^4 \equiv 1(\mathrm{mod}10)$

so units place of $(7^4)^26 =\equiv 1^{26}(\mathrm{mod}10)$

$(7)^{104} =\equiv 1(\mathrm{mod}10) \qquad (1)$

$7 \equiv 7(\mathrm{mod}10) \qquad (2)$

From (1) and (2),

$(7)^{104} \times 7 \equiv 1 \times 7(\mathrm{mod}10)$

i.e. $7^{105} \equiv 7(\mathrm{mod}10)$

Therefore Units place of 7^{105} is 7.

Problem 4.23.4. Find the remainder when $175 \times 113 \times 53$ is divided by 11 (VTU Model 2022)

Solution: $175 \equiv 10(\mathrm{mod}11)$

$113 \times 53 \equiv 3(\mathrm{mod}11)$

$53 \equiv 9(\mathrm{mod}11)$

$175 \times 113 \times 53 \equiv 10 \times 3 \times 9 = 270(\mathrm{mod}11)$

$\equiv 6(\mathrm{mod}11)$

Hence the remainder is 6

Problem 4.23.5. Find the remainder when the number 2^{1000} is divided by 13 (VTU Model 2022)

Solution: By inspection

$2 = 2;\ \ 2^2 = 4:\ \ 2^3 = 8,\ \ 2^4 = 16 \equiv 3(\mathrm{mod}13)$

$$2^5 = 32\ ,2^6 \equiv 64 \equiv -1(\mathrm{mod}13)$$

$(2^6)^{166} = (-1)^{166}(\text{mod}13)$

$2^{996}1(\text{mod}13) \qquad (1)$

$2^4 = 16 \equiv 3(\text{mod}13) \qquad (2)$

From (1) and (2)

$2^{996} \times 2^4 \equiv 1 \times 3(\text{mod}13)$

$2^{1000} \equiv 3(\text{mod}13)$

Hence the remainder is 3

Problem 4.23.6. Find the value of $11^{153}(\text{mod}12)$

Solution: a mod n is the smallest nonnegative integer b such that $a \equiv b(\text{mod}n)$.

Using division algorithm, we have

$153 = 2 \times 75 + 3$

$11^{153} = 11^{(2\times 75+3)}$

$\equiv (11^2)^{75} \times 11^3(\text{mod}12) \equiv (11)^{2+1}(\text{mod}12)$

$\equiv 11(\text{mod}12) \qquad (\because 11^2 \equiv 1(\text{mod}12))$

Hence $11^{153}(\text{mod}12) = 11$

Problem 4.23.7. Find the last digit of 13^{37} (VTU Model 2022)

Solution:

$13 \equiv 3(\text{mod}10)$

$13^2 \equiv 3^2(\text{mod}10)$

$13^2 \equiv 9(\text{mod}10)$

$\equiv -1(\text{mod}10)$

$13^4 \equiv (-1)^2(\text{mod}10)$

$13^4 \equiv 1(\text{mod}10)$

$(13^4)^9 \equiv 1(\text{mod}10)$

$(13)^{36} \equiv 1(\text{mod}10) \qquad (1)$

$12 \equiv 13 \equiv 13(\text{mod}10) \qquad (3)$

From (1) and (2)

$(13)^{36} \times 13 \equiv 13(\text{mod}10)$

$(13)^{37} \equiv 13(\text{mod}10) \equiv 3(\text{mod}10)$

Problem 4.23.8. Find $14^{100} \bmod 27$.

Solution: $14^{100} = 14^{64} \cdot 14^{32} \cdot 14^{4}$.

Compute $14^2 \bmod 27 = 7$.

$14^4 \bmod 27 = 7^2 \bmod 49 = 22$

$14^8 \bmod 27 = 22^2 \bmod 27 = (-5)^2 \bmod 27 = 25$

$14^{16} \bmod 27 = 25^2 \bmod 27 = (-2)^2 \bmod 27 = 4$

$14^{32} \bmod 27 = 4^2 \bmod 27 = 16$

$14^{64} \bmod 27 = 16^2 \bmod 27 = 13$

So $14^{100} mod 27 = 22 \times 16 \times 13 \pmod{27} = 13$

Problem 4.23.9. Find the last digit of 3^{1963}

Solution: Finding the last digit of a number is the same as finding the remainder when this number is divided by 10

Finding the last digit of 3^{1963} is equivalent to finding x so that $3^{1963} \equiv x \pmod{10}$

We have $3^2 \equiv (-1) \pmod{10}$

But $1963 = 196 \times 10 + 3$

Hence

$$\begin{aligned}
3^{1963} = 3^{196\times 10+3} &\equiv (-1)^{196\times 10+3} \pmod{10} \\
&\equiv (-1)^3 \pmod{10} \\
&\equiv (-1) \pmod{10} \\
&\equiv 9 \pmod{10}
\end{aligned}$$

Last digit $= 9$

Problem 4.23.10. Find the last digit of 7^{2013} (VTU Model 2022)

Solution :

$7^2 \equiv 9(\bmod 10)$

$7^4 \equiv 81 \equiv 1(\bmod 10)$

$(7^4)^{503} \equiv 1(\bmod 10)$

$7^{2012} \equiv 1(\bmod 10) \qquad \rightarrow (1)$

$7 \equiv 7(\bmod 10) \qquad \rightarrow (2)$

From (1) and (2)

$7^{2012} \times 7 \equiv 1 \times 7(\bmod 10)$

$\therefore 7^{2013} \equiv 7(\bmod 10)$

$\therefore$ last digit is 7

Problem 4.23.11. Find the last digit in 7^{118} (VTU Model 2022)

Solution :

$7^2 \equiv 9(\bmod 10)$

$7^4 \equiv 81 \equiv 1(\bmod 10)$

$(7^4)^{29} \equiv 1(\bmod 10)$

$7^{116} \equiv 1(\bmod 10) \qquad \rightarrow (1)$

$7^2 = 49 \equiv 9(\bmod 10) \qquad \rightarrow (2)$

From (1) and (2)

$7^{116} \times 7^2 \equiv 1 \times 9(\bmod 10)$

$7^{118} \equiv 9(\bmod 10)$

Hence the remainder is 9

i.e. 9 is the last digit in 7^{118}

Problem 4.23.12. Find the remainder when $3^{12} + 5^{12}$ is divided by 13.

Solution : Here, $3^{12} = (3^4)^3 \equiv (81)^3$

$3^3 \equiv 1(\bmod 13$

$(3^3)^4 \equiv 1^4(\bmod 13)$

$i.e. 3^{12} \equiv 1(\bmod 13) \qquad (1)$

Similarly

$5^2 \equiv -1(\bmod 13)$

$(5^2)^6 \equiv (-1)^6(\bmod 13)$

i.e. $5^{12} \equiv 1(\bmod 13)$ (2)

From (1) and (2)

$3^{12}+5^{12} \equiv 1+1(\bmod 13)=2(\bmod 13)$

Problem 4.23.13. Find the remainder divided 5^{439} by 6

Solution : By inspection, We can write

$$25 \equiv 1(\bmod 6)$$

$$5^2 \equiv 1(\bmod 6)$$

$$\Rightarrow \left(5^2\right)^{219} \equiv 1^{219}(\bmod 6); (\because \text{ We have} 2(219)=438, \text{ is nearest to 439})$$

$$\Rightarrow 5^{438} \equiv 1(\bmod 6) \quad \cdots(1)$$

$$5 \equiv 5(\bmod 6); \quad \cdots(2)$$

$$\Rightarrow 5^{438}\cdot 5 \equiv 1\cdot 5(\bmod 6) \quad \text{From (1) and (2)}$$

$$\Rightarrow 5^{438+1} \equiv 5(\bmod 6)$$

$$\Rightarrow 5^{439} \equiv 5(\bmod 6)$$

Hence Remainder is $Remainder=5$. (**Recall :** $a \equiv b(\bmod n) \Rightarrow$ b is the remainder when a is divided ny n)

Problem 4.23.14. Find the least positive value of x such that $5x+3 \equiv 7(\bmod 8)$.

Solution:

$5x+3 \equiv 7(\bmod 8)$

$$\Rightarrow 5x+3-7=8k, \quad k \in \mathbb{Z}$$

$$\Rightarrow 5x-4=8k, \quad k \in \mathbb{Z}$$

$$\Rightarrow 5x=8k+4, \quad k \in \mathbb{Z}$$

$$\Rightarrow x=\frac{8k+4}{5}$$

By inspection, we can check that least value of k to get integer value of x is $k=2$

Hence $x=\frac{8\times 2+4}{5}=4$

Problem 4.23.15. show that 41 divides $2^{20}-1$.

Solution : By inspection, we have $2^5=50 \equiv -9(\bmod 41)$,

$\left(2^5\right)^4 \equiv (-9)^4(\bmod 41)$

In other words, $2^{20} \equiv 81\cdot 81(\bmod 41)$.

But $81 \equiv -1(\mathrm{mod} 41)$,

and so $81 \cdot 81 \equiv 1(\mathrm{mod} 41)$.

$2^{20} \equiv 1(\mathrm{mod} 41) \qquad \cdots(1)$

and $1 \equiv 1(\mathrm{mod} 41) \qquad \cdots(2)$

From (1) and (2), $2^{20} - 1 \equiv 1 - 1(\mathrm{mod} 41)$

$2^{20} - 1 \equiv 0(\mathrm{mod} 41)$

$\Rightarrow$ Remainder =0

Thus, $41 \mid 2^{20} - 1$

Problem 4.23.16. What is the remainder of $15 \times 17 \times 19$ when divided by 7 ?

Solution : $15 \equiv 1(\mathrm{mod} 7)$

$17 \equiv 3(\mathrm{mod} 7)$

$19 \equiv 5(\mathrm{mod} 7)$

From these, we get $15 \times 17 \times 19 \bmod 7 = 1 \times 5 = 15(\mathrm{mod}7) = 1(\mathrm{mod}7)$ Hence the remainder is 1.

Problem 4.23.17. Find the remainder when 2^{81} is divided by 17.

Solution : We shall find x such that

$$2^{81} \equiv x(\mathrm{mod} 17)$$

we know $2^3 \equiv 8(\mathrm{mod} 17)$

$$(2^3)^3 \equiv 8^3(\mathrm{mod} 17)$$
$$\Rightarrow 2^9 \equiv 512(\mathrm{mod} 17)$$
$$\Rightarrow 2^9 \equiv 2(\mathrm{mod} 17)$$
$$\Rightarrow (2^9)^9 \equiv 2^9(\mathrm{mod} 17)$$
$$\Rightarrow 2^{81} \equiv 512(\mathrm{mod} 17)$$
$$\Rightarrow 2^{81} \equiv 2(\mathrm{mod} 17)$$

So when 2^{81} is divided by 17 , then the remainder is 2.

Problem 4.23.18. What is the remainder when 7^{121} is divided by 12?

Solution : $7^2 = 49 \equiv 1(\mathrm{mod} 12$

$(7^2)^60 \equiv 1^{60}(\mathrm{mod} 12$

$7^{120} \equiv 1(\text{mod}12)$ (1)

$7 \equiv 7(\text{mod}12)$ (2)

From (1) and (2)

$7^{120} \times 7 \equiv 1 \times 7(\text{mod}12)$

i.e. $7^{121} \equiv 7(\text{mod}12)$

Thus remainder is 7

4.24 LINEAR CONGRUENCES :

An equation of the form $ax \equiv b(\text{mod}n)$ where $a, b \in Z, n \in Z^+$ and $x \in Z$ is unknown, is called a linear congruence in one variable. By a solution of such an equation we mean an integer x_0 for which $ax_0 \equiv b(\text{mod}n)$.

This form of a linear congruence has at most n solutions.

Theorem: The linear congruence $ax \equiv b(\text{mod}n)$ has a solution if and only if $d \mid b$, where $d = \gcd(a, n)$. If $d \mid b$, then it has d mutually incongruent solutions modulo n.

Corollary: If $\gcd(a, n) = 1$, then the linear congruence $ax = b(\text{mod}n)$ has a unique solution modulo n.

Problem 4.24.1. Find the least positive values of x such that $71 \equiv x(\text{mod}8)$ (VTU Model 2022)

Solution:

$$71 = x(\text{mod}8)$$

is equivalent to

$$x = 71(\text{mod}8)$$

$$x = (71 - 8 \times 8)(\text{mod}8)$$

$$x = (71 - 64)(\text{mod}8)$$

$$x = 7(\text{mod}8)$$

yielding $x = 7$ the least positive value of x.

Problem 4.24.2. Find the least positive values of x such that $78 + x \equiv (\text{mod}5)$ (VTU Model 2022)

Solution: $78 + x \equiv 3(\text{mod}5)$

$78 + x - 3 = 5\mathrm{n}$ for some integer n

$75 + x = 5\mathrm{n}$

$75 + x$ is a multiple of 5

Hence the least positive value of x must be 5 , since so is the nearest value multiple of 5 more than 75 .

Problem 4.24.3. Find the least positive values of x such that $89 \equiv (x+3)(\bmod 4)$ (VTU Model 2022)

Solution: $89 \equiv (x+3)(\bmod 4)$

i.e. $89 - (x+3) = 4\mathrm{n}$ for some integer n

$86 - x = 4\mathrm{n}$

$86 - x$ is a multiple of 4

Therefore the least positive value of x must be 2 .

Since $86 - 2 = 84$ is the nearest multiple of 4 less than 86 .

Problem 4.24.4. Find the least positive values of x such that $96 \equiv \frac{x}{7}(\bmod 5)$

Solution: $96 \equiv \frac{x}{7}(\bmod 5)$

$96 - \frac{x}{7} = 5\mathrm{n}$ for some integer n

$75 + x = 5\mathrm{n}$

$96 - \frac{x}{7}$ is a multiple of 5

Therefore the least positive value of x must be 7 .

Since $96 - \frac{7}{7} = 96 - 1 = 95$ is the nearest multiple of 5 less than 96 .

Problem 4.24.5. Find the least positive values of x such that $5x \equiv 4(\bmod 6)$

Solution:$5x \equiv 4(\bmod 6)$

$5x - 4 = 6n$ for some integer n

$5x - 4$ is a multiple of 6 .

$\mathrm{x} = \frac{6\mathrm{n}+4}{5}$

When we put $\mathrm{n} = 1, 6, 11, 16 \ldots$ then $6\mathrm{n} + 4$ is divisible by 5 .

When $\mathrm{n} = 1, x = \frac{6\times1+4}{5} = 2$

Therefore the least positive value of x is 2 .

Problem 4.24.6. Find the least positive value of x such that $8x - 7 \equiv 5(\bmod 20)$.

Solution: $8x - 7 \equiv 5(\bmod 20)$

$\Rightarrow 8x - 7 - 5 = 20k \quad k \in \mathbb{Z}$

$8x - 12 = 20k$

$8x = 20k + 12$

$x = \frac{20k+12}{8}$

By inspection, we can check that least value of k to get integer value of x is $k = 1$

$x = \frac{20+12}{8} = 4$

Problem 4.24.7. Find all the solutions of the congruence $3x \equiv 12(\bmod 6)$.

Solution : Here $a = 3, b = 12, n = 6$

$d = \gcd(a, n) = \gcd(3, 6) = 3$

Now check whether $d \mid b$

Clearly $3 \mid 12$.

Thus there are $d = 3$ incongruent solutions modulo 6.

$3x \equiv 12(\bmod 6)$

$\Rightarrow 3x - 12 = 6k$

$\Rightarrow x = \frac{6k+12}{3}$

By inspection, the least value of k to get integer value of x is $k = 1$

Hence $x = \frac{6+12}{3} = 6$ is one of the solutions.

i.e. $x_0 = 6$

Other solutions are given by

$x = x_0 + \frac{n}{d}t, \quad$ where $t = 1, 2, \cdots, d - 1$

$x = 6 + \frac{6}{3} = 8(\bmod 6) = 2(\bmod 6)$ (When $t = 1$)

and $x = 6 + \frac{6}{3}(2) = 10 = 4(\bmod 6)$ (When $t = 2$)

Problem 4.24.8. Solve $9x \equiv 21(\bmod 30)$.

Solution : Here $a = 9, b = 21, n = 30$ Since $d = \gcd(a, n) = \gcd(9, 30) = 3$ and $3 \mid 21$ (i.e. $d \mid b$), there are be $d = 3$ incongruent solutions.

$$9x \equiv 21 (\bmod 30)$$
$$9x - 21 = 30k$$
$$9x = 30k + 21$$
$$x = \frac{30k + 21}{9}$$

By inspection, $k = 2$ gives the integer value of $x = 9$

Thus $x_0 = 9$

Other solutions are given by

$x = x_0 + \frac{n}{d}t = 9 + \frac{30}{3}t$, where $t = 1, 2, \cdots, d - 1$

$x = 9 + 10t$

Taking $t = 0, 1, 2$, in the formula we get

$x = 9, 19, 29$

i.e. $x \equiv 9(\bmod 30)$, $x \equiv 19(\bmod 30)$ and $x \equiv 29(\bmod 30)$ are the required three solutions of $9x \equiv 21(\bmod 30)$.

Problem 4.24.9. solve the linear congruence $2x \equiv 3(\bmod 7)$

Solution : Here $a = 2, b = 3, n = 7$

$d = \gcd(2, 7) = 1$,

Hence it has unique congruent solution and then,

$2x - 3 = 7k$ where $k \in \mathbb{Z}$

$x = \frac{7k+3}{2}$

By inspection, for $k = 1, x = 5$

Hence the $x = 5(\bmod 7)$

Problem 4.24.10. Solve $5x \equiv 6(\bmod 9)$

Solution : Here $a = 5, b = 6, n = 9$

$d = \gcd(a, n) = \gcd(5, 9) = 1$,

Hence it has unique congruent solution and solution is given by

$$x - 6 = 9k$$
$$x = \frac{9k+6}{5}$$
$$x = 3$$

i.e. $x \equiv 3(\text{mod}9)$

Problem 4.24.11. solve $4x \equiv 2(\text{mod}6)$

Solution : Here $a = 4, b = 2, n = 6$

$d = \gcd(4, 6) = 2$ and $d \mid b$, hence it has two solutions.

$4x \equiv 2(\text{mod}6) \Rightarrow 4x - 2 = 6k$

$\Rightarrow 4x = 6k + 2$

$\Rightarrow x = \frac{6k+2}{4} \Rightarrow x_0 = 2$ (for $k = 1$)

another solution is given by

$x = x_0 + \frac{n}{d}t, \;\; t = 0, 1, \cdots, d-1$

$x = 2 + \frac{6}{2}t, \quad x = 2 + 3(1)$ (since $d - 1 = 1$)

$x = 5$ or $x = 5(\text{mod}6$

Problem 4.24.12. Solve $3x \equiv 1(\text{mod}6)$

Solution : Here $a = 3, b = 1, n = 6$

$d = \gcd(3, 6) = 3$ and $3 \nmid 1$

Hence there is no solution.

Problem 4.24.13. Solve $18x \equiv 30(\text{mod}42)$

Solution : Here $d = \gcd(18, 42) = 6$ and 6 surely divides 30,

Hence there are exactly $d = 6$ solutions, which are in-congruent modulo 42.

$18x \equiv 30(\text{mod}42) \Rightarrow 18x - 30 = 42k$

$x = \frac{42k+30}{18}$ By inspection, one solution is found to be $x = 4$.

Other solutions are

$x = x_0 + \frac{n}{d}t, \;\; t = 0, 1, \cdots, d-1$

$$x = 4 + (42/6)t = 4 + 7t(\text{mod}42) \quad t = 0, 1, \ldots, 5$$

For $t = 0, 1, 2, 3, 4, 5$ we get

$$x \equiv 4, 11, 18, 25, 32, 39(\text{mod}42)$$

Problem 4.24.14. Find all non-congruence solutions modulo 15 for $27x \equiv 3(\text{mod}15)$.

Solution : Here $a = 27, b = 3, n = 15$

Clearly $d = \gcd(a, n) = \gcd(27, 15) = 3$

$d \mid b$

$27x \equiv 3(\text{mod}15) \Rightarrow 27x - 3 = 15k$

$\Rightarrow x = \frac{15k+3}{27}$

Clearly, $x = 4$ satisfies the congruence.

Hence the solutions are given by

$$x = x_0 + \frac{n}{d}t, \text{ where } 0 \le t \le d - 1$$

$$x = 4 + \frac{(15)t}{3} = 4 + 5t, \quad 0 \le t \le 2$$

$$x_1 = 4 + 5(0) = 4 \text{ for } t = 0$$

$$x_2 = 4 + 5(1) = 4 + 5 = 9 \text{ for } t = 1$$

$$x_3 = 4 + 5(2) = 4 + 10 = 14 \text{ for } t = 2$$

Problem 4.24.15. Solve $5x \equiv 4(\text{mod}6)$

Solution: $5x \equiv 4(\text{mod}6)$

$5x - 4 = 6n$ for some integer n

$5x - 4$ is a multiple of 6

$x = \frac{6n+4}{5}$

When we put $\text{n} = 1, 6, 11, 16 \ldots$ then $6n + 4$ is divisible by 5

If $\text{n} = 1$, then $x = \frac{6\times1+4}{5} = 2$

If $\text{n} = 6$, then $x = \frac{6\times6+4}{5} = 8$

Therefore the solutions are $2, 8, 14, 20 \ldots$

Problem 4.24.16. Solve $3x - 2 \equiv 0(\text{mod}11)$

Solution:$3x - 2 \equiv 0(\text{mod}11)$

$3x - 2 = 11n$ for some integer n

$3x - 2$ is a multiple of 11

$x = \frac{11n+2}{3}$

When we put $n = 2, 5, 8 \ldots$ is divisible by 3

If $n = 2, x = \frac{11\times2+2}{3} = 8$

If $n = 5, x = \frac{11\times5+2}{3} = 19$

Therefore the solutions are $8, 19, 30 \ldots$.

Problem 4.24.17. Find the solutions of the linear congruence $11x \equiv 4(\text{mod}25)$. (VTU Model 2022)

Solution: Given $11x \equiv 4(\text{mod}25)$

$a = 11, \quad b = 4, \quad n = 25$

$d = GCD(a, n) = GCD(11, 25) = 1$

$d \mid b$

Hence unique solution exists.

$11x \equiv 4(\text{mod}25) \Rightarrow 11x - 4 \equiv 25k$

$11x = 25k + 4$

$x = \dfrac{25k + 4}{11}$

By inspection, for $k = 6$ we get integer value of x

$\therefore x = \frac{25\times8+4}{11} = 14$

4.25 The Chinese Remainder Theorem :

Let $n_1, n_2, \ldots, n_r$ be positive integers such that gcd $(n_i, n_j) = 1$ for $i \neq j$. Then the system of linear congruences

$$x \equiv a_1 \, (\text{mod} n_1)$$
$$x \equiv a_2 \, (\text{mod} n_2)$$
$$\vdots$$
$$x \equiv a_r \, (\text{mod} n_r)$$

has a simultaneous solution $x \in Z$, which is unique modulo $N = n_1 n_2 \cdots n_r$ and the solution is given by

$$x = N_1 a_1 y_1 + N_2 a_2 y_2 + \cdots N_r a_r y_r$$

where $N_1 = \frac{N}{n_1}, N_2 = \frac{N}{n_2}, \cdots N_r = \frac{N}{n_r}$
$y_1, y_2, \cdots y_r$ are inverses of $N_1, N_2, \cdots N_r$ respectively satisfying $N_i y_i \equiv 1(\text{mod} n_i), \quad i = 1, 2, \cdots, r$

Problem 4.25.1. Solve the system

$$x \equiv 1 \pmod{2}$$
$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}.$$

Solution : Here $a_1 = 1, a_2 = 2, a_3 = 3, n_1 = 2, n_2 = 3, n_3 = 5$
Clearly, gcd $(n_i, n_j) = \gcd(2, 3) = \gcd(3, 5) = \gcd(2, 5) = 1$
We have $N = n_1 \times n_2 \times n_3 = 2 \cdot 3 \cdot 5 = 30.$
Also $N_1 = \frac{N}{n_1} = \frac{30}{2} = 15,$
$N_2 = \frac{N}{n_2} = \frac{30}{3} = 10,$
and $N_3 = \frac{N}{n_3} = \frac{30}{5} = 6$
So we have to solve now $15y_1 \equiv 1(\text{mod} 2)-$ a solution is $y_1 \equiv 1(\text{mod} 2)$.
In the same way, we find that $y_2 \equiv 1(\text{mod} 3)$ and $y_3 \equiv 1(\text{mod} 5)$.
Therefore $N_1 a_1 y_1 + N_2 a_2 y_2 + N_3 a_3 y_3$

$$x = 1 \cdot 15 \cdot 1 + 2 \cdot 10 \cdot 1 + 3 \cdot 6 \cdot 1 = 53 \equiv 23 \pmod{30}.$$

Problem 4.25.2. Solve the system of three congruences

$$x \equiv 2(\text{mod} 3)$$
$$x \equiv 3(\text{mod} 5)$$
$$x \equiv 2(\text{mod} 7)$$

Solution : Here $a_1 2, a_2 = 3, a_3 = 2, n_1 = 3, n_2 = 5, n_3 = 7$
we have $N = 3 \cdot 5 \cdot 7 = 105$ and

$$N_1 = \frac{n}{3} = 35 \quad N_2 = \frac{n}{5} = 21 \quad N_3 = \frac{n}{7} = 15$$

Now the linear congruences

$$35y_1 \equiv 1(\text{mod} 3) \quad 21y_2 \equiv 1(\text{mod} 5) \quad 15y_3 \equiv 1(\text{mod} 7)$$

are satisfied by $y_1 = 2, y_2 = 1, y_3 = 1$, respectively.

Thus, a solution of the system is given by

$$x = N_1a_1y_1 + N_2a_2y_2 + N_3a_3y_3 = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 = 233$$

Modulo 105, we get the unique solution $x = 233 \equiv 23 (\bmod 105)$.

Problem 4.25.3. Use the Chinese Remainder Theorem to find solutions of the system

$$x \equiv 3 \bmod 4$$

$$x \equiv 2 \bmod 3$$

$$x \equiv 4 \bmod 5.$$

Solution : Here, $a_1 = 3, a_2 = 2, a_3 = 4$, $m_1 = 4, n_2 = 3, n_3 = 5$,

and $N = 4 \cdot 3 \cdot 5 = 60$. Now $N_1 = N/n_1 = 60/4 = 15$

Similarly, $N_2 = 20$, and $N_3 = 12$.

Let us solve $N_iy_i \equiv 1 \bmod n_i, i = 1, 2, 3$.

In this problem, we need to solve

$$15y_1 \equiv 1 \bmod 4$$

$$20y_2 \equiv 1 \bmod 3$$

$$12y_3 \equiv 1 \bmod 5.$$

By inspection, we find that $y_1 = 3, y_2 = 2$, and $y_3 = 3$.

$$x \equiv a_1y_1N_1 + a_2y_2N_2 + a_3y_3N_3 (\bmod 60)$$

$$= 3 \cdot 3 \cdot 15 + 2 \cdot 2 \cdot 20 + 4 \cdot 3 \cdot 12$$

$$= 359$$

Hence the solution is $x \equiv 59 (\bmod 60)$.

Problem 4.25.4. Find the least positive integer such that:

$$x \equiv 1 (\bmod 3)$$

$$x \equiv 2 (\bmod 4)$$

$$x \equiv 3 (\bmod 5)$$

Solution : Here, $a_1 = 1, a_2 = 2, a_3 = 3, n_1 = 3, n_2 = 4, n_3 = 5$,

and $N = 3 \cdot 3 \cdot 5 = 60$.

Now $N_1 = N/n_1 = 60/3 = 20$

Similarly, $N_2 = 15$, and $N_3 = 12$.

Let us solve $N_i y_i \equiv 1 \bmod n_i, i = 1, 2, 3$.

In this problem, we need to solve

$$20y_1 \equiv 1(\text{mod} 3),$$

$$15y_2 \equiv 1(\text{mod} 4),$$

$$12y_3 \equiv 1(\text{mod} 5)$$

$$y_1 = 2, y_2 = 3, y_3 = 3$$

$$x = a_1y_1N_1 + a_2y_2N_2 + a_3y_3N_3(\text{mod} 60)$$

$$(1 \times 20 \times 2) + (2 \times 15 \times 3)$$

$$+(3 \times 12 \times 3)(\text{mod} 60)$$

$$= 238(\text{mod} 60)$$

$$= 58(\text{mod} 60)$$

Problem 4.25.5. Solve the system of congruences

$$x \equiv 1(\text{mod} 5)$$

$$x \equiv 4(\text{mod} 7)$$

$$x \equiv 2(\text{mod} 9)$$

Solution: $a_1 = 1, a_2 = 4, a_3 = 2, n_1 = 5, n_2 = 7, n_3 = 9$

$N = 5 \times 7 \times 9 = 315$

Let us solve $N_i y_i \equiv 1 \bmod n_i, i = 1, 2, 3$.

In this problem, we need to solve

$7 \cdot 9 \cdot y_1 \equiv 1(\text{mod} 5) \Rightarrow y_1 = 2$

$5 \cdot 9 \cdot y_2 \equiv 1(\text{mod} 7) \Rightarrow y_2 = 5$

$5 \cdot 7 \cdot y_3 \equiv 1(\text{mod} 9) \Rightarrow y_3 = 8$

So $x = a_1y_1N_1 + a_2y_2N_2 + a_3y_3N_3(\text{mod} N)$

$x = 1 \cdot 63 \cdot 2 + 4 \cdot 45 \cdot 5 + 2 \cdot 35 \cdot 8(\text{mod} 315)$

$x == 126 + 900 + 560 = 1586(\text{mod} 315) =\equiv 11(\text{mod} 315)$ is the solution.

Problem 4.25.6. Solve the system below using the Chinese remainder theorem:

$$x \equiv 3(\text{mod} 5)$$

$$x \equiv 5(\text{mod} 7)$$

Solution: Given data,

$$x \equiv 3(\bmod 5)$$

$$x \equiv 5(\bmod 7)$$

By the Chinese Remainder Theorem, We have

$$N = 5 \times 7 = 35$$
$$N_1 = \frac{35}{5} = 7$$
$$N_2 = \frac{35}{7} = 5$$

Now using relation,

$$N_i y_i \equiv 1\,(\bmod n_i)$$

$$7y_1 \equiv 1(\bmod 5)$$

$$\Rightarrow 2y_1 \equiv 1(\bmod 5)$$

$$6y_1 \equiv 3(\bmod 5)$$

$$y_1 = 3$$

Similarly,

$$5y_2 \equiv 1(\bmod 7)$$

$$15y_2 \equiv 3(\bmod 7)$$

$$y_2 = 3$$

Finally,

$$N = N_1 y_1 a_1 + N_2 y_2 a_2$$

$$7 \times 3 \times 3 + 5 \times 5 \times 3 = 138$$

$$\Rightarrow 138(\bmod 35)$$

$$\Rightarrow 33(\bmod 35)$$

Problem 4.25.7. Solve the system below using the Chinese remainder theorem:

$$x \equiv 1(\bmod 3)$$

$$x \equiv 2(\bmod 5)$$

$$x \equiv 3(\bmod 7)$$

Solution: we have $N = 3 \cdot 5 \cdot 7 = 105$, $N_1 = 105/3 = 35$, $N_2 = 105/5 = 21$, and $N_3 = 105/7 = 15$.

To determine y_1, we solve $35y_1 = 1(\text{mod}3)$, or equivalently, This yields $y_1 \equiv 2(\text{mod}3)$.

We find y_2 by solving $21y_2 \equiv 1(\text{mod}5)$; this immediately gives $y_2 \equiv 1(\text{mod}5)$.

Finally, we find y_3 by solving $15y_3 \equiv 1(\text{mod}7)$. This gives $y_3 \equiv 1(\text{mod}7)$.

Hence,

$$\begin{aligned} x &\equiv [1 \cdot 35 \cdot 2 + 2 \cdot 21 \cdot 1 + 3 \cdot 15 \cdot 1]\,(\text{mod}105) \\ &\equiv 157(\text{mod}105) \\ &\equiv 52(\text{mod}105) \end{aligned}$$

4.26 System of linear congruence's in two variables :

Theorem : The system of linear congruences

$ax + by = r(\text{mod}n)$

$cx + dy = s(\text{mod}n)$

has a unique solution modulo n whenever $\gcd(ad - bc, n) = 1$.

Problem 4.26.1. Find the solutions of the following systems of congruences:

$$5x + 3y = 1(\text{mod}7)$$

$$3x + 2y = 4(\text{mod}7)$$

Solution :

$$5x + 3y = 1(\text{mod}7) \qquad \cdots(1)$$

$$3x + 2y = 4(\text{mod}7). \qquad \cdots(2)$$

Here $a = 5, b = 3, r = 1, n = 7, c = 3, d = 2, s = 4$

$ad - bc = 10 - 9 = 1$

$\gcd(ad - bc, n) = \gcd(1, 7) = 1$

so the solution exists

Let us eliminate y from (1) and (2).

Multiplying (1) by 2 and (2) by 3 we get

$$10x + 6y = 2(\text{mod}7) \qquad \cdots(3)$$

$$9x + 6y = 12(\text{mod}7). \qquad \cdots(4)$$

(3)-(4) gives

$x = -10(\text{mod}7) \Rightarrow x + 10 = 7k \Rightarrow x = 7k - 10$

By inspection $x = 4$ under modulo 7

i.e. $x = -3 \equiv 4(\bmod 7)$

now Let us eliminate x from (1) and (2).

Multiplying (1) by 3 and (2) by 5 we get

$$15x + 9y = 3(\bmod 7) \qquad \cdots (5)$$

$$15x + 10y = 20(\bmod 7). \qquad \cdots (6)$$

(6)-(5) gives $y = 17(\bmod 7) \Rightarrow y - 17 = 7k \Rightarrow y = 7k + 17$

By inspection, $y = 24 \equiv 3(\bmod 7)$

Hence solution is $x \equiv 4(\bmod 7)$ and $y \equiv 3(\bmod 7)$

Problem 4.26.2. Find the solutions of the following systems of congruences: $7x + 3y = 6(\bmod 11)$, $4x + 2y = 9(\bmod 11)$.

Solution : Given

$$7x + 3y = 6(\bmod 11) \qquad \cdots (1)$$

$$4x + 2y = 9(\bmod 11) \qquad \cdots (2)$$

. Here $a = 7, b = 3, r = 6, n = 11, c = 4, d = 2, s = 9$

$ad - bc = 14 - 12 = 2$

$\gcd(ad - bc, n) = \gcd(2, 11) = 1$

so the solution exists

Let us eliminate y from (1) and (2).

Multiplying (1) by 2 and (2) by 3 we get

$$14x + 6y = 12(\bmod 11) \qquad \cdots (3)$$

$$12x + 6y = 27(\bmod 11) \qquad \cdots (4)$$

(3)-(4) gives

$$2x = -15(\bmod 11) \Rightarrow 2x + 15 = 11k \Rightarrow x = \frac{11k - 15}{2}$$

By inspection, we get $x = -2 \equiv 9(\bmod 11)$

Similarly, by eliminating x from (1) and (2), we can find $y \equiv 3(\bmod 11)$ (try this !)

4.27 Linear Diophantine Equation

An algebraic equation in one or more unknowns whose constants and variables are all integers is called a Diophantine equation.

The linear Diophantine equation in two unknowns is of the form

$$ax + by = c$$

where a, b, c are given integers and a, b are not both zero.

A given linear Diophantine equation can have a number of solutions, as is the case with $3x + 6y = 18$, where

$$3 \cdot 4 + 6 \cdot 1 = 18$$
$$3(-6) + 6 \cdot 6 = 18$$
$$3 \cdot 10 + 6(-2) = 18$$

Some Diophantine equations are not solvable. for example, there is no solution to the equation $2x + 10y = 17$.

The condition for solvability is given in the following theorem.

Theorem (condition for solvability of linear Diophantine equation):

The linear Diophantine equation $ax + by = c$ has a solution if and only if $d \mid c$, where $d = \gcd(a, b)$. If x_0, y_0 is any particular solution of this equation, then all other solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right)t \quad y = y_0 - \left(\frac{a}{d}\right)t, \quad t \in \mathbb{Z}$$

where t is an arbitrary integer.

Example 1: Consider the equation $3x + 4y = 7$.

Here, $a = 3, b = 4, c = 7$

Since $d = gcd(a, b) = gcd(3, 4) = 1 \mid 7$ there are infinitely many solutions;

By inspection, we can see that $x0 = y0 = 1$ is a particular solution.

Then all other solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right)t \quad y = y_0 - \left(\frac{a}{d}\right)t$$

where t is an arbitrary integer.

i.e. all the solutions are of the form $x = 1 + 4t, y = 1 - 3t, t \in Z$.

Example 2 : Consider the equation $12x + 18y = 50$.

Here $a = 12, b = 18, c = 50$

Since $d = gcd(12, 18) = 6 \nmid 50$ there are no solutions

Problem 4.27.1. Solve the Diophantine equation $56x + 72y = 40$.

Solution: First, let us find $d = gcd(56, 72)$ by division algorithm.

$$a = 56, b = 72$$

$$72 = 1 \times 56 + 16 \qquad \cdots (1)$$

$$56 = 3 \times 16 + 8 \qquad \cdots (2)$$

$$16 = 2 \times 8 + 0$$

$$d = (56, 72) = 8$$

Check whether d divides c.

Clearly $8 \mid 40$

Hence there exists a solution.

Now using reverse substitution method, from (2)

$$\begin{aligned} 8 &= 56 - 3 \times 16 \\ &= 56 - 3 \times (72 - 56) \\ &= 56 + 3 \times 56 - 3 \times 72 \\ &= 4 \times 56 - 3 \times 72 \end{aligned}$$

$$4(56) - 3(72) = 8$$

$$56 \times 4 + 72 \times (-3) = 8$$

This is in the form $56x + 72y = d = 8$

In the given equation, Constant term $c = 40$

Hence multiply the above equation by 5, we get

$$56(20) + 72(-15) = 40, \text{ This is in the form}$$

$$56x + 72y = 40$$

$$\Rightarrow x_0 = 20 \text{ and } y_0 = -15$$

is a solution for the given equation.

Then all other solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right) t \quad y = y_0 - \left(\frac{a}{d}\right) t$$

where t is an arbitrary integer.

Hence the general solution is
$$x = 20 + \left(\frac{72}{8}\right) t = 20 + 9t$$
$$y = -15 - \left(\frac{56}{8}\right) t = -15 - 7t$$

where $t \in \mathbb{Z}$

Problem 4.27.2. Solve the Diophantine equation $172x + 20y = 1000$

Solution: Consider the linear Diophantine equation

$$172x + 20y = 1000$$

Applying the Euclidean's Algorithm to the evaluation of $d = \gcd(172, 20)$, we find that

$$172 = 8 \cdot 20 + 12$$
$$20 = 1 \cdot 12 + 8$$
$$12 = 1 \cdot 8 + 4$$
$$8 = 2 \cdot 4$$

hence $d = \gcd(172, 20) = 4$.

Clearly $d \mid n$. i.e. $4 \mid 1000$, Hence a solution to this equation exists.

To obtain the integer 4 as a linear combination of 172 and 20 , we work backward through the previous calculations, as follows:

$$\begin{aligned} 4 &= 12 - 8 \\ &= 12 - (20 - 12) \\ &= 2 \cdot 12 - 20 \\ &= 2(172 - 8 \cdot 20) - 20 \\ &= 2 \cdot 172 + (-17)20 \end{aligned}$$

This is in the form $d = 4 = 172x + 20y$

To get the constant 1000, multiply this relation by 250 , we arrive at

$$250 \cdot 4 = 250[2 \cdot 172 + (-17)20]$$
$$1000 = 500 \cdot 172 + (-4250)20$$

so that $x = 500$ and $y = -4250$ provide one solution to the Diophantine equation in question. All other solutions are expressed by

$$x = 500 + (20/4)t = 500 + 5t$$

$$y = -4250 - (172/4)t = -4250 - 43t$$

for some integer t.

Problem 4.27.3. Solve $7x + 18y = 208$

Solution: Here $a = 7, b = 18, c = 108$ with $d = gcd(7, 18) = 1$ and $d \mid c$

Hence there exists a solution.

By Euclidian algorithm we have

$$18 = 7 \times 2 + 4$$

$$7 = 4 \times 1 + 3$$

$$4 = 3 \times 1 + 1$$

$$3 = 1 \times 3 + 0$$

Hence $d = gcd(7, 18) = 1$

To obtain the integer $d = 1$ as a linear combination of 7 and 18 , we work backward through the previous calculations, as follows:

$$\begin{aligned} 1 &= 4 - 3 \times 1 \\ &= 4 - (7 - 4) \times 1 \\ &= 4 - 7 \times 1 + 4 \times 1 \\ &= 4 \times 2 - 7 \times 1 \\ &= (18 - 7 \times 2) \times 2 - 7 \times 1 \\ &= 18 \times 2 - 7 \times 4 - 7 \times 1 \\ &= 18 \times 2 - 7 \times 5 \\ &= 7 \times -5 + 18 \times 2 \end{aligned}$$

This is in the form $d = 1 = 7x + 18y$

To get the constant 208, multiply this equation by 208,

$208 = 7 \times -5 \times 208 + 18 \times 2 \times 208$

$208 = 7 \times (-1040) + 18 \times 416$

Hence one solution is $x_0 = -1040, \quad y_0 = 416$ General solution is $x = x_0 + \left(\frac{b}{d}\right) t \quad y = y_0 - \left(\frac{a}{d}\right) t$

i.e. $x = -1040 + (18/1)t$ and $y = 416 + (7/1)t$

Problem 4.27.4. Solve $56x + 72y = 40$

Solution: Let us first find d $= gcd(56, 72)$

By Euclidian algorithm,

$$72 = 56 \times 1 + 16$$

$$56 = 16 + 8$$

$$16 = 8 \times 2 + 0$$

Hence $d = gcd(56, 72) = 8$

To obtain the integer $d = 8$ as a linear combination of 56 and 72 , we work backward through the previous calculations, as follows:

$$\begin{aligned} 8 &= 56 - 16 \times 3 \\ &= 56 - (72 - 56 \times 1) \times 3 \\ &= 56 - 72 \times 3 + 56 \times 3 \\ &= 56 \times 4 + 72 \times (-3) \end{aligned}$$

multiply by 5

$40 = 56 \times 20 + 72 \times -15$

Hence one solution is $x = 20$ and $y = -15$

General solution is $x = x_0 + \left(\frac{b}{d}\right) t \quad y = y_0 - \left(\frac{a}{d}\right) t$

$x = 20 + (72/8)t$ and $y = -15 + (56/8)t$

4.28 Polynomial congruence :

Let m be a positive integer, and let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

be any polynomial with integer coefficients. Then a high-order congruence or a polynomial congruence is a congruence of the form

$$f(x) \equiv 0 (\text{mod } n)$$

In order to find all the solutions of $f(x) \equiv 0(\bmod n)$, it suffices to substitute in the polynomial $f(x)$ the values $x = 0, \pm 1, \pm 2, \pm 3, \cdots, \pm(n-1)$ and check whether the integer so obtained is congruent to 0 modulo n or not.

Problem 4.28.1. Solve the congruence $x^3 + 2x^2 - 1 \equiv 0(\text{mod}5)$.

Solution : Here $n = 5$. In order to find all the solutions, it suffices to substitute in $f(x) = x^3 + 2x^2 - 1$ the values $x = 0, 1, 2, 3, 4$ and check whether the integer so obtained is congruent to 0 modulo 5 or not. So we have

x	0	1	2	3	4
$f(x)$	-1	2	$15 \equiv 0$	$44 \equiv 4$	$95 \equiv 0$

and the values of x that are solutions of $f(x) \equiv 0(\text{mod}5)$ are $x = 2$ and $x = 4$.

Note : In the above problem, considering the computations involved, it is simpler to choose $x = 0, 1, 2, -1, -2$, (Since value 3 is same as -2 and value 4 is same as -1 under modulo 5). Thus we get

x	0	1	2	-2	-1
$f(x)$	-1	2	$15 \equiv 0$	-1	0

. Thus, we find the solutions $x = 2$ and $x = -1 \equiv 4(\text{mod}5)$.

Problem 4.28.2. Find all the roots of the polynomial congruence

$$x^2 + 2x - 3 \equiv 0(\text{mod}5) \qquad \rightarrow (1)$$

Solution :let $f(x) = x^2 + 2x - 3$

$f(x) \equiv 0(\text{mod}5)$ if 5 divides $f(x)$

Note that 5 is a prime number $\therefore$ we need to test (1) for x in $\{0, 1, 2, 3, 4\}$ or in $\{0, \pm 1, \pm 2, \pm 3\}$

when $x = 0$: $f(x) = -3$ not divisible by 5

$x = 1; \quad f(x) = 0$ divisible by 5

$x = 2; \quad f(x) = 5$ divisible by 5

$x = 3 \quad (fx) = 12$ not divisible by 5

$x = 4 : f(x) = 21$ not divisible by 5

$x = 1, 2$ are the roots

Similarly $x = -4, -3$ are the roots.

In general $x = 1 + 5\text{k}$, $x = 2 + 5\text{k}$, $x = -3 + 5k$, $x = -4 + 5k$ are the roots.

Problem 4.28.3. Solve the congruence $x^3 + 2x^2 - 1 \equiv 0(\bmod 7)$.

Solution : Here $n = 7$, Let us compute $f(x) = x^3 + 2x^2 - 1$ for the values $x = 0, 1, 2, 3, 4, 5, 6$, or $x = 0, 1, 2, 3, -3, -2, -1$, modulo 7 :

x	0	1	2	3	-3	-2	-1
$f(x)$	-1	2	$15 \equiv 1$	$44 \equiv 2$	$-10 \equiv 4$	-1	0

The congruence admits the unique solution $x = -1$, that is, $x = 6$.

Lemma :

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of the positive integer n into distinct primes. Then the polynomial congruence

$$f(x) \equiv 0 \pmod{n}$$

is solvable if and only if each of the congruences $f(x) \equiv 0\,(\bmod p_i^{\alpha_i})$, is solvable, for $i = 1, 2, \ldots, k$.

Problem 4.28.4. Solve the congruence

$$f(x) = x^3 + 3x + 2 \equiv 0 \pmod{49}.$$

Solution : Here $n = 49$, We need not compute $f(x)$ for all $x = 0, 1, \ldots, 48$, by noticing that $n = 49 = 7^2$, where 7 is a prime number.

Hence a solution of $f(x) \equiv 0 \pmod{49}$ is clearly also a solution of $f(x) \equiv 0 \pmod{7}$

Let us first solve

$$f(x) = x^3 + 3x + 2 \equiv 0 \pmod{7}.$$

Let us compute $f(x) = x^3+3x+2$ for the values $x = 0, 1, 2, 3, 4, 5, 6$, or $x = 0, 1, 2, 3, -3, -2, -1$, mo

:

x	0	1	2	3	-3	-2	-1
$f(x)$	2	6	$16 \equiv 2$	$38 \equiv 3$	$-34 \equiv 1$	$-12 \equiv 2$	$-2 \equiv 5$

There is no value of x satisfying $f(x) \equiv 0 \pmod{7}$.

Hence there is no solutions for $f(x) \equiv 0 \pmod{7}$

and thus no solution for $f(x) \equiv 0 \pmod{49}$

Problem 4.28.5. Solve the congruence $x^4 + x + 3 \equiv 0(\bmod 25)$.

Solution : Here $n = 25$. We need not compute $f(x)$ for all $x = 0, 1, \ldots, 24$, by noticing that $n = 25 = 5^2$, where 5 is a prime number.

The solutions of $x^4 + x + 3 \equiv 0(\bmod 25)$ must also be solutions of $x^4 + x + 3 \equiv 0(\bmod 5)$.

This congruence $x^4 + x + 3 \equiv 0(\bmod 5)$ has the unique solution $x = 1$(check it !), which is not a solution of $x^4 + x + 3 \equiv 0(\bmod 25)$.

Problem 4.28.6. Solve $x^3 + 5x + 1 \equiv 0(\bmod 27)$

Solution : Here $n = 27 = 3^3$ where 3 is a prime number.

The solutions of $x^3 + 5x + 1 \equiv 0(\bmod 27)$ must also be solutions of $x^3 + 5x + 1 \equiv 0(\bmod 3)$.

We can check by inspection and no value of x in $\{0, \pm 1, \pm 2\}$ satisfy $x^3 + 5x + 1 \equiv 0(\bmod 3)$

$\therefore$ there is no solutions for $x^3 + 5x + 1 \equiv 0(\bmod 27)$

Problem 4.28.7. Solve the congruence $x^2 + 3x + 17 \equiv 0(\bmod 9)$

Solution : Here $n = 9$. clearly $9 = 3^2$ and 3 is a prime number.

So let us first find the solution of $x^2 + 3x + 17 \equiv 0(\bmod 3)$

By inspection, $\therefore$ $x = 1$ and $x = 2$ are the solutions.

4.29 EULER'S PHI-FUNCTION :

For $n \geq 1$, let $\phi(n)$ denote the number of positive integers not exceeding n (i.e. $< n$) that are relatively prime to n. The function ϕ is usually called the **Euler phi-function** or **Euler totient-function** or **Euler indicator-function**

Example 1 : Consider $n = 30$, then the positive integers that do not exceed 30, which are relatively prime to 30 are specifically, $1, 7, 11, 13, 17, 19, 23, 29$

Hence $\phi(30) = 8$; Similarly, for the first few positive integers, we can verify that $\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2, \phi(5) = 4, \phi(6) = 2, \phi(7) = 6, \ldots$

Note that $\phi(1) = 1$, because $gcd(l, 1) = 1$. In the event $n > 1$, then $gcd(n, n) = n \neq 1$.

Example 2 : Since 1 and 3 are the only two integers that are relatively prime to 4 and less than 4 , then $\phi(4) = 2$.

Example 3 : Since $1, 2, 3, 4, 5, 6$ are the integers that are relatively prime to 7 that are less than 7 , thus $\phi(7) = 6$.

Example 4 : If $n = 9$, then the integers that are relatively prime to 9 that are less 9 are $1, 2, 4, 5, 7, 8$.

Thus $\phi(9) = 6$.

Example 5 : If $n = 12$, then the integers that are relatively prime to 12 that are less 12 are $1, 5, 7, 11$.

Thus $\phi(12) = 4$.

Theorem : If p is a prime and $k > 0$, then

$$\phi\left(p^k\right) = p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right)$$

Note : If p is a prime, then $\phi(p) = p - 1$.

Example 1 : let $p = 3$ then

$$\phi(9) = \phi(3^2) = 3^2 - 3 = 6$$

The six integers less than and relatively prime to 9 being $1, 2, 4, 5, 7, 8$.

Example 1 : let $p = 2$, Then

$$\phi(16) = \phi(2^4) = 2^4 - 2^3 = 16 - 8 = 8$$

Lemma : Given integers $a, b, e, gcd(a, be) = 1$ if and only if $gcd(a, b) = 1$ and $gcd(a, e) = 1$.

Theorem : If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then

$$\begin{aligned}\phi(n) &= \left(p_1^{k_1} - p_1^{k_1-1}\right)\left(p_2^{k_2} - p_2^{k_2-1}\right)\cdots\left(p_r^{k_r} - p_r^{k_r-1}\right)\\ &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right)\end{aligned}$$

Example 1 : If

$$\begin{aligned}&n = 20 = 2^5 \times 5^1\\ &\Rightarrow \phi(20) = 20\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 20 \times \frac{1}{2} \times \frac{4}{5} = 8\end{aligned}$$

Example 2: If

$$\begin{aligned}&n = 72 = 2^3 \times 3^2\\ &\Rightarrow \phi(72) = 72\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 72 \times \frac{1}{2} \times \frac{2}{3} = 24\end{aligned}$$

Example 3 : If

$$\begin{aligned}&n = 42 = 2 \times 3 \times 7\\ &\Rightarrow \phi(42) = 42\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) = 42 \times \frac{1}{2} \times \frac{2}{3} \times \frac{6}{7} = 12\end{aligned}$$

Example 4 : Let us calculate the value $\phi(360)$. The prime-power decomposition of 360 is $2^3 \cdot 3^2 \cdot 5$, and Theorem tells us that

$$\begin{aligned}\phi(360) &= 360\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\\ &= 360\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5} = 96\end{aligned}$$

Problem 4.29.1. Find $\phi(10)$

Solution :

$$10 = 2\times 5 \Rightarrow n = 10, p_1 = 2, p_2 = 5$$

$$\begin{aligned}\phi(10) &= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\\ &= 10\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right)\\ &= 10\cdot\frac{1}{2}\cdot\frac{4}{5}\\ &= 4\end{aligned}$$

They are 1,3,7,9 (numbers which are relatively prime to 10.

Problem 4.29.2. Find $\phi(20)$

Solution :

$$\begin{aligned}\phi(20) &= \phi(2^2\times 5) \quad p_1 = 2, p_2 = 5\\ &= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\\ &= 20\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right)\\ &= 20\cdot\frac{1}{2}\cdot\frac{4}{5}\\ &= 8\end{aligned}$$

Problem 4.29.3. Find $\phi(540)$

Solution : Solution: By factorizing using prime numbers, we can check that

$$540 = 2^2\cdot 3^3\cdot 5$$

$$\begin{aligned}\phi(540) &= \phi\left(2^2\right)\phi\left(3^3\right)\phi(5)\\ &= 2(2-1)3^2(3-1)(5-1) = 144\end{aligned}$$

4.30 FERMAT'S LITTLE THEOREM

Let p be a prime and suppose that $p \nmid a$ (i.e. $\gcd(a, p) = 1$). Then

$$a^{p-1} \equiv 1(\bmod p)$$

Corollary:

$$\boxed{If p is a prime, then a^p \equiv a(\bmod p) for any integer a.}$$

Problem 4.30.1. Using Fermat's Little Theorem, show that $8^{30} - 1$ is divisible by 31 (VTU Model 2022)

Solution : Here, p=31 is a prime number and $a = 8$ and $p \nmid a$

From Fermat's little theorem ,

$a^{p-1} \equiv 1(\bmod\ p)$

$a^{31-1} \equiv 1(\bmod 31)$ if $(a, 31) = 1$

given $a = 8$

$\Rightarrow (8, 31) = 1$

$\Rightarrow 8^{31-1} \equiv 1 \bmod 31$

$8^{30} \equiv 1 \bmod 31$

$\Rightarrow 8^{30} - 1 \equiv (1 - 1) \bmod 31$

$8^{30} - 1 \equiv 0(\bmod 31)$

i.e. remainder is 0.

$8^{30} - 1$ is divisible by 31

Problem 4.30.2. Find the remainder when 72^{1001} is divided by 31 .

Solution: Let $p = 31$, which is a prime number

consider $72 \equiv 10 \pmod{31}$

$\Rightarrow 72^{1001} = 10^{1001}(\bmod 31) \qquad \cdots (1)$

Take $a = 10$ and $gcd(a, p) = (10, 31) = 1$, i.e. $p \nmid a$

From Fermat's little theorem : $a^{p-1} \equiv 1(\bmod p)$

$10^{31-1} \equiv 1(\bmod 31)$

$10^{30} \equiv 1(\text{mod} 31)$

$(10^{30})^{33} \equiv 1^{33}(\text{mod} 31)$

$10^{30\times 33} \equiv 1^{33}(\text{mod} 31)$

$10^{990} \equiv 1(\text{mod} 31) \qquad \cdots (2)$

$10^{1000} = 10^{990} \times 10^{8} \times 10^{2} \times 10$

Consider $10^2 \equiv 7(\text{mod} 31)$

$(10^2)^2 \equiv 7^2(\text{mod} 31)$

$10^4 \equiv 49(\text{mod} 31)$

$10^4 \equiv -13(\text{mod} 31)$

$(10^4)^2 \equiv (-13)^2(\text{mod} 31)$

$(10^8) \equiv 169(\text{mod} 31)$

$(10^8) \equiv 14(\text{mod} 31)$

From (1)

$72^{1001} \equiv 10^{1001}(\text{mod} 31)$

$\equiv 10^{990} \times 10^{8} \times 10^{2} \times 10(\text{mod} 31)$

$\equiv 1 \times 14 \times 7 \times 10(\text{mod} 31)$

$72^{1001} \equiv 14 \times 70(\text{mod} 31)$

$\equiv 14 \times 8(\text{mod} 31)$

$\equiv 112(\text{mod} 31)$

$\equiv 19(\text{mod} 31)$

Problem 4.30.3. What is the remainder when 2^{50} is divided by 23?

Solution : Let $a = 2$ and $p = 23$(prime number) and $p \nmid a$

From Fermat's little theorem ,

$a^{p-1} \equiv 1(\text{mod } p)$

$2^{23-1} \equiv 1(\text{mod } 23)$

$$2^{22} \equiv 1(\bmod 23)$$
$$\left(2^{22}\right)^2 \equiv (1)^2(\bmod 23)$$
$$2^{44} \equiv 1(\bmod 23) \qquad \cdots(1)$$
$$2^6 = 64 \equiv 18(\bmod 23) \qquad \cdots(2)$$

From (1) and (2)

$$2^6 \times 2^{44} \equiv 1 \times 18(\bmod 23)$$
$$2^{50} \equiv 18(\bmod 23)$$

Hence the remainder is 18.

Problem 4.30.4. Show that $5^{38} \equiv 4 \bmod 11$

Here $p = 11$ is a prime and $a = 5$ with $p \nmid a$

From Fermat's little theorem

$$a^{p-1} \equiv 1 \pmod p$$
$$5^{11-1} \equiv 1(\bmod 11)$$
$$5^{10} \equiv 1 \bmod 11$$
$$(5^{10})^3 \equiv 1 \bmod 11$$
$$5^{30} \equiv 1 \bmod 11 \qquad (1)$$
$$5^2 = 25 \equiv 3(\bmod 11)$$
$$(5^2)^4 \equiv 3^4 = 81(\bmod 11)$$
$$5^8 \equiv 4(\bmod 11) \qquad (2)$$

From (1) and (2)

$$5^{30} \times 5^8 \equiv 1 \times 4(\bmod 11)$$
$$5^{38} \equiv 4 \pmod{11}$$

Hence remainder is 4

Problem 4.30.5. What is the remainder when 5^{40} is divided by 11?

Solution: Here $a = 5$, $p = 11$ is a prime and $p \nmid a$,

From Fermat's little theorem $a^{p-1} \equiv 1 \pmod p$

$5^{10} \equiv 1(\bmod 11)$

Therefore, $5^{40} \equiv 1^4 = 1(\text{mod}11)$

Then, the remainder is 1 .

Problem 4.30.6. What is the remainder of 3^{247} divided by 17?

Solution: 3 and 17 are prime numbers. Hence by Fermat's little theorem.

$$3^{16} \equiv 1(\text{mod}17)$$
$$\implies (3^{16})^{15} \equiv (1)^{15}(\text{mod}17)$$
$$\implies 3^{240} \equiv 1(\text{mod}17)$$
$$\implies 3^{247} \equiv 3^7(\text{mod}17)$$
$$\implies 3^{247} \equiv 11(\text{mod}17)$$

Hence the required remainder is 11

Problem 4.30.7. Find the remainder when 24^{1947} is divided by 17

Solution : Let $p = 17$, prime number and $a = 24$

Clearly $gcd(a, p) = gcd(24, 17) = 1$

By FLT, $a^{p-1} \equiv 1(\text{mod}p)$

$24^{16} \equiv 1(\text{mod}17)$

$(24^{16})^{121} \equiv 1^{121}(\text{mod}17)$ $\quad$ ($\because$ $16 \times 121 = 1936$ is near to 1947)

$(24^{1936} \equiv 1(\text{mod}17)$ $\quad$ (1)

Consider $24 \equiv 7(\text{mod}17)$ $\quad$ (2)

$(24)^2 \equiv 49(\text{mod}17) = -2(\text{mod}17)$ $\quad$ (3)

$((24)^2)^4 \equiv (-2)^4(\text{mod}17)$

$24^8 \equiv 16(\text{mod}17) = -1(\text{mod}17)$ $\quad$ (3)

Consider $1947 = 1936 + 8 + 2 + 1$ From (1), (2), (3) and (4)

$24^{1936} \times 24^8 \times 24^2 \times 24^1 \equiv 1 \times -1 \times -2 \times 7(\text{mod}17)$

$24^{1947} \equiv 14(\text{mod}17)$

Hence the remainder is 14.

4.31 Euler's Theorem

$If\ n \geq 1$ and $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1(\text{mod}n)$.

Example 1: Let $n = 5$ and $a = 3$

$\gcd(a, n) = \gcd(3, 5) = 1$, and $\phi(n) = \phi(5) = 4$

Then $a^{\phi(n)} = 3^{\phi(5)} = 3^4 = 81 \equiv 1(\mathrm{mod}5)$

Example 2: Let $n = 9, a = 2$

$\gcd(a, n) = \gcd(2, 9) = 1$

Then $a^{\phi(n)} = 2^{\phi(9)} = 64 \equiv 1(\mathrm{mod}9)$

Problem 4.31.1. Use Euler's theorem to find the unit digit in 3^{100}.

Solution : Let $n = 10$ and $a = 3$

$\phi(n) = \phi(2 \times 5) = 1 \times 4 = 4$

Since 3 is relatively prime to 10, by Eulers theorem, we have

$$a^{\phi(n)} \equiv 1(\mathrm{mod}n)$$
$$3^{\phi(10)} \equiv 1(\mathrm{mod}10)$$
$$3^4 \equiv 1(\mathrm{mod}10)$$
$$\left(3^4\right)^{25} \equiv 1^{25}(\mathrm{mod}10)$$
$$3^{100} \equiv 1(\mathrm{mod}10)$$

$\therefore$ 1 is the unit digit.

4.32 Wilson's Theorem :

If p is a prime, then

$$(p-1)! \equiv -1(\mathrm{mod}p)$$

Note : Wilson's theorem and Fermat's theorem can be utilized to decrease huge numbers concerning a given modulus and to simplify the congruences.

Corollary : If p is a prime, then

$$(p-2)! \equiv 1(\mathrm{mod}p)$$
$$(p-3)! \equiv \frac{(p-1)}{2}(\mathrm{mod}p)$$

Problem 4.32.1. Find the remainder when $14!$ is divided by 17 (VTU Model 2022)

Solution: Let $p = 17$, a prime number

By Wilson's theorem

$(p-1)! \equiv -1(\mathrm{mod}p)$

$(p-2)! \equiv 1(\bmod p)$

$(p-3)! \equiv \frac{(p-1)}{2}(\bmod p)$

Using this, we can easily write

$(17-3)! \equiv \frac{17-1}{2}(\bmod 17)$

i.e. $(14!) \equiv 8(\bmod 17)$

Hence remainder is 8

Problem 4.32.2. Show that $10!+1$ is divisible by 11

Solution : Let $p=11$, prime number

By Wilsons theorem, we have

$(p-1)! \equiv -1(\bmod p)$

$10! \equiv -1(\bmod 11) \qquad (1)$

also $1 \equiv 1(\bmod 11) \qquad (2)$

From (1) and (2)

$10!+1 \equiv (-1)+1(\bmod 11) = 0(\bmod 11)$

i.e. remainder is 0

Hence $10!+1$ is divisible by 11.

4.33 CRYPTOGRAPHY

cryptography (from the Greek kryptos meaning hidden and graphein meaning to write), is the science of making communications unintelligible to all except authorized parties.

Cryptography is the only known practical means for protecting information transmitted through public communications networks, such as those using telephone lines, microwaves, or satellites.

In the language of cryptography, where message in terms of codes is called **cipher**, the information to be concealed is called **plaintext**(i.e. message in its actual original (and final) form is called the plaintext). After transformation to a secret form, a message is called **ciphertext**. The process of converting from plaintext to ciphertext is said to be encrypting (or enciphering), whereas the reverse process of changing from ciphertext back to recover the plaintext is called decrypting (or deciphering).

Additional information (some of which is) used in encryption and (some of) which is necessary for successful decryption is called a **key**.

Any plaintext is first expressed numerically by translating the characters of the text into digits by means of some correspondence such as the following:

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12

N	0	P	Q	R	s	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

RSA Algorithm

RSA algorithm is an asymmetric cryptography algorithm. Asymmetric actually means that it works on two different keys i.e. Public Key and Private Key. As the name describes that the Public Key is given to everyone and the Private key is kept private.

- RSA stands for Rivest Shamir Adleman.
- It can be described as an encryption algorithm, which is used to securely transfer the message with the help of internet.
- RSA is a type of public-key cryptography. Public key cryptography requires two keys. One key will be used to decrypt the message, and the second key will be used to encrypt the message. This means that the one key will be made **public key**, and the second key will be secret from everyone. The second key can also be called a **private key**.
- A client (for example browser) sends its public key to the server and requests some data.
- The server encrypts the data using the client's public key and sends the encrypted data.
- The client receives this data and decrypts it.

Let us learn the mechanism behind the RSA algorithm :

- Given a plain text. choose two very large primes p and q.
- Their product $n = pq$ is known as the corresponding RSA modulus.
- Compute $\phi(n) = (p-1) \times (q-1)$, where $\phi(n)$ is the Euler Totient function.
- choose a number e, $1 < e < \phi(n)$ such that $\gcd(e, \phi(n)) = 1$. This number is called the RSA exponent.

- The RSA public [encryption] key consists of the pair (n, e);
- The RSA private [decryption] key consists of the pair (n, d), where

  $d \equiv e^{-1}(\mathrm{mod}\,\phi(n))$,

  i.e.

$$de \equiv 1(\mathrm{mod}\,\phi(n))$$

- Encryption cipher is given by $c =$ (message $)^e$ mod n

  $c \equiv M^e \ (\mathrm{mod}\, n)$. where M is the numeric form of the message.

- Decryption data is given by

$$\text{Decryption data } \equiv (\text{ cipher })^d \quad (\mathrm{mod}\, n) \equiv c^d \quad (\mathrm{mod}\, n).$$

All together, the above is called the RSA cryptosystem.**Example :** If $p = 3, \ q = 11$ and private key $d = 7$, find the public key using RSA algorithm and hence encrypt the number 19.

Choose $\mathrm{p} = 3$ and $\mathrm{q} = 11$

Compute $\mathrm{n = p * q = 3 * 11 = 33}$

Compute $\phi(\mathrm{n}) = (\mathrm{p} - 1) * (\mathrm{q} - 1) = 2 * 10 = 20$

Choose e such that $1 < \mathrm{e} < \phi(\mathrm{n})$ and e and $\phi(\mathrm{n})$ are relatively prime.

Since $d = 7$ is the given private key, we can find the public key e using $de \equiv 1(\mathrm{mod}\,\phi(n))$

i.e. $7e \equiv 1(\mathrm{mod}\,20)$

One solution is $\mathrm{e} = 3$ since $21 \equiv 1(\mathrm{mod}\,\phi(n))$

Public key is $(\mathrm{e, n}) \Rightarrow (3, 33)$

Private key is $(\mathrm{d, n}) = (7, 33)$

Encryption cipher is given by $c =$ (message $)^e$ mod n

The numeric form of message is, 19 (given)

Convert numbers to letters where letters should take up single digits; A is 0, B is 1, $\cdots$ Z is 25, i.e. using

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12
N	0	P	Q	R	s	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

message=19=BJ

Encryption cipher is given by $c =$ (message $)^e$ mod n

$c = 19^3 \pmod{33}$

$$c = -2 \times 19 \pmod{33} \quad (\because 19^2 \equiv -2 \pmod{33})$$

$$c = -38 \pmod{33}$$

$$= \equiv -5 \pmod{33}$$

$$c = 28 \bmod 33$$

$$c = 28$$

Problem 4.33.1. Encrypt the message '**HI**' using RSA with key $(3127, 3)$ using the prime numbers 53 and 59

Solution: Generating Public Key:

Denote the given the prime numbers by $P = 53$ and $Q = 59$.

Now First part of the Public key : $n = P \times Q = 3127$.

$\phi(n) = (P - 1)(Q - 1) = 3016$

Second part of the Public key is $e = 3$ (given)

Clearly e satisfies the condition $1 < e < \phi(n)$ and $\gcd(e, \phi(n)) = 1$

The RSA private [decryption] key consists of the pair (n, d), where

$d = e^{-1} \pmod{\phi(n)}$,

$de \equiv 1(\bmod \phi(n))$

$d3 \equiv 1(\bmod 3016)$

$3d - 1 = 3016k$

$3d = 3016k + 1$

$d = \frac{3016k+1}{3}$

For k = 2, value of d is 2011.

Private Key $(n, d) = (3127, 2011)$

Now we will encrypt "HI" :

Convert letters to numbers where letters should take up single digits; A is 0, B is 1, $\cdots$ Z is 25,

Thus H = 8 and I = 9

Hence message = HI = 89

Encryption cipher is given by $c =$ (message $)^e$ mod n

$c = 89^3 (\mathrm{mod} 3127)$.

Thus our Encrypted Data comes out to be $c = 1394$

(check this by finding the remainder by dividing 89^3 by 3127)

Note :

Let us check the decrypted data of 1394,

Decrypted Data =(cipher $)^d (\mathrm{mod} n)$

= $c^d (\mathrm{mod} n) = 1394^{2011} (\mathrm{mod} 3127) = 89$.

Thus our Decrypted Data comes out to be 89

But 8 = H and 9=I. i.e. "HI".

Problem 4.33.2. Encrypt the message **STOP** using RSA with key $(2537, 13)$ using the prime numbers 43 and 59

Here, $n = 2537$, $e = 13$, $p = 43$, and $q = 59$.

1. We have to first determine n with the following formula:

$$n = p * q$$

$$n = 43 * 59 = 2537$$

$\Phi(n) = (p - 1) \times (q - 1) = 2436$

2. After that, we have to convert each letter into a number in $[0, 25]$.

$$S = 18, T = 19, O = 14, P = 15$$

3. Now, we have to write the numeric form of message. i.e.

$$\text{STOP} = 18191415$$

4. Now, we will encrypt each block with the help of formula: Encryption cipher $c =$ (message $)^e$ mod n

5. Now, let us encrypt the first block 1819 like this:

$$c_1 = 1819^{13} (\mathrm{mod} 2537) = 2081$$

6. let us encrypt the second block 1415 like this:

$$c_2 = 1415^{13} (\mathrm{mod} 2537) = 2182$$

So the Ciphertext $c = \mathbf{20812182}$

and the encryption = 'UHBVHC'

(Since 20=U, 8=H, 1=B, 21=V, 8=H, 2=C)

Problem 4.33.3. Encrypt the plaintext message GOLD MEDAL using the RSA algorithm with key (2561; 3).

Ans : 13. 2014 1231 1263 0508 1106 1541 1331

Question Bank-Module 4

1. Find the least positive values of x such that $71 \equiv x(\text{mod}8)$ (VTU Model 2022)
2. Find the least positive values of x such that $78 + x \equiv (\text{mod}5)$ (VTU Model 2022)
3. Find the least positive values of x such that $89 \equiv (x+3)(\text{mod}4)$ (VTU Model 2022)
4. Find the least positive value of x such that $8x - 7 \equiv 5(\text{mod}20)$.
5. Find the least positive values of x such that $5x \equiv 4(\text{mod}6)$

   Ans : 2
6. Find the least positive value of x such that $8x - 7 \equiv 5(\text{mod}20)$. Ans : 4
7. Find the remainder when $(349 \times 74 \times 36)$ is divided by 3 (VTU Model 2022)
8. Find the remainder when 2^{50} is divided by 7 .

   Ans : 4
9. Find the remainder when 41^{65} is divided by 7 .

   Ans : 6
10. Solve $2x + 6y \equiv 1(\text{mod}7), 4x + 3y \equiv 2(\text{mod}7)$

    (VTU Model 2022)

    Ans : $x = 4(\text{mod } 7), y = 0(\text{mod}7)$
11. Solve $llx + 5y = 7(\text{mod}20),\ \ 6x + 3y = 8(mod20)$

    Ans : $x = 7(\text{mod}20), y = 2(\text{mod}20)$

12. Find the solutions of the system of congruences:

$3x + 4y = 5(\bmod 13)$

$2x + 5y = 7(\bmod 13)$

Ans : $x = 7, y = 9(\bmod 13)$

13. Find the solutions of the system of congruences:

$7x + 3y = 10(\bmod 16)$

$2x + 5y = 9(\bmod 16)$

Ans : $x = 3, y = 7(\bmod 16)$

14. Find the solutions of the linear congruence $11x \equiv 4(\bmod 25)$. (VTU Model 2022)

15. Solve the following linear congruences:

(a) $25x \equiv l5(\bmod 29)$.

Ans : $x = 18(\bmod 29)$

(b) $5x = 2(\bmod 26)$

Ans : $x = 16(mod 26)$

(c) $6x = l5(\bmod 21)$

Ans : $x = 6, 13, and 20(mod 21)$

(d) $36x = 8(\bmod 102)$

Ans : No solutions

(e) $34x = 60(\bmod 98)$

Ans : $x = 45, and 94(\bmod 98)$

16. Find all the solutions of the congruence $3x \equiv 12(\bmod 6)$

Ans : $x = 6, 2, 4(\bmod 6)$

17. Solve $9x \equiv 21(\bmod 30)$

Ans : $x \equiv 29(\bmod 30)$

18. Solve the Diophantine equation $56x + 72y = 40$.

Ans : $x = 20 + 9t, y = -15 - 7t$

19. Solve the Diophantine equation $24x + 138y = 18$.

Ans : $x = 18 + 23t, \quad y = -3 - 4t$.

20. Solve $x^3 + 5x + 1 \equiv 0(\text{mod}27)$ (VTU Model 2022)

21. Find the last digit of 7^{2013} (VTU Model 2022)

22. Find the last digit of 13^{37} (VTU Model 2022)

23. Find the last digit in 7^{118} (VTU Model 2022)

24. Find the remainder when the number 2^{1000} is divided by 13 (VTU Model 2022)

25. Find the remainder when 2^{23} is divided by 47. (VTU Model 2022)

    Ans : 1

26. Find the remainder when $175 \times 113 \times 53$ is divided by 11 (VTU Model 2022)

27. Solve the system of linear congruence

$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 6$$
$$x \equiv 4 \pmod 7.$$

    using Remainder Theorem. (VTU Model 2022)

28. Solve the system

$$x \equiv 1 \pmod 2$$
$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5.$$

    using Chinese Remainder theorem. Ans : $\equiv 23 \pmod{30}$

29. Solve the system of three congruences

$$x \equiv 2(\text{mod}3)$$
$$x \equiv 3(\text{mod}5)$$
$$x \equiv 2(\text{mod}7)$$

    using Chinese Remainder theorem. Ans : $x \equiv 23(\text{mod}105)$

30. Solve the system below using the Chinese remainder theorem:

$$x \equiv 1(\text{mod}3)$$
$$x \equiv 2(\text{mod}5)$$
$$x \equiv 3(\text{mod}7)$$

    using Chinese Remainder theorem. Ans : $x \equiv 52(\text{mod}105)$

31. Calculate $\phi(1001)$ Ans : 720

32. Encrypt the message **STOP** using RSA with key $(2537, 13)$ using the prime numbers 43 and 59 (VTU Model 2022)

33. In RSA algorithm, if $p = 7$, $q = 11$ and $e = 13$,, then what will be the value of d? Ans : $d = 37(\mathrm{mod}60)$

Module - 5 Linear Algebra

4.1 Elementary row transformation

An elementary row transformation is an operation of any one of the following types.

(i) Interchange of two rows i.e. $R_i \longleftrightarrow R_j$

(ii) The multiplication of the elements of rows by a non-zero number. $R_i \longrightarrow cR_i$ for the multiplication of ith row by a non zero constant c.

(iii) The addition to the elements of a row, the corresponding elements of a row multiplied by any number. $R_i \longrightarrow R_i + cR_j$ for the addition to the ith row to the products of the jth row by c.

Note : A row of only zero entries is called a zero row

4.1.1 The Row - Echelon Form

A rectangular matrix is in row echelon form if it has the following three properties:

(i) All zero rows are at the bottom of the matrix (i.e.If there is a row where every entry is zero, then this row lies below any other row that contains a non zero entry.)

(ii) In any non zero row the leading entry (i.e. first non-zero entry) is 1

(iii) The leading entry of each nonzero row after the first row occurs to the right of the leading entry of the previous row

Problem 4.1.1. The following matrix is in row echelon form: $A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note :

In echelon form the leftmost non zero entry of a non zero row is a leading 1. A column containing a leading 1 will be called a **pivot column**.

Minor : Let A be a matrix of order $m \times n$. Let B be a submatrix of A of order r obtained by deleting few rows and columns from A. Then the determinant of this matrix B is called a minor of A of order r.

4.1.2 Rank of a Matrix:

Definition: A matrix is said to be of rank r if

(i) It has at least one non-zero minor of order r and

(ii) Every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix. The rank of matrix A is denoted by $\rho(A)$.

Note : Using row-operations if we can reduce a matrix to echelon-form, then rank of the matrix is given by

Rank = Number of non-zero rows in echelon form.

Problem 4.1.2. Find the rank of the matrix $A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$

Solution :

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$$

using row operation$R_1 \longleftrightarrow R_2$

$$A \simeq \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

$$A \simeq \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$A \simeq \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & 4 \end{bmatrix}$$

This matrix is in Echelon form and number of non-zero rows =3

Hence$\rho(A) = 3$

Problem 4.1.3. Find the rank of $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (VTU Jan 2017, Jun 2012)

Sol :

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -1 \end{bmatrix}$$

Using $R_1 \leftrightarrow R_2$,

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Using $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\sim A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Using$R_3 \to 5R_3 - 4R_2$

$R_4 \to 5R_4 - 9R_2$

$$\sim A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

Using $R_4 \to R_4 - R_3$

$$\sim A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 3$

Problem 4.1.4. Find the rank of $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ (VTU Jan 2019, June 2019, Jan 2015, Jan 2013)

SOl: Given $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$R_2 : R_2 - 2R_1$$
$$R_3 : R_3 + R_1$$
$$R_4 : R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to \frac{1}{3} R_3$$

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sim$ (Echelom form) $\rho(A) = \text{Number of non} - \text{zero rows} = 4$

Problem 4.1.5. Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (VTU Jan 2020, Jun 2014)

Sol : Given $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Applying, $R_2 \to R_2 - 2R_1, \;\; R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Applying, $R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly number of non-zero rows=2, then $\rho(A) = 2$

Augmented Matrix : Suppose we have a system of m equations in n variables, with coefficient matrix A and constants B. Then the augmented matrix of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column (i.e. (n+1)th column) is the column matrix B. This matrix will be written as $[A|B]$

Row-Equivalent Matrices Two matrices, A and B, are row-equivalent if one can be obtained from the other by a sequence of row operations.

4.2 Solution of system of linear equations - Consistency and Gauss-Elimination method

Gauss Elimination method applied to three linear equations : First we explain this method applied to a particular systems of order three given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of linear equations can be written as $AX = B$ where A is the coefficient matrix given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ X is a column vector(column-matrix) containing the variables, i.e. $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and B is the matrix of right hand side constants. i.e. $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ The co-efficient matrix A written along with a constant matrix B is called an Augmented matrix and is denoted by [A:B].

The augmented matrix of the above system is $[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Following Steps are used to solve the above system of equations by using Gauss-Elimintion Method :

Step 1: Use the element $a_{11} \neq 0$ to make the remaining elements $a_{21} = 0 and a_{31} = 0$ by elementary row transformation operation.

i.e.$R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1, R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$

Then $[A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & a'_{22} & a'_{23} & : b'_2 \\ 0 & a'_{32} & a'_{33} & : b'_3 \end{bmatrix}$

Step 2: Use the element $a'_{22} \neq 0$ to make the remaining elements $a'_{32} = 0$ by elementary row transformation operation.

i.e.$R_3 \to R_3 - \frac{a'_{32}}{a'_{22}} R_2$

Then $[A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & a'_{22} & a'_{23} & : b'_2 \\ 0 & 0 & a''_{33} & : b''_3 \end{bmatrix}$ (*)

There are 3 possibilities:

(i) If **Rank of A = Rank of the Augmented Matrix [A:B] = no.of unknowns**, then the system is consistent and has a unique solution.

(ii) If **Rank of A = Rank of the Augmented Matrix [A:B] ¡ no.of unknowns**, then the system is consistent and has infinite number of solutions.

(iii) If **Rank of A $\neq$ Rank of the Augmented Matrix [A:B]**, then there is no solution. i.e. the system is inconsistent.

If the system is consistent, then (*) can be written as,

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a'_{22}x_2 + a'_{23}x_3 = b'_2$

$a'_{33}x_3 = b''_3$

By back substitution we get the values of x_1, x_2, x_3

Problem 4.2.1. Test for consistency and solve the following system of equation by using Gauss-elimination method.

$2x + y + 4z = 12; 4x + 11y - z = 33; 8x - 3y + 2z = 20$

Solution : The Augmented matrix is given by,

$$[A : B] = \begin{bmatrix} 2 & 1 & 4 : 12 \\ 4 & 11 & -1 : 33 \\ 8 & -3 & 2 : 20 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 2 & 1 & 4 : 12 \\ 0 & 9 & -9 : 9 \\ 0 & -7 & -14 : -28 \end{bmatrix}$$

$$R_2 \to R_2 \div 9, R_3 \to R_3 \div -7$$

$$\sim \begin{bmatrix} 2 & 1 & 4 : 12 \\ 0 & 1 & -1 : 1 \\ 0 & 1 & 2 : 4 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 4 : 12 \\ 0 & 1 & -1 : 1 \\ 0 & 0 & 3 : 3 \end{bmatrix}$$

We can observe that Rank of A = Rank of the Augmented Matrix $[A : B]$ = no.of unknowns, and hence the system is consistent and has a unique solution.

$\therefore$ the system of equation reduces to $2x + y + 4z = 12; y - z = 1; 3z = 3$

By using back substitution we get $z = 1, y = 2, x = 3$

Problem 4.2.2. Solve by using Gauss Elimination method, $x+y+z=9, 2x+y-z=0, 2x+5y+7z=52$ (VTU Jan 2019)

Sol :

$$(A,B)=\begin{bmatrix} 1 & 1 & 1 & :9 \\ 2 & 1 & -1 & :0 \\ 2 & 5 & 7 & :52 \end{bmatrix}$$

$$R_2 : R_2 - 2R_1$$

$$R_3 : R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & :9 \\ 0 & -1 & -3 & :-18 \\ 0 & 0 & -4 & :-20 \end{bmatrix}$$

$$R_3 : R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & :9 \\ 0 & 1 & 3 & :18 \\ 0 & 0 & 1 & :5 \end{bmatrix}$$

$\therefore \rho(A) = \rho(A,B) = 3$ Number of unknown

Hence system is consistent and unique solution exists.

From 3rd row, $z=5$ From 2nd row, $y+3z=18 \Rightarrow y+3(5)=3 \Rightarrow y=3$

from first row, $x+3+5=9 \Rightarrow x=9-8 \Rightarrow x=1$

$\therefore x=1, y=3, z=5$

Problem 4.2.3. Find the values of λ and μ so that the equations $2x+3y+5z=9, 7x+3y-2z=8, 2x+3y+\lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. (VTU July 2021)

Sol : The augmented matrix $[A, B]$ is

$$[A, B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$R_2 \to 2R_2 - 7R_1$$

$$R_3 \to R_3 - R_1$$

$$[A, B] \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix}$$

The above matrix is in echelon form (i) When $\lambda = 5, \mu \neq 9$,

$$\rho(A) = 2, \rho(A, B) = 3 \text{ (i.e) } \rho(A, B) \neq p(A)$$

The system is inconsistent and it has no solution. (ii) When $\lambda \neq 5, \mu \in R$,

$\rho(A, B) = \rho(A) = 3 =$ number of unknowns

The system is consistent with a unique solution.

(iii) When $\lambda = 5, \mu = 9$, then $\rho(A, B) = \rho(A) = 2 <$ number of unknowns

The system is consistent with infinite number of solutions.

Problem 4.2.4. Find the values of λ and μ so that the equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. (VTU Jan 2021)

Solution : The given system of linear equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The Augmented matrix is :

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

Applying elementary row transformations $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ we get

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix}$$

Using $R_3 \rightarrow R_3 - R_1$

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix},$$

This is in echelon form.

(i) For no solution, we must have $\rho(A) \neq \rho(A, B)$

i.e $\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 \neq 0 \Rightarrow \mu \neq 10$

(ii) for unique solution, we must have $\rho(A) = \rho(A, B) = 3$

i.e. $\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$

and $\mu - 10 \neq 0 \Rightarrow \mu \neq 10$

(iii) for infinite solutions, we must have $\rho(A) = \rho(A, B) < 3$

i.e. $\lambda - 3 = 0 \Rightarrow \lambda = 3$ and $\mu - 10 = 0 \Rightarrow \mu = 10$

4.3 Gauss-Jordan method

This method aims in reducing the coefficient matrix A to a diagonal matrix by using only row-operations.

Problem 4.3.1. Solve the following system of equation by using Gauss- Jordan method

$$2x + y + 4z = 12; 4x + 11y - z = 33; 8x - 3y + 2z = 20$$

Solution : The Augmented matrix is given by,

$$[A:B]=\begin{bmatrix}2 & 1 & 4:12\\ 4 & 11 & -1:33\\ 8 & -3 & 2:20\end{bmatrix}$$

$$R_2\to R_2-2R_1, R_3\to R_3-4R_1$$

$$\sim\begin{bmatrix}2 & 1 & 4:12\\ 0 & 9 & -9:9\\ 0 & -7 & -14:-28\end{bmatrix}$$

$$R_2\to R_2\div 9, R_3\to R_3\div -7$$

$$\sim\begin{bmatrix}2 & 1 & 4:12\\ 0 & 1 & -1:1\\ 0 & 1 & 2:4\end{bmatrix}$$

$$R_1\to R_1-R_2; R_3\to R_3-R_2$$

$$\sim\begin{bmatrix}2 & 0 & 5 & :11\\ 0 & 1 & -1 & :1\\ 0 & 0 & 3 & :3\end{bmatrix}$$

$$R_3\to R_3\div 3$$

$$\sim\begin{bmatrix}2 & 0 & 5 & :11\\ 0 & 1 & -1 & :1\\ 0 & 0 & 1 & :1\end{bmatrix}$$

$$R_1\to R_1-5R_3; R_2\to R_2+R_3$$

$$\sim\begin{bmatrix}2 & 0 & 0 & :6\\ 0 & 1 & 0 & :2\\ 0 & 0 & 1 & :1\end{bmatrix}$$

$\therefore$ the system of equation reduces to $2x=6; y=2; z=1$

By solving , we get $x=3, y=2, z=1$.

Problem 4.3.2. Solve by using Gauss Jordan, $x+y+z=9, 2x+y-z=0, 2x+5y+7z=52$ (VTU June 2019)

$$[A,B]=\begin{bmatrix}1&1&1&:9\\2&1&-1&:0\\2&5&7&:52\end{bmatrix}$$

$R_2 : R_2 - 2R_1$

$R_3 - 2R_1$

Solution :

$$[A,B]\sim\begin{bmatrix}1&1&1&:9\\0&-1&-3&:-18\\0&3&5&:34\end{bmatrix}$$

$R_1 \to R_1 + R_2 : R_3 + 3R_2$

$$[A,B]\sim\begin{bmatrix}1&0&-2&:-9\\0&-1&-3&:-18\\0&0&-4&:-20\end{bmatrix}$$

$R_3 : \left(\frac{1}{4}\right) R_3$

$$\sim\begin{bmatrix}1&0&-2&-9\\0&-1&-3&:-18\\0&0&1&:\quad 5\end{bmatrix}$$

$R_1 : R_1 + 2R_3$

$R_2 : R_2 + 3R_3$

$$\sim\begin{bmatrix}1&0&0&:1\\0&-1&0&:-3\\0&0&1&:5\end{bmatrix}$$

Rewriting the equations, we get

$x = 1, \quad -y = -3 \Rightarrow y = 3$ and $z = 5$

Problem 4.3.3. Apply Gauss - Jordan method to solve the system of equations, $2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16$ (VTU Jan 2015)

Solution : The Augmented matrix of the system is

$$[A \mid B]=\begin{bmatrix}2&1&1&10\\3&2&3&18\\1&4&9&16\end{bmatrix}$$

Applying $R_2 \to R_2 - \frac{3}{2}R_1$ and $R_3 \to R_3 - \frac{1}{2}R_1$, we get

$$[A \mid B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{bmatrix}$$

Applying $R_3 \to R_3 - 7R_2$, we get

$$[A \mid B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix}$$

Applying $R_1 \to R_1 + \frac{1}{2}R_3$, $R_2 \to R_2 + \frac{3}{4}R_3$ and $R_1 \to R_1 - 2R_2$ we get

$$[A \mid B] \sim \begin{bmatrix} 2 & 0 & 0 & 14 \\ 0 & 1/2 & 0 & -9/2 \\ 0 & 0 & -2 & -10 \end{bmatrix}$$

We see that A is reduced to diagonal form.

From this, we get $2x = 14 \Rightarrow x = 7$;

$\frac{1}{2}y = \frac{-9}{2} \Rightarrow y = -9$;

$-2z = -10 \Rightarrow z = 5$

Thus $x = 7, y = -9, z = 5$ is the solution.

Problem 4.3.4. Solve by Gauss Jordan method : $x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix}$$

$$R_2 \longrightarrow R_2 - 2R_1 \text{ and}$$

$$R_3 \longrightarrow R_3 - 3R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 9 & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}$$

$$R_1 \longrightarrow 5R_1 + R_2 \text{ and}$$

Sol :

$$R_3 \longrightarrow 5R_3 + R_2$$

$$[A:B] \sim \begin{bmatrix} 5 & 0 & 7 & : & 40 \\ 0 & -5 & 2 & : & -5 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$$R_3 \Rightarrow \frac{R_3}{12}$$

$$[A:B] \sim \begin{bmatrix} 5 & 0 & 7 & : & 40 \\ 0 & -5 & 2 & : & -5 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_2 \Rightarrow R_2 - 2R_3 \text{ and } R_1 \Rightarrow R_1 - 7R_3 [A:B] \sim \begin{bmatrix} 5 & 0 & 0 & : & 5 \\ 0 & -5 & 0 & : & -15 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$5x = 5 \Rightarrow x = 1$$

$$-5y = -15 \Rightarrow y = 3$$

$$z = 5$$

4.4 Gauss-Seidel Iteration Method:

Iterative methods provide an alternative approach. An iterative method starts with an approximate solution, and uses it by means of a recurrence formula to provide another approximate solution; by repeatedly applying the formula, a sequence of solutions is obtained which converges to the exact solution.

First we explain this method applied to a particular systems of order three given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and third equation for x_3 and obtain the rewritten equations :

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3]$$
$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3]$$
$$x_3 = \frac{1}{a_{33}}[b_3 - a_{31}x_1 + a_{32}x_2]$$

Then make an initial guess of the solution $x_1^{(0)} = y_1^{(0)} = z_1^{(0)} = 0$ Substitute these values into the right hand side the of the rewritten equations to obtain the first approximations as

$$x_1^{(1)} = \frac{1}{a_{11}}[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}]$$
$$x_2^{(1)} = \frac{1}{a_{22}}[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}]$$
$$x_3^{(1)} = \frac{1}{a_{33}}[b_3 - a_{31}x_1^{(1)} + a_{32}x_2^{(1)}]$$

In each iteration we use the new values(i.e.latest values) as soon as they are known. i.e. we calculate $x_1^{(1)}$ from the first equation, its value is then used in the second equation to obtain the new $X_2^{(1)}$, and these new values of $X_1^{(1)}$ and $X_2^{(1)}$ are then used in the third equation to obtain the new $X_3^{(1)}$. To find the second approximations we use the above set of equations by replacing $x_1^{(0)}, y_1^{(0)}, z_1^{(0)}$ by $x_1^{(1)}, y_1^{(1)}, z_1^{(1)}$. We continue like this by obtaining each successive approximation, using a previous approximation until the values of x_1, x_2 and x_3 obtained in two successive approximations are equal.

Note : The following condition is essential to use Gauss-Seidel Iteration Method.

• The Coefficient matrix of given system of equations must be **diagonally dominant.**

(A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row.

Problem 4.4.1. Solve the following system of equation by using Gauss- Seidel method. $x + y + 54z = 110; 27x + 6y - z = 85; 6x + 15y + 2z = 72$

Solution : The system of equation is not in diagonal form by interchanging the rows.

$27x + 6y - z = 85; 6x + 15y + 2z = 72; x + y + 54z = 110$

We can observe , $|27| > +|-1|; |15| > |6| + |2|; |54| > |1| + |1|$

$\therefore$ the system of equation is diagonally dominant.

$$\text{From}(1) x = \frac{1}{27}[85 - 6y + z]$$
$$\text{From}(2) y = \frac{1}{15}[72 - 6x - 2z]$$
$$\text{From}(3) z = \frac{1}{54}[110 - x - y]$$

First Iteration :

$$x^{(1)} = \frac{1}{27}[85 - 0 + 0] = \frac{85}{27} = 3.1481$$
$$y^{(1)} = \frac{1}{15}[72 - 6x^{(1)} - 2z] = \frac{1}{15}[72 - 6(3.1481) - 0] = 3.5481$$
$$z^{(1)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = 1.913$$

Second Iteration :

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = 2.432$$
$$y^{(2)} = \frac{1}{15}[72 - 6x^{(2)} - 2z^{(1)}] = 3.572$$
$$z^{(2)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = 1.925$$

Third Iteration :

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = 2.4256$$
$$y^{(3)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(2)}] = 3.5730$$
$$z^{(3)} = \frac{1}{54}[110 - x^{(3)} - y^{(3)}] = 1.925$$

Third Iteration :

$$x^{(4)} = \frac{1}{27}[85 - 6y^{(3)} + z^{(3)}] = 2.425$$
$$y^{(4)} = \frac{1}{15}[72 - 6x^{(4)} - 2z^{(3)}] = 3.573$$
$$z^{(4)} = \frac{1}{54}[110 - x^{(4)} - y^{(4)}] = 1.925$$

Since $x^{(3)} \approx x^{(4)} = 2.425$, $y^{(3)} \approx y^{(4)} = 3.573$, $z^{(3)} \approx z^{(4)} = 1.925$, the solution is $x = 2.425, y = 3.573, z = 1.925$

Problem 4.4.2. Solve the following system of equations by Gauss-Seidal iteration method, $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$ (VTU Jan 2020, June 2019, Jan 2018, July 2017, Jan 2017, Model 2015)

Given, $2x + y - 2z = 17 \longrightarrow (1)$

$3x + 20y - z = -18 \longrightarrow (2)$

$2x - 3y + 2z = 25 \longrightarrow (3)$

$\therefore$ The equation are diagonally dominant.

$$(1) \Rightarrow x = \frac{1}{20}[17 - y + 2z]$$
$$(2) \Rightarrow y = \frac{1}{20}[-18 - 3x + z]$$
$$(3) \Rightarrow z = \frac{1}{20}[25 - 2x + 3y]$$

Let $x = 0, y = 0, z = 0$

First Iteration :

$$x^{(1)} = \frac{1}{20}[17 - 0 + 0] = 0.85$$
$$y^{(1)} = \frac{1}{20}[-18 - 2.55 + 0] = -1.0275$$
$$z^{(1)} = \frac{1}{20}[20 - 1.7 - 3(0.825)] = 1.0109$$

Second iteration :

$$x^{(2)} = \frac{1}{20}[17 + 1.025 - 12(1.0109)] = 1.0025$$
$$y^{(2)} = \frac{1}{20}[-18 - 3(1.0025) + 1.0109] = -0.999$$
$$z^{(2)} = \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998$$

Third iteration :

$$x^{(3)} = \frac{1}{20}[17 + 0.9998 + 2(0.9998)] = 1$$
$$y^{(3)} = \frac{1}{20}[-18 - 3(1) + 0.9998] = -1$$
$$z^{(3)} = \frac{1}{20}[25 - 2(1) + 3(-1)] = 1$$

$\therefore x = 1, y = -1, z = 1$

Problem 4.4.3. Solve the system of equations $12x + y + z = 31, 2x + 8y - z = 24, 3x + 4y + 10z = 58$ using Gauss Seidal method. (VTU Jan 2019)

Sol : $\Rightarrow$ Given equation are diagonally dorninant.

$$x = \frac{1}{12}[31 - y - z]$$
$$y = \frac{1}{8}[24 - 2x + z]$$
$$z = \frac{1}{10}[58 - 3x - 4y]$$

$$x^{(1)} = \frac{1}{12}[31 - 0 - 0] = \frac{31}{12} = 2.583$$
$$y^{(1)} = \frac{1}{8}[24 - 2(2.583) + 0] = 2.354$$
$$z^{(1)} = \frac{1}{10}[58 - 3(2.583) - 4(2.354)] = 4.0835$$

$$x^{(2)} = \frac{1}{12}[31 - (2.354) - (4.0835)] = 2.04.6$$
$$y^{(2)} = \frac{1}{8}[24 - 2(2.046) + 4.0835] = 2.99 = 3$$
$$z^{(2)} = \frac{1}{10}[58 - 3(2.046) - 4(3)] = 3.98 = 4$$

$$x^{(3)} = \frac{1}{12}[31 - (3) - (4)] = 2$$
$$y^{(3)} = \frac{1}{8}[24 - 2(2) + 4] = 3 \qquad \therefore x = 2, y = 3, z = 4$$
$$z^{(3)} = \frac{1}{10}[58 - 3(2) - 4(3)] = 4$$

Problem 4.4.4. Solve the system of equations $83x + 11y - 4z = 95, 7x + 52y + 13z = 104, 3x + 8y + 29z = 71$ using Gauss Seidal method. (VTU Model 2018)

Sol:

The given equation can be written in the iteration form as

$$x = \frac{1}{83}(95 - 11y + 4z)$$
$$y = \frac{1}{52}(104 - 7x - 13z)$$
$$z = \frac{1}{29}(71 - 3x - 8y)$$

Taking first $x^{(1)} = 0, y^{(1)} = 0, z^{(1)} = 0$ and put these values in (i), we get

$$x^{(2)} = \frac{1}{83}\left(95 - 11y^{(1)} - 4z^{(1)}\right)$$
$$= \frac{1}{83}(95 - 11 \times 0 + 4 \times 0) = \frac{95}{83} = 1.14$$

$x^{(2)} = 1.14, y^{(1)} = 0, z^{(1)} = 0$, in (ii), we get

$$y^{(2)} = \frac{1}{52}\left(104 - 7x^{(2)} - 13z^{(1)}\right) = \frac{1}{52}(104 - 7 \times 1.14 - 13 \times 0)$$
$$= \frac{96.02}{52} = 1.85$$

Put $x^{(2)} = 1.14, y^{(2)} = 1.85, z^{(1)} = 0$ in (iii), we get

$$\begin{aligned} z^{(2)} &= \frac{1}{29}\left(71 - 3x^{(2)} - 8y^{(2)}\right) \\ &= \frac{1}{29}(71 - 3 \times 1.14 - 8 \times 1.85) = \frac{52.78}{29} = 1.82 \end{aligned}$$

Now put $x^{(2)} = 1.14, y^{(2)} = 1.85, z^{(2)} = 1.82$ in (i), we get

$$\begin{aligned} x^{(3)} &= \frac{1}{83}\left(95 - 11y^{(2)} + 4z^{(2)}\right] \\ x^{(3)} &= \frac{1}{83}[95 - 11 \times 1.85 + 4 \times 1.82] \\ &= \frac{81.93}{83} = 0.99 \end{aligned}$$

$x^{(3)} = 0.99, y^{(2)} = 1.85, z^{(2)} = 1.82$ in (ii), we get

$$\begin{aligned} y^{(3)} &= \frac{1}{52}\left[104 - 7x^{(3)} - 13z^{(3)}\right] = \frac{1}{52}[104 - 7 \times 0.99 - 13 \times 1.82] \\ &= \frac{73.41}{52} = 1.41. \end{aligned}$$

Put $x^{(3)} = 0.99, y^{(3)} = 1.41, z^{(2)} = 1.82$ in (iii), we get

$$\begin{aligned} z^{(3)} &= \frac{1}{29}\left(71 - 3x^{(3)} - 8y^{(3)}\right) \\ &= \frac{1}{29}(71 - 3 \times 0.99 - 8 \times 1.41) \\ &= \frac{56.75}{29} = 1.95 \end{aligned}$$

Now put $x^{(3)} = 0.99, y^{(3)} = 1.41, z^{(3)} = 1.95$ in (i), we get

$$\begin{aligned} x^{(4)} &= \frac{1}{83}\left(95 - 11y^{(3)} + 4z^{(3)}\right) \\ &= \frac{1}{83}(95 - 11 \times 1.41 + 4 \times 1.95) \\ &= \frac{1}{83}(87.29) = 1.05 \end{aligned}$$

$= \frac{1}{83}(95 - 11 \times 1.41 + 4 \times 1.94.59) = 1.05 = \frac{1}{83}(87.29)$ $x^{(4)} = 1.05, y^{(3)} = 1.41, z^{(3)} = 1.95$ in (ii), we get $y^{(4)} = \frac{1}{52}\left(104 - 7x^{(3)} - 13z^{(3)}\right)$ Put

$$\begin{aligned} &= \frac{1}{52}(104 - 7 \times 1.05 - 13 \times 1.95) \\ &= \frac{71.3}{52} = 1.37 \end{aligned}$$

Put $x^{(4)} = 1.05, y^{(4)} = 1.37, z^{(3)} = 1.95$ in (iii), we get

$$\begin{aligned} z^{(4)} &= \frac{1}{29}\left(71 - 3x^{(4)} - 8y^{(4)}\right) \\ &= \frac{1}{29}(71 - 3 \times 1.05 - 8 \times 1.37) \\ &= \frac{56.89}{29} = 1.96 \end{aligned}$$

$$\begin{aligned} z^{(4)} &= \frac{1}{29}\left(71 - 3x^{(4)} - 8y^{(4)}\right) \\ &= \frac{1}{29}(71 - 3 \times 1.05 - 8 \times 1.37) \\ &= \frac{56.89}{29} = 1.96 \end{aligned}$$

Here The values are sufficiently close to $x^{(3)}, y^{(3)}, z^{(3)}$

$x^{(4)} = 1.05, y^{(4)} = 1.37, z^{(4)} = 1.96$
respectively. Hence the solution is

$$x = 1.05, y = 1.37, z = 1.96.$$

4.5 Eigen values and Eigen Vectors :

Let A be an $n \times n$ matrix. Then a real number λ is called an eigenvalue of the matrix A, if and only if, there is a n-dimensional nonzero vector, X for which

$$AX = \lambda X \qquad \cdots (1)$$

Any such vector, X is called an **eigenvector** of the matrix A, associated with the eigenvalue λ.

Procedure for calculating eigenvalues and eigenvectors :

If A is any square matrix of size $n \times n$ and λ is an associated eigenvalue, rewrite equation (1) as

$$AX = \lambda IX$$

where I is the identity matrix of size $n \times n$ The above equation reduces to

$$(A - \lambda I)X = 0$$

which is a homogenous system of 'n' equations in 'n' variables and has a nonzero solution if and only if its coefficient matrix satisfies the equation, $|A - \lambda I| = 0$. The equation $|A - \lambda I| = 0$ is called the **characteristic equation** of matrix A. The solutions λ of the characteristic equation are called eigen values or characteristic roots. FOr each value of λ, solve the system of equations

$$(A - \lambda I)X = 0$$

to find the corresponding eigen vector.

Problem 4.5.1. Find the Eigen values and Eigen Vectors for $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution : Let

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Characteristic equation is

$$|A - \lambda I| = 0$$

$$i.e. \begin{vmatrix} (5-\lambda) & 4 \\ 1 & (2-\lambda) \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

By solving $\lambda = 1, 6$ we get the eigen values as $\lambda = 1, 6$

The system of equation can be written as ,

$$(5-\lambda)x + 4y = 0$$

$$x + (2-\lambda)y = 0$$

case 1: put $\lambda = 1$ then

$$4x + 4y = 0$$

$$x + y = 0$$

$$x = -y$$

$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to eigen value $\lambda = 1$. **case 2:** put $\lambda = 6$ then

$$-x + 4y = 0$$
$$x - 4y = 0$$
$$x = 4y$$

$\therefore X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to eigenvalue $\lambda = 6$.

Problem 4.5.2. Find the Eigen values of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

SOl : The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$$

$$\Rightarrow (6-\lambda)[(3-\lambda-1)(3-\lambda+1)] + 4[-3+\lambda+1] + 4[1-3+\lambda] = 0$$

$$\Rightarrow (6-\lambda)[(2-\lambda)(4-\lambda)] + 4[\lambda-2] + 4[\lambda-2] = 0$$

$$\Rightarrow (\lambda-2)[-(6-\lambda)(4-\lambda) + 4 + 4] = 0$$

$$-(\lambda-2)\left[\lambda^2 - 10\lambda + 16\right] = 0$$
$$-(\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$\lambda = 8, 2, 2$

4.6 Rayleigh's power method

To find largest eigen value and corresponding eigen vector of a given square matrix A, we use the following steps.

Step1. To find the largest eigen value and corresponding eigen vector of a given square matrix 'A' of order 3 ,we initially assume the eigen vector in the form $X^(0) = [1 \quad 0 \quad 0]^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

or$X^{(0)} = [0 \quad 0 \quad 1]^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$
or $X^{(0)} = [0 \quad 1 \quad 0]^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

Step2. Evaluate the matrix product $AX^{(0)}$,which is a column matrix of order 3.

Step3. Take out the numerically largest value and write it in the form $\lambda^{(1)}X^{(1)}$

Step4. Compute $AX^{(1)}$ and re-write it in the form $\lambda^{(2)}X^{(2)}$

Step5. Similarly Compute $AX^{(2)} = \lambda^{(3)}X^{(3)}$ and so on.

Continue this iterative process until we get two consecutive values of λ and X with desired accuracy.

The values so obtained are largest eigen value and corresponding eigen vectors respectively for the given square matrix A .

Problem 4.6.1. Use power method to find the largest eigen value and the corresponding eigen vectors of the matrix A by using power method. Take $[1\ 0\ 0]'$ as the initial eigen vector. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(Apply 4 iterations) (VTU Jan 2020, June 2019, June 2018, Jan 2017, Jan 2016)

SOl : Given,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Ax^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)}x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)}x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.9286 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9228 \end{bmatrix} = \begin{bmatrix} 2.9286 \\ 0 \\ 2.8572 \end{bmatrix} = 2.9286 \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \lambda^{(4)} (x)^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \begin{bmatrix} 2.9756 \\ 0 \\ 2.9512 \end{bmatrix} = 2.9756 \begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix}$$

$$Ax^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix} = \begin{bmatrix} 2.9918 \\ 0 \\ 2.9836 \end{bmatrix} = 2.9918 \begin{bmatrix} 1 \\ 0 \\ 0.9973 \end{bmatrix}$$

$\therefore$ The large Given value $\lambda = 2.9918 \sim 3$ and $\lambda^{(6)} x^{(6)}$

it's eigen value $\begin{bmatrix} 1 \\ 0 \\ 0.9973 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Problem 4.6.2. Find the Dominant eigen value and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by power method taking the initial eigen vector as $(1\ 1\ 1)'$ (VTU Jan 2018, July 2017, Jun 2015)

Sol : Given,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}, x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ax^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} \lambda^{(1)} x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 7.34 \\ -2.67 \\ 4.01 \end{bmatrix} = 7.34 \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \lambda_2 x_2$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \begin{bmatrix} (7.82) \\ -3.63 \\ 4.01 \end{bmatrix} = 7.82 \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \lambda_3 X_3$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.94 \\ -3.89 \\ 3.99 \end{bmatrix} = 7.94 \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \lambda_4 X_4$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 7.98 \\ -3.97 \\ 3.99 \end{bmatrix} = 7.98 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda_5 x_5$$

$$\begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda_6 x_6$$

$\therefore$ The large Given value of $\lambda = 7.9970 \sim 8$ and it's eigen value is $\begin{bmatrix} 1 \\ -0.4994 \\ 0.5002 \end{bmatrix}$

Problem 4.6.3. Determine the largest eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ using Rayleigh's power method. (**VTU** Model 2014)

Sol :

$$AX^0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

Repeating the above process, we get

$$AX^{(2)} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} X^{(3)};$$

$$AX^{(3)} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} X^{(5)};$$

$$AX^{(5)} = 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)} X^{(6)};$$

$$AX^{(6)} = 3.41 \begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen vector is $[0.74, -1, 0.67]'$.

Problem 4.6.4. Find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ by using Power method.

Solution : Let $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$, $and X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Ist iteration :

$$AX^{(0)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix}$$

$$= 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}$$

$$= \lambda^{(1)} X^{(1)}$$

2nd iteration :

$$AX^{(1)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}$$

$$= \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix}$$

$$= 25.2 \begin{bmatrix} 1 \\ 0.04 \\ 0.0666 \end{bmatrix}$$

$$= \lambda^{(2)} X^{(2)}$$

3^{rd} iteration:

$$AX^{(2)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.0666 \end{bmatrix}$$

$$= \begin{bmatrix} 25.173 \\ 1.12 \\ 1.7336 \end{bmatrix}$$

$$= 25.173 \begin{bmatrix} 1 \\ 0.0444 \\ 0.0689 \end{bmatrix}$$

$$= \lambda^{(3)} X^{(3)}$$

4^{th} iteration :

$$AX^{(3)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.0689 \end{bmatrix}$$

$$= \begin{bmatrix} 25.1822 \\ 1.1332 \\ 1.7244 \end{bmatrix}$$

$$= 25.1822 \begin{bmatrix} 1 \\ 0.045 \\ 0.069 \end{bmatrix}$$

$$= \lambda^{(4)} X^{(4)}$$

5^{th} iteration:

$$AX^{(4)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.045 \\ 0.069 \end{bmatrix}$$

$$= \begin{bmatrix} 25.183 \\ 1.135 \\ 1.724 \end{bmatrix}$$

$$= 25.183 \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix}$$

$$= \lambda^{(5)} X^{(5)}$$

6^{th} iteration :

$$AX^{(5)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.045 \\ 0.0685 \end{bmatrix}$$

$$= \begin{bmatrix} 25.182 \\ 1.135 \\ 1.726 \end{bmatrix}$$

$$= 25.182 \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix}$$

$$= \lambda^{(6)} X^{(6)}$$

Here $\lambda^{(5)} = \lambda^{(6)} = 25.183$ and $X^{(5)} = X^{(6)} = \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix}$

Hence largest eigen value is $\lambda = 25.183$ and largest eigen vector is $X = \begin{bmatrix} 1 \\ 0.0450 \\ 0.0689 \end{bmatrix}$

4.7 Question Bank- Module 5

Rank of a matrix-echelon form.

1. Find the rank of $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ (VTU Model CS 2022)

2. Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 3 & 2 & 1 \\ 2 & 4 & 6 & 2 \end{bmatrix}$ (VTU Model E 2022)

3. Find the rank of $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ (VTU Jun 2013, Jun 2011)

4. Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$ (VTU Jan 2016) **Ans :** 3

5. Find the rank of $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (VTUModel E 2022, June 2018, Jun 2014, Dec 2011)

   Ans : 2

6. Find the rank of $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$ (VTU June 2018, Jan 2016) **Ans :** 3

7. Find the rank of $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (VTU Model CS 2022, Jan 2017, Jun 2012)

8. Find the rank of $\begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \\ 95 & 96 & 97 & 98 & 99 \end{bmatrix}$ (VTU July 2014)

9. Find the rank of $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$ (VTU Model E 2022)

10. Find the rank of $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ (VTU Jan 2019, June 2019, Jan 2015, Jan 2013)

11. Find the rank of $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ (VTU Jan 2018, Jan 2015)

12. Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (VTU Jan 2020, Jun 2014)

13. Find the rank of $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ (VTU Jan 2014)

14. Find the rank of $\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ (VTU June 2018, Jan 2018)

15. Find the rank of $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$ (VTU Model E 2022)

16. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ **Ans :** 2

Consistency and Gauss-Elimination method

1. Find the values of λ and μ so that the equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. (VTU Model 2022, Jan 2021)

2. Test for consistency and hence solve $5x+3y+7z = 4, 3x+26y+2z = 9, 7x+2y+10z = 5$ (VTU ModelE 2022)

3. Test for consistency and hence solve $4x - 2y + 6z = 8, x + y - 3z = -1, 15x–3y + 9z = 21$ Ans : $x = 1, y = 3k - 2, z = k$

4. Solve $x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0$ Ans : $x = \frac{-10}{7}k, y = \frac{8}{7}k, z = k$

5. Solve: $x + 3y + 2z = 0, 2x - y + 3z = 0, 3x - 5y + 4z = 0, x + 17y + 4z = 0$ Ans ; $x = 11k, y = k, z = -7k$

6. Find the values of k for which the equations $x + y + z = 1, 2x + y + 4z = k$ and $4x + y + 10z = k^2$ has a solution and solve them in each case. Ans : $k = 1, x = -3n, y = 2n + 1, z = n$ and when $k = 2, x = 1 - 3m, y = 2m, z = m$

7. Find the values of λ for which the system $x+y+z=1, x+2y+4z=\lambda, x+4y+10z=\lambda^2$ has a solution. Solve it in each case. Ans : when $\lambda=1, x=2k_1+1, y=-3k_1, z=k_1$ and when $\lambda=2, x=2k_2, y=1-3k_2, z=k_2$

8. Find the values of λ and μ so that the equations $2x+3y+5z=9, 7x+3y-2z=8, 2x+3y+\lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. Ans: (i) $\lambda=5, \mu\neq 9$, (ii) $\lambda\neq 5$ (iii) $\lambda=5$ and $\mu=9$

9. Solve by using Gauss Elimination method, $x+y+z=9, x-2y+3z=8, 2x+y-z=3$.

   (VTU Model CS 2022)

10. Solve by using Gauss Elimination method, $x+2y+z=3, 2x+3y+2z=5, 3x-5y+5z=2$.

    (VTU July 2017)

11. Solve by using Gauss Elimination method, $3x-y+2z=12, x+2y+3z=11, 2x-2y-z=2$

    (VTU Jan 2015)

12. Solve by using Gauss Elimination method, $5x_1+x_2+x_3+x_4=4, x_1+7x_2+x_3+x_4=12, x_1+x_2+6x_3+x_4=-5, x_1+x_2+x_3+4x_4=-6$ (VTU Jun 2015)

13. Solve by using Gauss Elimination method, $x+2y+z=3, 2x+3y+3z=10, 3x-y+3z=13$

    (VTU Model 2015)

14. Solve by using Gauss Elimination method, $x_1+x_2+x_3=4, 2x_1+x_2-x_3=1, x_1-x_2+2x_3=2$

    (VTU Jan 2014)

15. Solve by using Gauss Elimination method, $2x_1-x_2+3x_3=1, -3x_1+4x_2-5x_3=0, x_1+3x_2-6x_3=0$ (VTU jun 2014, Jun 2012)

16. Solve by using Gauss Elimination method,

    $x + 4y - z = -5, x + y - 6z = -12, 3x - y - z = 4$

    (VTU Model 2014)

17. Solve by using Gauss Elimination method,

    $2x + 3y - z = 5, 4x + 4y - 3z = 3, 2x - 3y + 2z = 2$

    (VTU Jan 2015, July 2014)

18. Solve by using Gauss Elimination method,

    $2x - 3y + 4z = 7, 5x - 2y + 2z = 7, 6x - 3y + 10z = 23$

    (VTU July 2017)

19. Solve by using Gauss Elimination method,

    $2x + y + 4z = 12, 4x + 11y - z = 33, 8x - 3y + 2z = 20$ (VTU Model 2022, June 2018)

20. Test the consistency and solve,

    $x + 2y + 2z = 1, 2x + y + z = 2, 3x + 2y + 2z = 3, x + z = 0$ **Ans :** $x = 1, y = -k, z = k$

21. Solve by using Gauss Elimination method,

    $x + y + z = 9, 2x + y - z = 0, 2x + 5y + 7z = 52$ (VTU Jan 2019)

22. Solve by using Gauss Elimination method, $x - 2y + 3z = 2$, $3x - y + 4z = 4$, $2x + y - 2z = 5$ (VTU Model 2022, Jan 2020)

23. Solve by using Gauss Elimination method, $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$ (VTU Model E 2022)

Gauss-Jordan method

1. Apply Gauss - Jordan method to solve the system of equations, $x+y+z = 10, 2x-y+3z = 19, x + 2y + 3z = 22$ (VTU Model CS 2022)

2. Apply Gauss - Jordan method to solve the system of equations, $x+y+z = 11, 3x-y+2z = 12, 2x + y - z = 3$ (VTU Model E 2022)

3. Apply Gauss - Jordan method to solve the system of equations, $2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16$ (VTU Jan 2015)

4. $x + y + z = 9, x - 2y + 3z = 8, 2x + y - z = 3$ (**VTU** Jan 2018, July 2017)

5. Solve by using Gauss Jordan method, $2x+y+z = 10, 3x+2y+3z = 18, x+4y+9z = 16$ (VTU Jan 2017, Jan 2015)

6. Solve by using Gauss Jordan method, $2x + 5y + 7z = 52, 2x + y - z = 0, x + y + z = 9$ (VTU Model 2022, June 2019, Jan 2016, June 2014)

7. Solve by using Gauss Jordan method, $x+2y+z = 3, 2x+3y+3z = 10, 3x-y+3z = 13$ (VTU Jan 2018)

8. Solve by using Gauss Jordan method,
   $2x_1 + x_2 + 3x_3 = 1, 4x_1 + 4x_2 + 7x_3 = 1, 2x_1 + 5x_2 + 9x_3 = 3$ (VTU Jan 2020)

Gauss-Seidel Iteration Method:

1. Solve the following system of equations by Gauss-Seidal method. $10x + 2y + z = 9, x + 10y - z = -22, -2x + 3y + 10z = 22$ (VTU July 2017) (Ans : $x = 0.997 \approx 1$, $y = -1.99 \approx 2$, $z = 2.99 \approx 3$)

2. Solve the following system of equations by Gauss-Seidal iteration method, $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$ (VTU Model 2022, July 2021, Jan 2020, June 2019, July 2012, Jan 2018, July 2017, Jan 2017, Model 2015)

3. Solve the following system of equation by using Gauss- Seidel method. $x + y + 54z = 110; 27x + 6y - z = 85; 6x + 15y + 2z = 72$ (VTU Model M 2022)

4. Solve the system of equations
   $83x + 11y - 4z = 95, 7x + 52y + 13z = 104, 3x + 8y + 29z = 71$ using Gauss Seidal method. (VTU ModelE 2022, Jan 2021, Model 2018)

5. Solve the system of equations
   $12x + y + z = 31, 2x + 8y - z = 24, 3x + 4y + 10z = 58$ using Gauss Seidal method. (VTU Jan 2019)

6. Solve the following system of equations by Gauss-Seidal method.
   $10x + 2y + z = 9, x + 10y - z = -22, -2x + 3y + 10z = 22$ (VTU July 2017)

7. Solve the system of equations $10x + y + z = 12, x + 10y + z = 12, x + y + 10z = 12$ using Gauss Seidal method. (VTU Model 2022)

   Ans : $x = 1, y = 1, z = 1$

8. $5x + 2y + z = 12, x + 4y + 2z = 15, x + 2y + 5z = 20$ (Carry out 3 iterations taking initial approximation as (1,0,3). **Ans :** $x = 0.9987, y = 2.0131, z = 2.995$

9. Use Gauss-Seidel iteration method to solve the system.

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14$$

Eigen values and Eigen Vectors - Rayleigh's power method

1. Use power method to find the largest eigen value and the corresponding eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ taking $[0 \quad 1]'$ as the initial eigen vector. (VTU Model 2015)

2. Find the Dominant eigen value and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by power method taking the initial eigen vector as $(1\ 1\ 1)'$ (VTUModel 2022, Jan 2018, July 2017, Jun 2015)

3. Find the Dominant eigen value and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ by power method taking the initial eigen vector as $(1\ 1\ 1)'$ (VTUModelE 2022)

4. Determine the largest eigen value and the corresponding eigen vectors of the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ taking $[0 \quad 0 \quad 1]'$ as the initial eigen vector. Perform 5 iterations. (VTU Jan 2015)

5. Use power method to find the largest eigen value and the corresponding eigen vectors of the

matrix A by using power method. Take $[1\ 0\ 0]'$ as the initial eigen vector. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(Apply 4 iterations) (VTU Model 2022, Jan 2020, June 2019, June 2018, Jan 2017, Jan 2016)

6. Determine the largest eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ using Rayleigh's power method. (VTU June 2018, Model 2014) Ans : 3.41 and $[0.74, -1, 0.67]^T$

7. Determine the largest eigen value and the corresponding eigen vector by Rayleigh's power method, performing five iterations, with $x^{(0)} = [1,\ 1,\ 1]^T$ for $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8. Find the Dominant eigen value and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by Rayleigh's power method taking the initial eigen vector as $(1\ 0\ 0)^T$

(VTU Model 2018)

9. Find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ by using Power method.

10. Use power method to find the largest eigen value and the corresponding eigen vectors of the matrix A by using power method. Given $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ Take $[1\ 0\ 0]'$ as the initial eigen vector. (VTU Model E 2022)

Ans : 5.994, $[1, 0.999, -0, 999]'$