



Lecture Notes

BMATS201

Mathematics-II for CSE Stream

Module 1: Integral Calculus

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Module 1

Integral Calculus

Syllabus :

Introduction to Integral Calculus in Computer Science & Engineering.

Multiple Integrals: Evaluation of double and triple integrals, evaluation of double integrals by change of order of integration, changing into polar coordinates. Applications to find Area and Volume by double integral. Problems.

Beta and Gamma functions: Definitions, properties, relation between Beta and Gamma functions. Problems.

Self-Study: Center of gravity, Duplication formula. **Applications:** Antenna and wave propagation, Calculation of optimum value in various geometries. Analysis of probabilistic models.

(RBT Levels: L1, L2 and L3)

1.1 Evaluation of double integrals

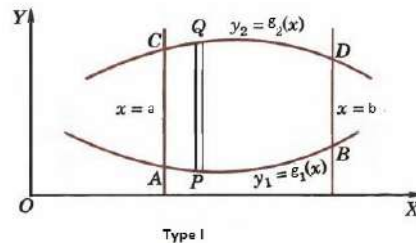
Let us consider a function $f(x, y)$ defined in Cartesian region. If this region is denoted by R , then double integral is defined as

$$\int \int_R f(x, y) dx dy$$

where R is called the region of integration and is a region in the (x, y) plane.

There are two types of regions of double integrals. These are as follows:

Type I :

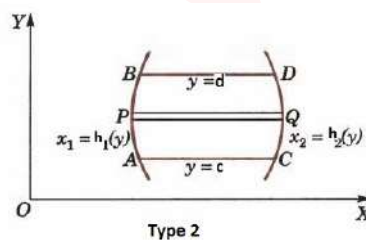


The region is defined as $R = (x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$ and the integral is defined as,

$$\int \int_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

We call $g_1(x)$ and $g_2(x)$ as the **inner limits** of integration and we call a and b as the **outer limits** of integration. Here, inner limits are functions of x and outer limits are constants. To evaluate this double integral, we first integrate $f(x, y)$ w.r.to y treating x as a constant, between the limits $y = g_1(x)$ to $y = g_2(x)$ and then the resulting expression is integrated w.r.to x between the outer limits.

Type II :



The region is defined as $R = (x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$ and the integral is defined as,

$$\int \int_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

We call $h_1(y)$ and $h_2(y)$ as the **inner limits** of integration and we call c and d as the **outer limits** of integration. Here, inner limits are functions of y and outer limits

are constants. To evaluate this double integral, we first integrate w.r.to x treating y as a constant, between the limits $y = h_1(y)$ to $y = h_2(y)$ and then the resulting expression is integrated w.r.to y between the outer limits.

Remember:

- The outer limits of integration are always constants.
- Both inner and outer limits of integration are constant if and only if the region of integration R is a rectangle.
- When the double integral is evaluated, the result is a real number, not a variable function.

Problem 1.1.1. Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y dx dy$ (VTU Jan 2017, Dec 2011)

Solution:

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y dx dy &= \int_0^1 y \int_0^{\sqrt{1-y^2}} x^3 dx dy \\ &= \frac{1}{4} \int_0^1 y [x^4]_0^{\sqrt{1-y^2}} dy \\ &= \frac{1}{4} \int_0^1 y [(1-y^2)^2] dy \\ &= \frac{1}{4} \int_0^1 [y + y^5 - 2y^3] dy \\ &= \frac{1}{4} \left[\frac{y^2}{2} + \frac{y^6}{6} - \frac{y^4}{2} \right]_0^1 \\ &= \frac{1}{24} \end{aligned}$$

Problem 1.1.2. Evaluate: $\int_0^1 \int_x^{\sqrt{x}} (x^2 y + xy^2) dy dx$

Solution: Let

$$\begin{aligned}
 I &= \int_0^1 \left[\int_x^{\sqrt{x}} (x^2y + xy^2) dy dx \right] \\
 &= \int_0^1 \left[\int_x^{\sqrt{x}} (x^2y + xy^2) dy dx \right] \\
 &= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3} \right)_x^{\sqrt{x}} dx \\
 &= \int_0^1 \left[\left(\frac{x^3}{2} + \frac{x \cdot xy^{3/2}}{3} \right) - \left(\frac{x^4}{2} + \frac{x^4}{3} \right) \right] dx \\
 &= \left[\frac{x^4}{8} + \frac{x^{7/2}}{(7/2)(3)} - \frac{5}{6} \left(\frac{x^5}{5} \right) \right]_0^1 \\
 &= \left(\frac{1}{8} + \frac{2}{21} + \frac{1}{6} \right) - (0) \\
 &= \left(\frac{21 + 16 - 28}{168} \right) \\
 &= \frac{9}{168} = \frac{3}{56}
 \end{aligned}$$

1.2 Evaluation of Triple Integrals

Let $f(x, y, z)$ be a function of three variable defined in a three dimensional region R. The triple integral of $f(x, y, z)$ over R is denoted by

$$\int \int \int_R f(x, y, z) dx dy dz$$

To evaluate a triple integral in the form $\int_a^b \int_{f(x)}^{g(x)} \int_{f(x,y)}^{g(x,y)} f(x, y, z) dx dy dz$, we first integrate the function w.r.t 'z' between the limits $f(x, y)$ to $g(x, y)$ keeping x and y fixed. The resulting expression is integrated w.r.t 'y' between the limits $f(x)$ to $g(x)$, keeping x constant. The result just obtained is finally integrated w.r.t x from a to b.

Thus

$$\int_a^b \int_{f(x)}^{g(x)} \int_{f(x,y)}^{g(x,y)} f(x, y, z) dx dy dz = \int_a^b \left\{ \int_{f(x)}^{g(x)} \left[\int_{f(x,y)}^{g(x,y)} f(x, y, z) dz \right] dy \right\} dz$$

Problem 1.2.1. Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z) dx dy dz$ (VTU Jan 2019, Jan 2017, Jan 2016, Jan 2015, July 2011, July 2009)

Solution: Integrating first w.r.t. y keeping x and z constant, we have

$$\begin{aligned}
 I &= \int_{-1}^1 \int_0^z \left[xy + \frac{y^2}{2} + yz \right]_{x-z}^{x+z} dx dz \\
 &= \int_{-1}^1 \int_0^z \left[(x+z)(2z) + \frac{1}{2}4xz \right] dx dz \\
 &= 2 \int_{-1}^1 \left[\frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right]_0^z dz \\
 &= 2 \int_{-1}^1 \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2} \right) dz \\
 &= 4 \left[\frac{z^4}{4} \right]_{-1}^1 \\
 &= 0.
 \end{aligned}$$

Problem 1.2.2. $\int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dz dy dx$ (VTU June 2019, Model 2018, July 2017, Jan 2016, July 2014, July 2013)

Solution: Let

$$\begin{aligned}
 I &= \int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dz dy dx \\
 &= \int_{-c}^c \left[\int_{-b}^b \left(ax^2 + ay^2 + \frac{z^3}{3} \right) \right]_{-a}^a dy dx \\
 &= \int_{-c}^c \int_{-b}^b \left(2ax^2 + 2ay^2 + \frac{2a^3}{3} \right) dy dx \\
 &= \int_{-c}^c \left[\left(2axy^2 + \frac{2ay^3}{3} + \frac{2ya^3}{3} \right) \right]_{-b}^b dx \\
 &= \int_{-c}^c \left(4abx^2 + \frac{4ab^3}{3} + \frac{4ba^3}{3} \right) dx \\
 &= 4ab \left[\frac{x^3}{3} + \frac{xb^2}{3} + \frac{xa^2}{3} \right]_{-c}^c \\
 \therefore I &= \frac{8abc}{3} (a^2 + b^2 + c^2)
 \end{aligned}$$

Problem 1.2.3. Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dz dy dx$ (VTU Jan 2020, June 2018, Jan 2013, Dec 2010, Jan 2008)

Solution: The given triple integral is

$$\begin{aligned}
 I &= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy dx \\
 &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2} xy (1-x^2-y^2) dy dx \\
 &= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \frac{1}{2} x (y - x^2 y - y^3) dy \right] dx \\
 &= \frac{1}{2} \int_0^1 x \left[\frac{y^2}{2} - x^2 \frac{y^2}{2} - \frac{y^4}{4} \right]_0^{\sqrt{1-x^2}} dx \\
 &= \frac{1}{2} \int_0^1 x \left[\frac{(1-x^2)}{2} - x^2 \frac{(1-x^2)}{2} - \frac{(1-x^2)^2}{4} \right] dx \\
 &= \frac{1}{2} \int_0^1 x \left[\frac{2(1-x^2) - 2x^2(1-x^2) - (1-x^2)^2}{4} \right] dx \\
 &= \frac{1}{8} \int_0^1 x (1-x^2)^2 dx \quad (\text{on simplification}) \\
 &= \frac{1}{8} \int_0^1 x (1 - 2x^2 + x^4) dx \\
 &= \frac{1}{8} \int_0^1 (x - 2x^3 + x^5) dx \\
 &= \frac{1}{8} \left(\frac{x^2}{2} - 2 \frac{x^4}{4} + \frac{x^6}{6} \right)_0^1 \\
 &= \frac{1}{48} \quad (\text{After simplification})
 \end{aligned}$$

Problem 1.2.4. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$ (VTU July 2016, Dec 2010, Jan 2010, July 2007)

Solution: We have

$$\begin{aligned}
 & \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx. \\
 &= \int_0^a \int_0^x e^{x+y+z} \Big|_0^{x+y} dy dx \\
 &= \int_0^a \int_0^x [e^{2(x+y)} - e^{x+y}] dy dx \\
 &= \int_0^a \left[\int_0^x e^{2(x+y)} dy - \int_0^x e^{x+y} dy \right] dx \\
 &= \int_0^a \left\{ \left[\frac{e^{2(x+y)}}{2} \right]_0^x - [e^{x+y}]_0^x \right\} dx \\
 &= \int_0^a \left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x \right] dx \\
 &= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{4} - \frac{e^{2x}}{2} + e^x \right]_0^a \\
 &= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a.
 \end{aligned}$$

1.3 Evaluation of Double integral when Region is Given

Let $f(x, y)$ be a function defined over a domain D (or Region R) in xy plane and we need to find the double integral of $f(x, y)$. Draw the figure from the given data to identify the specific region R.

If we divide, the required region into vertical stripes and carefully find the end points for x and y i.e. the limits of the region, then we can use the formula Then express I in any one of the following form. $I = \int_a^b \int_{f(x)}^{g(x)} f(x, y) dy dx$ If we divide, the required region into vertical stripes and carefully find the end points for x and y.

$$I = \int_c^d \int_{f(y)}^{g(y)} f(x, y) dx dy$$

Problem 1.3.1. Evaluate $\int \int xy(x + y) dx dy$ over the region between $y = x$ and $y = x^2$ (VTU Jan 2015, Dec 2010, Jan 2010).

Solution: Let $I = \int \int xy(x + y) dx dy$ Since $y = x^2$ and $y = x$
 $x^2 = x$ which gives us point of intersection $(0, 0)$ and $(1, 1)$

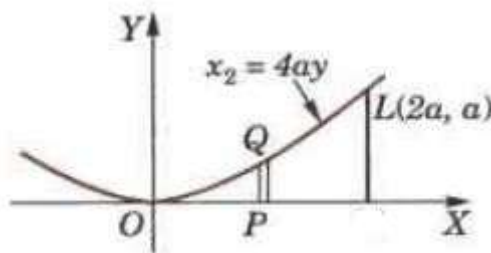
Clearly x varies from 0 to 1

and y varies from x^2 to x .

$$\begin{aligned} \therefore I &= \int_0^1 \int_{x^2}^x (x^2y + xy^2) dy dx \\ &= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^x dx \\ &= \int_0^1 \left[\left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3} \right) \right] dx \\ &= \left[\left(\frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24} \right) \right]_0^1 \\ &= \frac{3}{56} \end{aligned}$$

Problem 1.3.2. Evaluate $\int \int_R xy dx dy$, where R is the region bounded by x - axis, the ordinate $x = 2a$ and $x^2 = 4ay$ (VTU Jan 2016)

Solution: The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at L .



When $x = 2a$ and $x^2 = 4ay \therefore 4a^2 = 4ay \Rightarrow y = a \therefore$ The point of intersection of $x = 2a$ and $x^2 = 4ay$ is $L(2a, a)$

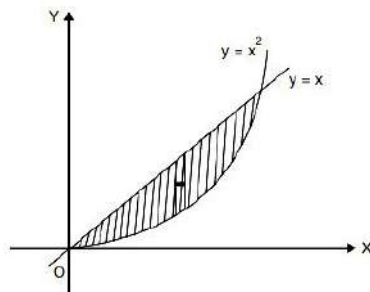
Integrating first over a vertical strip PQ , i.e., w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$

on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\begin{aligned}\iint_A xy dx dy &= \int_{x=0}^{2a} \int_{y=0}^{\frac{x^2}{4a}} xy dy dx \\ &= \int_0^{2a} x \left[\frac{y^2}{2} \right]_0^{\frac{x^2}{4a}} dx \\ &= \int_0^{2a} \frac{x^5}{32a^2} dx \\ &= \left[\frac{x^6}{32a^2 \times 6} \right]_0^{2a} \\ &= \frac{a^4}{3}\end{aligned}$$

Problem 1.3.3. Evaluate $\int \int xy(x+y) dx dy$ over the region between $y = x$ and $y = x^2$
(VTU June 2018, Jan 2015, Dec 2010, Jan 2010)

Solution: The bounded curves are $y = x^2$ and $y = x$.



The common points are given by solving the two equations. So, we have

$$x^2 = x = x(x - 1) = 0$$

$$\Rightarrow x = 0 \text{ or } 1$$

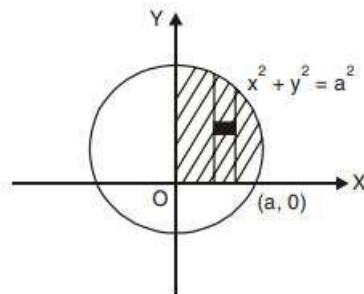
when $x = 0$, we have $y = 0$ and when $x = 1$, $y = 1$ (from $y = x$) Considering a vertical strip y varies from $y = x^2$ to $y = x$ and then we slide the strip from

$x = 0$ to $x = 1$

$$\begin{aligned}
 \therefore \iint_R xy(x+y) dx dy &= \int_{x=0}^1 \int_{y=x^2}^x xy(x+y) dy dx \\
 &= \int_0^1 x \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{x^2}^x dx \\
 &= \int_0^1 x \left\{ x \left(\frac{x^2}{2} - \frac{x^4}{2} \right) + \left(\frac{x^3}{3} - \frac{x^6}{3} \right) \right\} dx \\
 &= \int_0^1 \left(\frac{5}{6}x^4 - \frac{x^6}{2} - \frac{x^7}{3} \right) dx \\
 &= \left[\frac{5x^5}{6 \cdot 5} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1 \\
 &= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}.
 \end{aligned}$$

Problem 1.3.4. Evaluate $\iint xy dx dy$ over the area in the first quadrant bounded by the circle $x^2 + y^2 = a^2$

Solution:



$$\begin{aligned}
\iint xy \, dx \, dy &= \int_{x=0}^a \left[\int_{y=0}^{\sqrt{a^2-x^2}} xy \, dy \right] dx \\
\{ \because x^2 + y^2 &= a^2 \\
\Rightarrow y^2 &= a^2 - x^2 \\
y &= \sqrt{a^2 - x^2} \\
&= \int_0^a x \cdot \left[\frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} dx \\
&= \int_0^a x \left(\frac{a^2 - x^2}{2} \right) dx \\
&= \frac{1}{2} \int_0^a (a^2x - x^3) dx \\
&= \frac{1}{2} \left[a^2 \frac{x^2}{2} - \frac{x^4}{4} \right]_0^a \\
&= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] \\
&= \frac{a^4}{8}
\end{aligned}$$

1.4 Evaluation of Double integral by changing the order of integration

In a double integral with variable limits, the change of order of integration changes the limits of integration. i.e. The double integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) \, dy \, dx$ takes the equivalent form $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy$ or vice versa. This process of converting a given double integral into its equivalent double integral by changing the order of integration is often called change of order of integration or reversing the order of integration. Some of the problems connected with double integrals, which are complicated, can be made easy to evaluate by a change in the order of integration. one such double integral is, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx \, dy$. Here it is difficult to integrate $\frac{e^{-y}}{y}$ first w.r.to y.

The following steps are used in change of order of integration.

- Use the limits of given integral and draw a rough sketch of the region of integration.
- Then use the sketch to figure out how to write the limits of integration in the reverse order.
- In order to do that, you will have to write down on your sketch the equations of the various sides of the sketch, written in a suitable form depending on whether you are viewing the region as Type I or Type II.
- Note that this process does not change the function, which you are integrating. It changes only the limits of integration.
- Sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits.

Problem 1.4.1. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dx dy$ by changing the order of integration.

Solution: Given region:

x varies from 0 to 1

y varies from 0 to $\sqrt{1-x^2}$.

After changing the order :

x varies from 0 to $\sqrt{1-y^2}$

y varies from 0 to 1.

$$\begin{aligned}
 I &= \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 dx dy \\
 &= \int_0^1 [y^2 x]_0^{\sqrt{1-y^2}} dy \\
 &= \int_0^1 [y^2 \sqrt{1-y^2}] dy
 \end{aligned}$$

$$y = \sin \theta \implies dy = \cos \theta d\theta$$

$$\theta \text{ varies from } 0 \text{ to } \frac{\pi}{2}$$

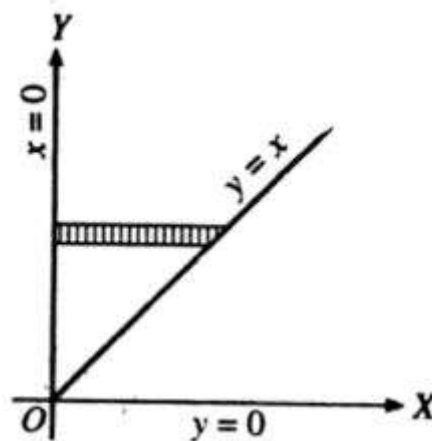
$$\therefore I = \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta$$

$$I = \frac{\pi}{16}$$

Problem 1.4.2. Change the order of integration and evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx dy$

Solution: In the given integral the limits of integration area given by the lines $y = x$, $y = \infty$, $x = 0$ and $x = \infty$ and the first integration is w.r. to y .

Therefore the region of integration is bounded $y = x$, $y = \infty$, $x = 0$ and $x = \infty$



If we want to reverse the order of integration, we have to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y . This is done by considering strip parallel to the x -axis.

Starting from the line $x = 0$ and terminating on the line $x = y$ (new inner limits)

we slide the strip from $y = 0$ to $y = \infty$ (new outer limits).

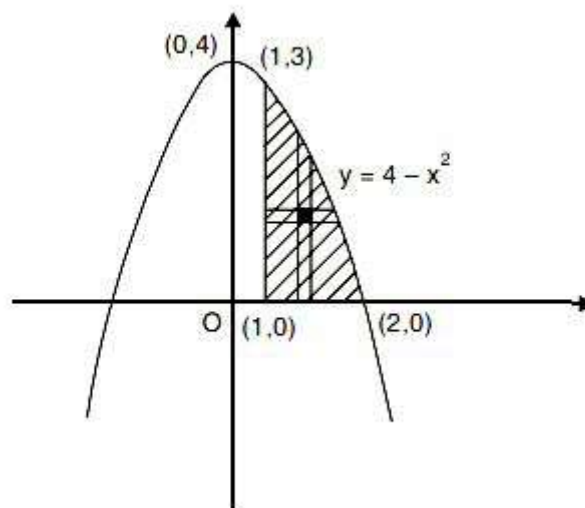
$$\begin{aligned}
 \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dx dy &= \int_0^{\infty} \int_0^y \frac{e^{-y}}{y} dy dx \\
 &= \int_0^{\infty} \frac{e^{-y}}{y} [x]_0^y dy \\
 &= \int_0^{\infty} \frac{e^{-y}}{y} \cdot y dy \\
 &= \int_0^{\infty} e^{-y} dy \\
 &= \left[\frac{e^{-y}}{-1} \right]_0^{\infty} \\
 &= 1
 \end{aligned}$$

Problem 1.4.3. Change the order of integration and hence evaluate

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy.$$

Solution: Here, the limits for x are $x = 1$ and $x = \sqrt{4-y}$ and limits for y are 0 and 3.

$x = \sqrt{4-y} \Rightarrow x^2 = 4-y \Rightarrow y = 4-x^2$, which is a parabola.



When $x = 1$, on $y = 4 - x^2$, $y = 3$

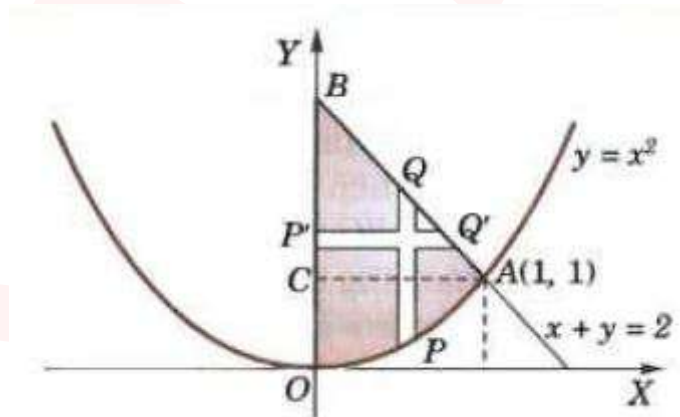
On changing the order, considering a horizontal strip, new inner limits are $y = 0$ to

$y = 4 - x^2$ and new outer limits are $y = 0$ to $y = 3$

$$\begin{aligned}
 \text{Now, } \int_{y=0}^3 \int_{x=1}^{\sqrt{4-y}} (x+y) dx dy &= \int_{x=1}^2 \int_{y=0}^{4-x^2} (x+y) dy dx \\
 &= \int_1^2 [xy + y^2/2]_0^{4-x^2} dx \\
 &= \int_1^2 \left(4x - x^3 + 8 - 4x^2 + \frac{x^4}{2} \right) dx \\
 &= \left[2x^2 - \frac{x^4}{4} + 8x - \frac{4}{3}x^3 + \frac{x^5}{10} \right]_1^2 \\
 &= 6 - \frac{15}{4} + 8 - \frac{28}{3} + \frac{31}{10} \\
 &= \frac{241}{60}
 \end{aligned}$$

Problem 1.4.4. Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy \, dx dy$ and hence evaluate the same.

Solution: Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as in the figure.



On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$),

For the region OAC , the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region

OAC is

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy dx$$

For the region ABC , the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy dx$$

Hence, on reversing the order of integration,

$$\begin{aligned} I &= \int_0^1 dy \int_0^{\sqrt{y}} xy dx + \int_1^2 dy \int_0^{2-y} xy dx \\ &= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y} \\ &= \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 y(2-y)^2 dy \\ &= \frac{1}{6} + \frac{5}{24} \\ &= \frac{3}{8}. \end{aligned}$$

Problem 1.4.5. Evaluate the integral $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy$ by changing the order of integration.

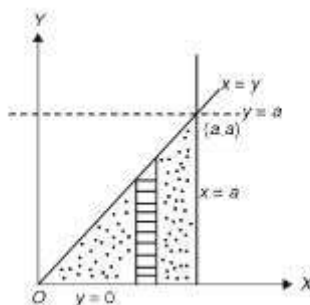
Solution: The given integral is

$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy.$$

Here inner limits are $x = y$ to $x = a$ and outer limits are $y = 0$ to $y = a$, and the first integration is w.r.to y

The region of integration is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$.

Thus the region of integration is shown in the figure below:



On changing the order of integration, we first integrate with respect to y along the strip parallel to y axis. The strip extends from $y = 0$ to $y = x$ (new inner limits). To cover the whole region, we then integrate with respect to x from $x = 0$ to $x = a$ (new outer limits).

Hence

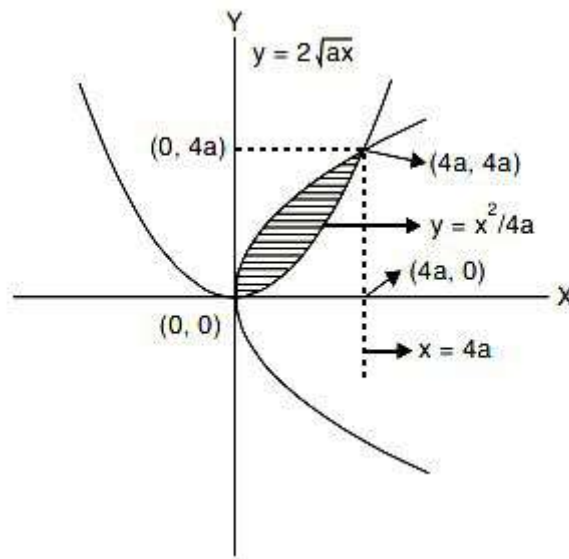
$$\begin{aligned} I &= \int_0^a \int_0^x \frac{x}{x^2 + y^2} dx dy = \int_a^a x \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_0^x dx \\ &= \int_0^a \frac{\pi}{4} dx = \frac{\pi}{4} a. \end{aligned}$$

Problem 1.4.6. Change the order of integration and hence evaluate

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy dy dx$$

(VTU Jan 2017, Model 2015)

Solution:



Let us first find the point of intersection :

We have $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$

or $\frac{x^2}{4a} = 2\sqrt{ax}$

i.e. $x^4 = 64a^3x$

i.e., $x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0$ and $x = 4a$

From $y = x^2/4a$, we get $y = 0$ and $y = 4a$

Thus the points of intersection of the parabola $y = x^2/4a$ and $y = 2\sqrt{ax}$ are $(0, 0)$ and $(4a, 4a)$

on changing the order of integration we take a horizontal strip and first integrate w.r.to x

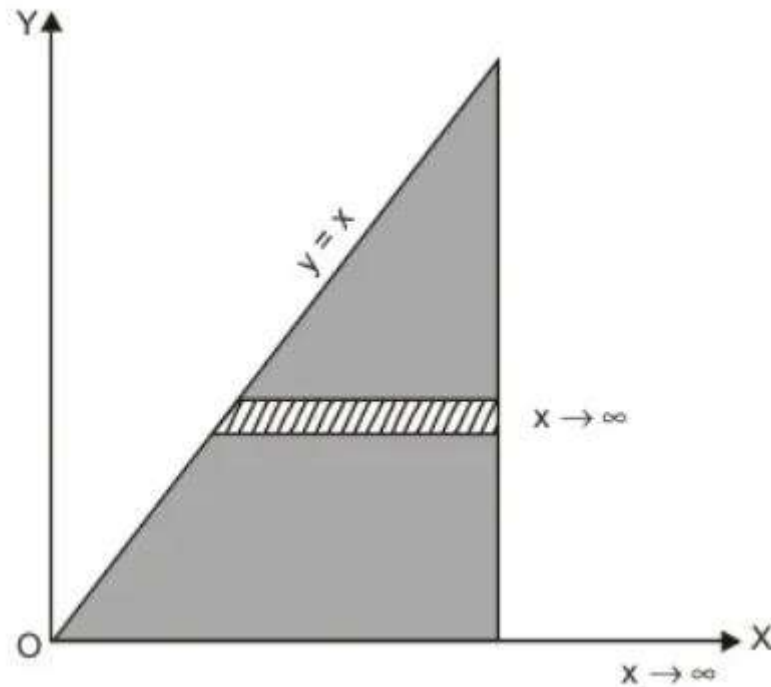
we have x varying from $y^2/4a$ to $2\sqrt{ay}$ (new inner limits) and y varying from 0 to $4a$ (new outer limits)

Thus

$$\begin{aligned}
 I &= \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} xy \, dx \, dy \\
 &= \int_{y=0}^{4a} y \cdot \left[\frac{x^2}{2} \right]_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy \\
 &= \frac{1}{2} \int_{y=0}^{4a} y \left[4ay - \frac{y^4}{16a^2} \right] dy \\
 &= \frac{1}{2} \int_{y=0}^{4a} \left(4ay^2 - \frac{y^5}{16a^2} \right) dy \\
 &= \frac{1}{2} \left[\frac{4ay^3}{3} - \frac{1}{16a^2} \cdot \frac{y^6}{6} \right]_{y=0}^{4a} \\
 &= \frac{1}{2} \left[4a \left(\frac{64a^3}{3} \right) - \frac{1}{96a^2} (4096a^6) \right] \\
 &= \frac{1}{2} \left[\frac{256a^3}{3} - \frac{128a^4}{3} \right] \\
 &= \frac{64a^4}{3}
 \end{aligned}$$

Problem 1.4.7. Evaluate the integral $\int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx dy$ by changing the order of integration. (VTU July 2016, July 2009, Jan 2009, July 2008, July 2007)

Solution: Here $y = 0$ and $y = x$, $x = 0$ and $x = \infty$



On changing the order, we consider a horizontal strip.

Here x start from $x = y$ and goes to $x \rightarrow \infty$ and y varies from $y = 0$ to $y \rightarrow \infty$

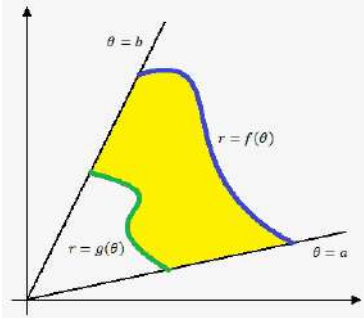
$$\begin{aligned} \therefore \int_0^{\infty} \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx dy &= \int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-\frac{x^2}{y}} \cdot dy dx \end{aligned}$$

Put $e^{-\frac{x^2}{y}} = t \Rightarrow -\frac{2x}{y} e^{-\frac{x^2}{y}} dx = dt$

$$\begin{aligned} \therefore \int_0^{\infty} \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx dy &= \int_0^{\infty} \left[-\frac{y}{2} e^{-\frac{x^2}{y}} \right]_y^{\infty} dy \\ &= \int_0^{\infty} \left[0 + \frac{y}{2} e^{-\frac{y^2}{y}} \right] dy = \int_0^{\infty} \frac{y}{2} e^{-y} dy \\ &= \left[\frac{y}{2} (-e^{-y}) - \frac{1}{2} (e^{-y}) \right]_0^{\infty} \\ &= (0 - 0) + \left(0 + \frac{1}{2} \right) \\ &= \frac{1}{2}. \end{aligned}$$

1.5 Double Integrals in Polar Coordinates

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta) dr d\theta$, we first integrate w.r.to. r between the limits $r = r_1$ and $r = r_2$, treating θ as a constant and then the resulting expression is integrated w.r.to. θ from θ_1 to θ_2 . In this integral r_1 and r_2 are functions of θ and θ_1 and θ_2 are constants.



1.6 Evaluation of Double integrals by changing in to polar form

Let (r, θ) be the polar co-ordinates of the point (x, y) . Where $x = r \cos \theta$, $y = r \sin \theta$ and $r^2 = x^2 + y^2$

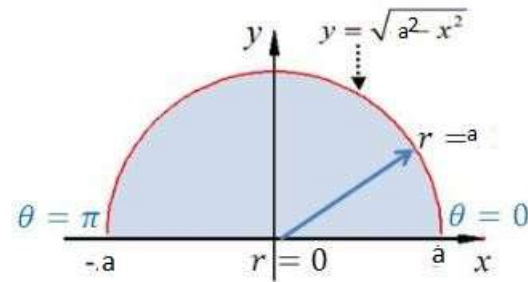
$$\text{Then } \int \int_R dx dy = \int \int_R r dr d\theta$$

$$\therefore dx dy = J dr d\theta$$

$$\text{Where } J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

Problem 1.6.1. Evaluate $\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2 + y^2} dx dy$ by changing in to polar form.

Solution: In the given Region,
 y varies from 0 to $\sqrt{a^2 - x^2}$.
 x varies from $-a$ to a



After changing to polar form :

r varies from 0 to a

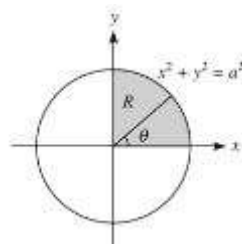
θ varies from 0 to π . Where $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$, $dx dy = r dr d\theta$

$$\begin{aligned}
 \therefore I &= \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2 + y^2} dx dy \\
 &= \int_0^a \int_0^{\pi} \sqrt{r^2} r dr d\theta \\
 &= \int_0^a \int_0^{\pi} r^2 dr d\theta \\
 &= \int_0^a \pi r^2 dr \\
 &= \frac{\pi a^3}{3}
 \end{aligned}$$

Problem 1.6.2. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2 + y^2} dy dx$.

Solution: Let $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2 + y^2} dy dx$

Here y varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$ i.e. $x^2 + y^2 = a^2$ and x varies from $x = 0$ to $x = a$ (see the Figure)



Putting $x = r \cos \theta$, $y = r \sin \theta$, we get $r = a$.

Therefore in the given region R (which is first quadrant) r varies from $r = 0$ to a and θ varies from $\theta = 0$ to $\frac{\pi}{2}$. Hence

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^a r^2 \sin^2 \theta r r dr d\theta \\ &= \int_0^{\pi/2} \int_0^a r^4 \sin^2 \theta dr d\theta \\ &= \frac{a^5}{5} \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2} \right) \theta \\ &= \frac{a^5 \pi}{20} \end{aligned}$$

Problem 1.6.3. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ using polar coordinates

Solution:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad dx dy = r dr d\theta$$

are the polar coordinates for the above integral.

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta \\ &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \frac{1}{2} d(r^2) d\theta \\ &= \int_0^{\pi/2} \frac{1}{2} \left[-e^{-r^2} \right]_0^\infty d\theta \\ &= -\frac{1}{2} \int_0^{\pi/2} \left[e^{-r^2} \right]_0^\infty d\theta \\ &= -\frac{1}{2} \int_0^{\pi/2} [0 - 1] d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} d\theta \\ &= \frac{1}{2} [\theta]_0^{\pi/2} \\ &= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right] \\ &= \frac{\pi}{4} \end{aligned}$$

1.7 Application of multiple integrals to find area and volume

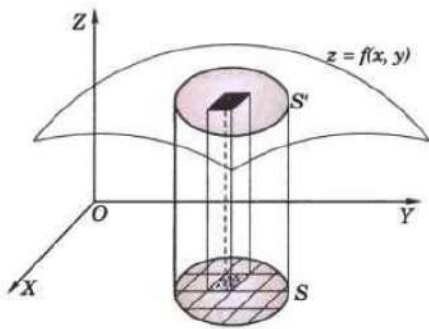
Area enclosed by a plane curve in Cartesian form:

$$\text{Area} = \iint_R dx dy$$

Area enclosed by a plane curve in polar form:

$$\text{Area} = \iint_R r dr d\theta$$

Consider a surface $z = f(x, y)$. Let the orthogonal projection of this surface on XY -plane be the area S

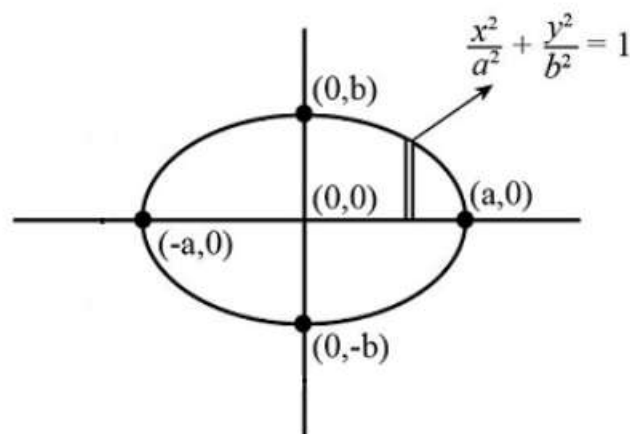


Then the volume of the solid bounded by $z = f(x, y)$ and its projection R on the XY plane is given by

$$\text{Volume} = \iiint z dx dy \quad \text{or} \quad \iint f(x, y) dx dy$$

Problem 1.7.1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

solution This ellipse is symmetric w.r.t x - axis and y - axis.



\therefore Area of the ellipse = 4 times Area of the region in the first quadrant.

In this region x varies from 0 to a and y varies from 0 to $\frac{b\sqrt{a^2-x^2}}{a}$

$$\therefore \text{Area of ellipse} = 4 \left[\int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{a}} dy dx \right]$$

$$= 4 \left[\int_0^a \frac{b\sqrt{a^2-x^2}}{a} dx \right]$$

$$\text{put } x = a \sin \theta \therefore dx = a \cos \theta d\theta$$

$$\text{when } x = 0, \theta = 0 \text{ and when } x = a, \theta = \frac{\pi}{2}$$

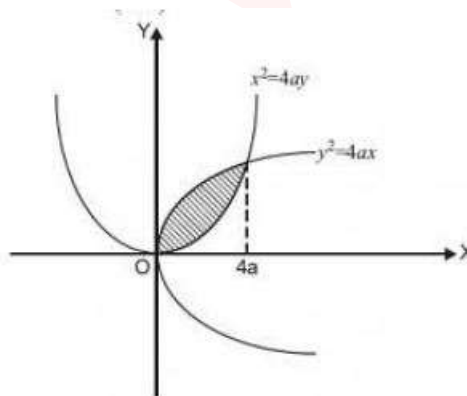
$$\therefore \text{Area of ellipse} = 4 \left[\int_0^{\frac{\pi}{2}} \left[\frac{b\sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta}{a} \right] d\theta \right]$$

$$A = 4ab \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$A = 4ab \frac{\pi}{4} = \pi ab \text{ sq. units.}$$

Problem 1.7.2. Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Solution:



Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at $O(0, 0)$ and $A(4a, 4a)$. As such for the shaded area between these parabolas, considering vertical strip PQ, y varies from $y = x^2/4a$ to $y = 2\sqrt{ax}$

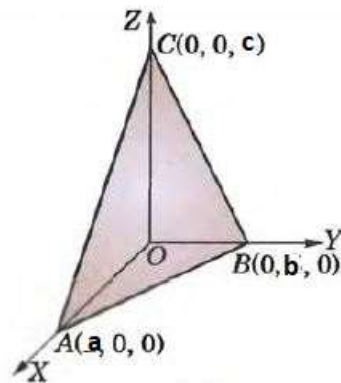
and x varies from 0 to $4a$.

Hence the required area is

$$\begin{aligned}
 \text{Area} &= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx \\
 &= \int_0^{4a} \left(2\sqrt{ax} - x^2/4a \right) dx \\
 &= \left[2\sqrt{a} \cdot \frac{2}{3} x^{3/2} - \frac{1}{4a} \cdot \frac{x^3}{3} \right]_0^{4a} \\
 &= \frac{32}{3} a^2 - \frac{16}{3} a^2 \\
 &= \frac{16}{3} a^2.
 \end{aligned}$$

Problem 1.7.3. Find the volume of the tetrahedron bounded by the coordinate surfaces $x = 0$, $y = 0$ and $z = 0$ and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

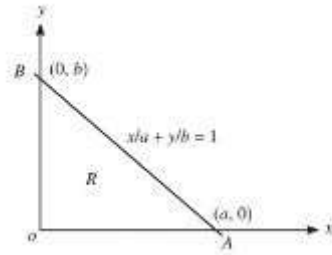
Solution:



The volume V of the tetrahedron is given by

$$V = \iint_R z dx dy$$

where R is the triangular region in the xy -plane (Figure) whose sides are $x = 0$, $y = 0$ and $x/a + y/b = 1$.



Therefore,

$$\begin{aligned}
 V &= \iint_R z \, dx \, dy \\
 &= \int_0^a \int_0^{b(1-\frac{x}{a})} \left(1 - \frac{y}{b} - \frac{x}{a}\right) \, dy \, dx \\
 &= \int_0^a \left[y - \frac{y^2}{2b} - \frac{xy}{a} \right]_0^{b-\frac{bx}{a}} \, dx \\
 &= \int_0^a \left(\frac{bx^2}{2a^2} - \frac{bx}{a} + \frac{b}{2} \right) \, dx \\
 &= \frac{abc}{6} \quad (\text{after simplification})
 \end{aligned}$$

1.8 Beta and Gamma functions

Beta function : An expression of the form

$$B(m, n) = \int_{x=0}^1 x^{(m-1)} (1-x)^{(n-1)} \, dx$$

where $m, n > 0$ is called an Beta function.

Alternative definition of Beta function:

$$B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta$$

Gamma Function : An expression of the form

$$\Gamma(n) = \int_{x=0}^{\infty} e^{-x} x^{n-1} \, dx$$

where $n > 0$ is called an Gamma function.

Alternative definition of Gamma function:

$$\Gamma(n) = 2 \int_0^{\infty} e^{-t^2} t^{2n-1} dt$$

Properties:

$$B(m, n) = B(n, m)$$

$$\Gamma(n + 1) = n\Gamma(n)$$

$$\Gamma(n + 1) = n!, \quad \text{when } n \text{ is an integer}$$

Also

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Note: When n is a -ve fraction :

We have $\Gamma(n + 1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n + 1)}{n}$$

Problem 1.8.1. Evaluate $\Gamma(4)$

Solution: Gamma integral $\Gamma(n + 1) = n!$ where n is a positive integer.

$$\therefore \Gamma(4) = (4 - 1)! = 3! = 3 \times 2 = 6$$

Problem 1.8.2. Evaluate $\Gamma\left(\frac{9}{2}\right)$

Solution: We know $\Gamma(n + 1) = n\Gamma(n)$

$$\begin{aligned}
 \therefore \Gamma\left(\frac{9}{2}\right) &= \Gamma\left(\frac{7}{2} + 1\right) \\
 &= \frac{7}{2}\Gamma\left(\frac{7}{2}\right) \\
 &= \frac{7}{2}\Gamma\left(\frac{5}{2} + 1\right) \\
 &= \frac{7}{2} \times \frac{5}{2} \times \Gamma\left(\frac{5}{2}\right) \\
 &= \frac{7}{2} \times \frac{5}{2} \times \Gamma\left(\frac{3}{2} + 1\right) \\
 &= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{3}{2}\right) \\
 &= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{1}{2} + 1\right) \\
 &= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \\
 &= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \quad (\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}) \\
 &= \frac{105}{16}\sqrt{\pi}
 \end{aligned}$$

Problem 1.8.3. Compute $\Gamma\left(-\frac{1}{2}\right)$

Solution: We have $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

$$\begin{aligned}
 \text{Put } n &= \left(\frac{-1}{2}\right) \\
 \Gamma\left(-\frac{1}{2}\right) &= \frac{\Gamma(1/2)}{\frac{-1}{2}} = -2\sqrt{\pi}
 \end{aligned}$$

Relation Between Beta and Gamma function :

Problem 1.8.4. Prove that $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (VTU Jan 2017, July 2016, July 2013, July 2011, July 2008)

Proof: We have

$$B(m, n) = 2 \int_0^{\frac{\pi}{2}} [\sin^{2m-1} \theta \cos^{2n-1} \theta d(\theta)]$$

$$\Gamma(n) = 2 \int_0^{\infty} [e^{-x^2} x^{2n-1} dx]$$

$$\Gamma(m) = 2 \int_0^{\infty} [e^{-y^2} y^{2m-1} dy]$$

$$\Gamma(m+n) = 2 \int_0^{\infty} [e^{-r^2} r^{2(m+n)-1} dr]$$

$$\begin{aligned} \text{Now } \Gamma(m)\Gamma(n) &= 2 \int_0^{\infty} [e^{-x^2} x^{2n-1} dx] \times 2 \int_0^{\infty} [e^{-y^2} y^{2m-1} dy] \\ &= 4 \int_0^{\infty} \int_0^{\infty} [e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dx dy] \end{aligned}$$

Let us evaluate this double integral by changing in to polar co-ordinate systems.

put $x = r \cos(\theta)$, $y = r \sin(\theta)$, and $r^2 = x^2 + y^2$, $dx dy = r dr d(\theta)$

In the region of integration, r varies from 0 to ∞

θ varies from 0 to $\frac{\pi}{2}$.

$$\begin{aligned} \Gamma m \Gamma n &= 4 \int_0^{\infty} \int_0^{\frac{\pi}{2}} [e^{-r^2} r^{2n-1} r^{2m-1} \sin^{2m-1} \theta \cos^{2n-1} \theta r dr d(\theta)] \\ &= 2 \int_0^{\infty} [e^{-r^2} r^{2(m+n)-1} dr] \cdot 2 \int_0^{\frac{\pi}{2}} [\sin^{2m-1} \theta \cos^{2n-1} \theta d(\theta)] \\ &= \Gamma(m+n) B(m, n) \end{aligned}$$

$$i.e. B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Hence the proof.

Problem 1.8.5. S.T $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Solution: We know that $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m > 0$, $n > 0$

Taking $m = n = \frac{1}{2}$, we have

$$\begin{aligned} B\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} \\ &= \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \dots\dots (1) \quad [\because \Gamma(1) = 1] \end{aligned}$$

But using

$$B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d(\theta)$$

we can write

$$\begin{aligned} B\left(\frac{1}{2}, \frac{1}{2}\right) &= 2 \int_0^{\frac{\pi}{2}} \sin^{2 \times \frac{1}{2} - 1} \theta \cos^{2 \times \frac{1}{2} - 1} \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \sin^0 \theta \cos^0 \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} d\theta \\ &= 2[\theta]_0^{\frac{\pi}{2}} \\ &= 2\left[\frac{\pi}{2} - 0\right] \\ &= \pi \quad \dots\dots (2) \end{aligned}$$

From Equation (1) and (2),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

1.9 Evaluation of Integrals Using Beta and Gamma functions :

Some Useful Formulae :

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d(\theta) = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

In particular,

$$\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$$

Problem 1.9.1. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\cot \theta} d\theta = \frac{\pi}{\sqrt{2}}$

Solution: We know that,

$$\begin{aligned}
 \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta &= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \\
 &= \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \\
 \int_0^{\pi/2} \sqrt{\cot \theta} d\theta &= \int_0^{\pi/2} \frac{\cos^{1/2} \theta}{\sin^{1/2} \theta} d\theta \\
 &= \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta d\theta \\
 &= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} \\
 &= \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} \\
 &= \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) \\
 &= \frac{1}{2} \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} \\
 &\quad \left(\text{using } \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}, \quad n = \frac{1}{4}\right) \\
 &= \frac{1}{2} \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} \\
 &= \frac{\pi}{\sqrt{2}}
 \end{aligned}$$

Problem 1.9.2. Evaluate $\int_0^2 (4-x^2)^{\frac{3}{2}} dx$ using Beta and Gamma function.

Solution: Let $I = \int_0^2 (4-x^2)^{\frac{3}{2}} dx$

Put $x = 2\sin\theta$ $dx = 2\cos\theta d\theta$

when $x = 0, \theta = 0$

and when $x = 2, \theta = \frac{\pi}{2}$

$$\begin{aligned}
 \therefore I &= \int_0^{\frac{\pi}{2}} [4 - 4 \sin^2 \theta]^{\frac{3}{2}} 2 \cos \theta d\theta \\
 &= 16 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta \\
 &= 16 \int_0^{\frac{\pi}{2}} \sin^0 \theta \cos^4 \theta d\theta \\
 &= 16 \frac{1}{2} B\left(\frac{1}{2}, \frac{5}{2}\right) \\
 &= 8 \left[\frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right) + \left(\frac{5}{2}\right)} \right] \\
 &= 8 \frac{3\pi}{\Gamma 3} \\
 &= \frac{24\pi}{8} = 3\pi
 \end{aligned}$$

Problem 1.9.3. S.T. $\Gamma(n) = \int_0^1 (\log 1/x)^{n-1} dx, n > 0$

Solution: We have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

Putting $x = \log \frac{1}{y} = -\log y$

(or) $y = e^{-x}$ so that $dy = -e^{-x} dx$

$$dx = \frac{-1}{y} dy$$

Equation (1) becomes

$$\begin{aligned}
 \Gamma(n) &= - \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} \cdot y \cdot \frac{1}{y} dy \\
 &= \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy \\
 \therefore \Gamma(n) &= \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx
 \end{aligned}$$

Problem 1.9.4. Prove that

$$B(m, n) = B(m+1, n) + B(m, n+1)$$

Solution:

$$\begin{aligned}
 \text{R.H.S.} &= B(m+1, n) + B(m, n+1) \\
 &= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} \\
 &= \frac{m\Gamma(m) \cdot \Gamma(n) + \Gamma(m)n\Gamma(n)}{\Gamma(m+n+1)} \\
 &= \frac{\Gamma(m)\Gamma(n)(m+n)}{(m+n)\Gamma(m+n)} \\
 &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \\
 &= B(m, n) \\
 &= \text{L.H.S.}
 \end{aligned}$$

Problem 1.9.5. Prove that $\Gamma(n)\Gamma(n+1/2) = 2^{1-2n}\sqrt{\pi}\Gamma(2n)$.

Solution:

By the definition of beta function, we have

$$\begin{aligned}
 B(n, n) &= 2 \int_0^{\pi/2} (\sin \theta)^{2n-1} (\cos \theta)^{2n-1} d\theta \\
 &= 2 \int_0^{\pi/2} (\sin \theta \cos \theta)^{2n-1} d\theta \\
 &= 2 \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2} \right)^{2n-1} d\theta \\
 &= \frac{2}{2^{2n-1}} \int_0^{\pi/2} (\sin 2\theta)^{2n-1} d\theta \\
 &\quad \text{put } 2\theta = \phi \\
 &= 2^{1-2n} \int_0^{\pi} (\sin \phi)^{2n-1} d\phi \\
 B(n, n) &= 2^{1-2n} \cdot 2 \int_0^{\pi/2} (\sin \phi)^{2n-1} d\phi \\
 &= 2^{1-2n} \cdot B(n, 1/2)
 \end{aligned}$$

Converting both sides to Gamma function

$$\begin{aligned}
 \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} &= 2^{1-2n} \cdot B(n, 1/2) \\
 &= 2^{1-2n} \frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n + 1/2)}
 \end{aligned}$$

rearranging the terms

$$\Gamma(n)\Gamma(n + 1/2) = 2^{1-2n} \sqrt{\pi} \Gamma(2n)$$

Problem 1.9.6. Express the following in terms of gamma functions $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$

Solution:

$$\begin{aligned}
\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta &= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \\
&= \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \\
\therefore \int_0^{\pi/2} \sqrt{\tan \theta} d\theta &= \int_0^{\pi/2} \sin^{1/2} \theta \cos^{-1/2} \theta d\theta \\
&= \frac{\Gamma\left(\frac{1/2+1}{2}\right) \Gamma\left(\frac{-1/2+1}{2}\right)}{2\Gamma\left(\frac{1/2-1/2+2}{2}\right)} \\
&= \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} \\
&= \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) \\
&= \frac{1}{2} \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} \\
&\quad \left(\text{using } \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}, \quad n = \frac{1}{4}\right) \\
&= \frac{1}{2} \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} \\
&= \frac{\pi}{\sqrt{2}}
\end{aligned}$$

Problem 1.9.7. Prove that $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

Solution:

$$\begin{aligned}
I_1 &= \int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \\
&= \int_0^{\pi/2} \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{2\sqrt{(\sin \theta)}} d\theta
\end{aligned}$$

$$\left(\text{Putting } x^2 = \sin \theta, dx = \frac{\cos \theta d\theta}{2\sqrt{(\sin \theta)}} \right)$$

$$\begin{aligned} \therefore I_1 &= \frac{1}{2} \int_0^{\pi/2} \sqrt{(\sin \theta)} d\theta \\ &= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right) \\ &= \frac{1}{4} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)} \\ &= \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(1/4)} \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} \\ &= \int_0^{\pi/4} \frac{\sec^2 \theta d\theta}{2\sqrt{(\tan \theta)} \sec \theta} \quad \left(\text{Putting } x^2 = \tan \theta, dx = \frac{\sec^2 \theta d\theta}{2\sqrt{(\tan \theta)}} \right) \\ &= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin 2\theta)}} \\ &= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} \phi d\phi \quad \left(\text{Putting } 2\theta = \phi, d\theta = \frac{1}{2} d\phi \right) \\ &= \frac{1}{4\sqrt{2}} B\left(\frac{1}{4}, \frac{1}{2}\right) \\ &= \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} \\ \therefore \int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} &= \frac{1}{4\sqrt{2}} \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \frac{\pi}{4\sqrt{2}} \end{aligned}$$

1.10 Question Bank : Module 1

Evaluation of double integrals

- 1) $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y dx dy$ (VTU Jan 2017, Dec 2011) Ans: $\frac{1}{24}$
- 2) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dx dy$ (VTU July 2011) Ans: $\frac{3}{35}$

Triple Integrals

- 1) $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z) dy dx dz$ (VTU Jan 2019, Jan 2017, Jan 2016, Jan 2015, July 2011, July 2009) **Ans: 0**
- 2) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$ (VTU Model 2023, July 2016, Dec 2010, Jan 2010, July 2007) **Ans: $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$**
- 3) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz dz dy dx$ (VTU July 2016) **Ans:**
- 4) $\int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dx dy dz$ (VTU Nov 2023, Model 2023, June 2019, Model 2018, July 2017, Jan 2016, July 2014, July 2013) **Ans: $\frac{8abc}{3}(a^2 + b^2 + c^2)$**
- 5) $\int_0^4 \int_0^{\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy dx dz$ (VTU Jan 2017) **Ans: 8π**
- 6) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz dy dx$ (VTU July 2016, July 2008) **Ans:**
- 7) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dz dy dx$ (VTU June 2023, Jan 2020, June 2018, Jan 2013, Dec 2010, Jan 2008) **Ans: $\frac{1}{48}$**
- 8) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz dz dy dx$ (VTU Jan 2018) **Ans:**
- 9) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{a^2-x^2-y^2-z^2} dz dy dx$ (VTU July 2017) **Ans:**
- 10) Evaluate $\int_1^3 \int_{y=\frac{1}{x}}^1 \int_{z=0}^{\sqrt{x}} xyz dz dy dx$ (VTU July 2017) **Ans:**
- 11) $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z dz dx dy$ (VTU July 2017) **Ans: $\frac{1}{4}(e^2 - 8e + 13)$**
- 12) $\int_0^{\frac{\pi}{2}} \int_0^{a \sin \theta} \int_0^{\frac{a^2-r^2}{a}} r dr d\theta dz$ (VTU July 2009)

Evaluation of Double integral when Region is Given

- 1) Evaluate $\int \int_R xy dx dy$, where R is the region bounded by x - axis, the ordinate $x = 2a$ and $x^2 = 4ay$ (VTU Jan 2016) **Ans: $\frac{a^4}{3}$**

- 2) $\int \int xy(x+y)dx dy$ over the region between $y = x$ and $y = x^2$ (VTU June 2018, Jan 2015, Dec 2010, Jan 2010) **Ans:** $\frac{3}{56}$
- 3) Evaluate $\int \int y dx dy$ over the region bounded by the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (VTU Jan 2020, Jan 2013)
- 4) $\int \int_R xy dx dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. **Ans:** $\frac{a^4}{8}$
- 5) Evaluate $\int \int_R x^2 y dx dy$ where R is the region bounded by the lines $y = x, y + x = 2$ & $y = 0$. **Ans:** $\frac{11}{30}$

Changing the order of integration

Change the order of integration of the following Integrals & hence evaluate them :

- $\int_0^1 \int_x^{\sqrt{x}} xy dy dx$ (VTU Nov 2023, Model 2023, Jan 2019, July 2016, June 2010, Jan 2008)
- $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dx dy$ (VTU July 2011) **Ans:** $\frac{\pi}{16}$
- $\int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy$ (VTU July 2016) **Ans:** $\frac{241}{60}$
- $\int_0^1 \int_{x^2}^{2-x} xy dy dx$ (VTU July 2017, Jan 2016, Dec 2011) **Ans:** $\frac{3}{8}$
- $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy dy dx$ (VTU Jan 2017, Model 2015) **Ans:** $\frac{64a^4}{3}$
- $\int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{ax}} dy dx$ (VTU July 2014)
- $\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}} dx dy$ (VTU July 2015) **Ans:** $\frac{1}{2}(\sqrt{2} - 1)$
- $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} dy dx$ (VTU July 2016, July 2009, Jan 2009, July 2008, July 2007) **Ans:** $\frac{1}{2}$
- $\int_0^1 \int_{x^2}^{2-x} xy dx dy$ (VTU Jan 2012, July 2008)
- $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$ (VTU June 2018, Dec 2010) **Ans:** 1

11. $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy$ (VTU June 2023, Model 2023, Model 2018, Jan 2018, July 2017, Jan 2008) **Ans:** $\frac{\pi a}{4}$
12. $\int_0^a \int_0^{2\sqrt{ax}} x^2 dy dx$ (VTU June 2019, Jan 2013)
13. $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2) dy dx$ (VTU Jan 2014)
14. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dx dy$ (VTU July 2012) **Ans:** $\frac{\pi}{16}$
15. $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} (2-x) dy dx$ (VTU Jan 2020)
16. Change the order of integration in $\int_0^1 \int_x^{2-x} xy dx dy$ and hence evaluate the same.
Ans: $\frac{3}{8}$
17. $\int_0^1 \int_{\sqrt{y}}^{2-y} xy dx dy$ **Ans:** $\frac{7}{24}$

Change of Variables

Evaluate by changing to polar coordinates.

1. $\int_0^a \int_0^{\sqrt{a^2-y^2}} y \sqrt{x^2 + y^2} dx dy$ (VTU Jan 2018) **Ans:** $a^4/4$
2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ (VTU Jan 2019, Model 2018, July 2017, Jan 2017, July 2016, Jan 2016, July 2013, Dec 2011)
3. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2 + y^2} dx dy$ (VTU July 2015) **Ans:** $\frac{a^5 \pi}{20}$
4. $\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{x^2 + y^2} dx dy$ (VTU Model 2023)
5. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ using polar coordinates (VTU Nov 2023, Model 2023)
6. Evaluate $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} (2-x) dx dy$ by changing in to polar form. (VTU June 2023) **Ans:** 2π
7. Evaluate $\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2 + y^2} dx dy$ by changing in to polar form. **Ans:** $\frac{\pi a^3}{3}$

Area and Volume

- By using double integral, find the area bounded by the coordinate axes and the line $x + y = 1$
(VTU July 2017)
- Show that area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$
(VTU Model 2023, June 2018, Jan 2016)
- Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant (VTU Model 2023, July 2016) **Ans: $\frac{\pi ab}{4}$**
- Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration. (VTU June 2019, Jan 2018, July 2016) **Ans: πab**
- Find the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$
(VTU Jan 2017)
- Find the area enclosed by the curve $r = a(1 + \cos \theta)$ above the initial line (or between $\theta = 0$ and $\theta = \pi$) (VTU Jan 2020, July 2016) **Ans: $\frac{3}{4}a^2\pi$**
- Using Double integration, find the area of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$ (VTU Model 2018) **Ans: a^2**
- Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
(VTU July 2016) **Ans : $\frac{abc}{6}$**
- Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, x + y + z = 1$ and $z = 0$
(VTU July 2015)
- A pyramid is bounded by three coordinate planes and the plane $x + 2y + 3z = 6$. Compute the volume by double integration.
(VTU July 2015)

11. Find the area bounded by the circle $x^2 + y^2$ and the line $x + y = a$
(VTU June 2019)

Beta and Gamma functions

- 1) Prove that $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
(VTU Nov 2023, June 2023, Model 2023, Jan 2020, June 2019, Jan 2019, June 2018, Model 2018, Jan 2018, July 2017, Jan 2017, July 2016, July 2013, July 2011, July 2008)
- 2) Prove that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} d\theta = \pi$
(VTU Jan 2020, June 2019, Jan 2019, Model 2018, Jan 2018, July 2017, July 2016, Jan 2015, Dec 2010, Jan 2008)
- 3) Evaluate $\int_0^a x^4 \sqrt{a^2 - x^2} dx$ using Beta and Gamma function. (VTU July 2017, July 2016, July 2011)
- 4) Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^4}} dx$ using beta and Gamma functions. (VTU July 2015)
- 5) Define Beta and Gamma function. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (VTU Model 2023, June 2018, June 2017, Jan 2016)
- 6) For m and n positive, prove that $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$ (VTU Jan 2017, Dec 2011, Jan 2008)
- 7) Evaluate $\int_0^a y^4 \sqrt{a^2 - y^2} dy$ using Beta and Gamma function. (VTU Jan 2017)
- 8) Prove that $B(m, \frac{1}{2}) = 2^{2m-1} B(m, m)$ (VTU Jan 2016)
- 9) Prove that $\int_0^\infty x e^{-x^8} dx \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$ (VTU Jan 2013)
- 10) $\int_{-1}^1 (1+x)^{m-1} (1-x)^{n-1} dx = 2^{m+n-1} B(m, n)$ (VTU July 2011)
- 11) Prove that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$ (VTU Dec 2011)
- 12) Prove that $\sqrt{\pi} \Gamma(2m) = 2^{2m-1} \Gamma(m) \Gamma\left(m + \frac{1}{2}\right)$ (VTU June 2010)

- 13) Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cot \theta} d\theta$ expressing in terms of Gamma Function. (VTU Jan 2010, July 2009)
- 14) Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \cdot \int_0^1 \frac{1}{\sqrt{1-x^4}} dx = \frac{\pi}{4}$ (VTU July 2016)
- 15) Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \cdot \int_0^1 \frac{1}{\sqrt{1+x^4}} dx = \frac{\pi}{4\sqrt{2}}$ (VTU July 2015, July 2014)
- 16) Show that $\int_0^{\frac{\pi}{2}} \sqrt{\cot \theta} d\theta = \frac{\pi}{\sqrt{2}}$ (VTU Nov 2023)



Lecture Notes

BMATS201

Mathematics-II for CSE Stream

Module 2: Vector Calculus

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Module 2

Vector Calculus

Syllabus :

Introduction to Vector Calculus in Computer Science & Engineering: Scalar and vector fields. Gradient, directional derivative, curl and divergence - physical interpretation, solenoidal and irrotational vector fields. Problems.

Curvilinear coordinates: Scale factors, base vectors, Cylindrical polar coordinates, Spherical polar coordinates, transformation between cartesian and curvilinear systems, orthogonality. Problems.

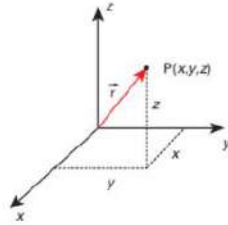
Self-Study: Vector integration and Vector line integral.

Applications: Conservation of laws, Electrostatics, Analysis of streamlines.

2.1 Recall of basic Concepts

- **Vector :** A vector is a physical quantity having both magnitude and direction. Example : Force, Acceleration, Velocity etc.
- A vector is represented by a directed line segment with an arrow over it. i.e. \vec{PQ} is a vector with initial point P and terminal point Q whose direction is from P to Q and magnitude is the length PQ.

- **Position vector of a point P** :Position vector is a vector which represents the position of a point in a space with respect to the origin. The position vector of a point P(x,y,z) in space is given by $\vec{r} = xi + yj + zk$



- **Note** : If x, y, z are functions of a single variable 't' then the vector \vec{r} is said to be vector point function of 't' (i.e. vector equation of a curve at any point 't'). i.e. $\vec{r} = r(\vec{t}) = x(t)i + y(t)j + z(t)k$.
- **Magnitude** of a vector \vec{r} denoted by $|\vec{r}|$ or r is defined as $r = \sqrt{x^2 + y^2 + z^2}$

- **Dot product** : If $\vec{A} = a_1i + a_2j + a_3k$ and $\vec{B} = b_1i + b_2j + b_3k$ are two vectors, then the dot product is given by the formula,

$$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$$

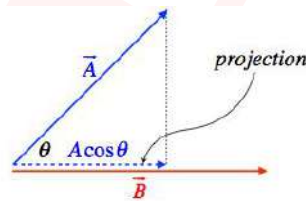
Sometimes the dot product is called the scalar product.

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- $i \cdot i = j \cdot j = k \cdot k = 1$
- $i \cdot j = j \cdot k = k \cdot i = 0$
- If $\vec{A} = a_1i + a_2j + a_3k$ and $\vec{B} = b_1i + b_2j + b_3k$ then $\vec{A} \times \vec{B} =$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
- $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$
- $i \times i = j \times j = k \times k = 0$
- $i \times j = k, j \times k = i, k \times i = j$

- $j \times i = -k, k \times j = -i, i \times k = -j$
- If θ is the angle between two vectors \vec{A} and \vec{B} , then $\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$
- If $\vec{A} \cdot \vec{B} = 0$, then \vec{A} and \vec{B} are perpendicular.
- If $\vec{A} \times \vec{B} = 0$, then \vec{A} and \vec{B} are parallel.
- **Component of a vector along a given direction:** A component of a vector \vec{A} along a given direction \vec{B} is the resolved part of \vec{A} and is given by

$$\vec{A} \cdot \hat{n} \quad \text{where} \quad \hat{n} = \frac{\vec{B}}{|\vec{B}|}$$



2.2 Vector differential operator

The vector differential operator is denoted by ∇ (Read it as **del** or **nabla**) and is defined as

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

2.3 The gradient of the scalar point function:

The gradient of the scalar point function Φ denoted by $grad\Phi$ or $\nabla\Phi$ is a vector function defined by

$$\begin{aligned} grad\Phi &= \nabla\Phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \Phi \\ &= \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k \\ &= \sum \frac{\partial\Phi}{\partial x}i \end{aligned}$$

Note : Gradient of a scalar function is a vector quantity

2.4 Divergence of a vector point function :

The divergence of a vector point function $\vec{F} = f_1i + f_2j + f_3k$ denoted by $divF$ is defined by,

$$\begin{aligned} div\vec{F} &= \nabla \cdot \vec{F} \\ &= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \right) \cdot (f_1i + f_2j + f_3k) \\ &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ &= \sum \frac{\partial f_1}{\partial x} \end{aligned}$$

Note :Divergence of a vector point function is a scalar quantity

2.5 Curl of a vector point function

The curl of a continuously differentiable vector point function $\vec{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ denoted by $\text{curl}\vec{F}$ is defined by

$$\begin{aligned}\text{curl}\vec{F} &= \nabla \times \vec{F} \\ &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times (f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \\ &= \sum \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) i\end{aligned}$$

Note : Curl of a vector point function is a vector quantity.

Problem 2.5.1. Find grad Φ when $\Phi = 3x^2y - y^3z^2$ at the point $(1, -2, -1)$

Solution:

$$\begin{aligned}\nabla\phi &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) (3x^2y - y^3z^2) \\ &= \frac{\partial}{\partial x} (3x^2y - y^3z^2) \mathbf{i} + \frac{\partial}{\partial y} (3x^2y - y^3z^2) \mathbf{j} \\ &\quad + \frac{\partial}{\partial z} (3x^2y - y^3z^2) \mathbf{k} \\ &= (6xy)\mathbf{i} + (3x^2 - 3y^2z^2) \mathbf{j} + (-2y^3z) \mathbf{k} \quad \dots (1)\end{aligned}$$

Thus $\nabla\phi$ at the point $(1, -2, -1)$ is obtained by putting these values of x, y, z in

R.H.S. of (1) Thus

$$\begin{aligned}\nabla\phi |_{(1,-2,-1)} &= 6(1)(-2)\mathbf{i} + [3 - 3(-2)^2(-1)^2]\mathbf{j} + [-2(-2)^3(-1)]\mathbf{k} \\ &= -12\mathbf{i} + \mathbf{j}(3 - 12) + \mathbf{k}(-16) \\ &= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}\end{aligned}$$

Problem 2.5.2. If $\Phi = \log(x^2 + y^2 + z^2)$, then find $\nabla\Phi$ & $|\nabla\Phi|$ at $(1, 2, 1)$

Solution: Given

$$\phi(x, y, z) = \log(x^2 + y^2 + z^2)$$

$$\begin{aligned} \text{grad } \phi &= \nabla \phi \\ &= \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k \end{aligned}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get,

$$\frac{\partial \phi}{\partial x} = \frac{1}{x^2 + y^2 + z^2} \cdot 2x$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2 + y^2 + z^2} \cdot 2y$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{x^2 + y^2 + z^2} \cdot 2z$$

At the point (1, 2, 1)

$$\frac{\partial \phi}{\partial x} = \frac{2 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{\partial \phi}{\partial y} = \frac{2 \cdot 2}{1^2 + 2^2 + 1^2} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{\partial \phi}{\partial z} = \frac{2 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\begin{aligned} \therefore \text{grad } \phi |_{(1,2,1)} &= \frac{1}{3}i + \frac{2}{3}j + \frac{1}{3}k \\ &= \frac{1}{3}[i + 2j + k] \end{aligned}$$

$$\begin{aligned} | \text{grad } \phi | &= \sqrt{\frac{1^2 + 2^2 + 1^2}{9}} \\ &= \frac{\sqrt{6}}{3} \end{aligned}$$

Problem 2.5.3. Find $\text{div}F$ and $\text{curl}F$ where $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

Solution: Let $\phi = x^3 + y^3 + z^3 - 3xyz$

The given expression can be written as

$$\begin{aligned}\vec{F} &= \text{grad}\phi = \nabla\phi \\ &= \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k \\ &= \frac{\partial(x^3 + y^3 + z^3 - 3xyz)}{\partial x}i + \frac{\partial(x^3 + y^3 + z^3 - 3xyz)}{\partial y}j \\ &\quad + \frac{\partial(x^3 + y^3 + z^3 - 3xyz)}{\partial z}k\end{aligned}$$

$$\vec{F} = (3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k$$

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \right) \cdot ((3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k)$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z)$$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz) \right]$$

$$- j \left[\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz) \right]$$

$$+ k \left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz) \right]$$

$$= i(-3x + 3x) - j(-3y + 3y) + k(-3z + 3z)$$

$$= 0i - 0j + 0k = \vec{0}$$

Problem 2.5.4. If $\Phi = 2x^3y^2z^4$, find $\text{div}(\text{grad}\Phi)$ at $(1,1,1)$

Solution:

$$\begin{aligned}\text{grad } \phi &= \nabla \phi \\ &= i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \\ &= i \frac{\partial}{\partial x} (2x^3y^2z^4) + j \frac{\partial}{\partial y} (2x^3y^2z^4) + k \frac{\partial}{\partial z} (2x^3y^2z^4)\end{aligned}$$

and

$$\text{grad } \phi = i (6x^2y^2z^4) + j (4x^3yz^4) + k (8x^3y^2z^3)$$

$$\text{div}(\text{grad } \phi) = \nabla \cdot \nabla \phi$$

$$= \frac{\partial}{\partial x} (6x^2y^2z^4)$$

$$+ \frac{\partial}{\partial y} (4x^3yz^4)$$

$$+ \frac{\partial}{\partial z} (8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\text{div}(\text{grad } \phi) |_{(1,1,1)} = 12 + 4 + 24 = 40$$

Problem 2.5.5. Find $\text{div } \vec{F}$ and $\text{curl } F$ at the point $(1,2,3)$ if $\vec{F} = \text{grad}(x^3y + y^3z + z^3x - x^2y^2z^2)$

Solution: Let $\Phi = x^3y + y^3z + z^3x - x^2y^2z^2$

$$\text{Given } \vec{F} = \text{grad} (x^3y + y^3z + z^3x - x^2y^2z^2)$$

$$= \nabla(\Phi)$$

$$= \frac{\partial}{\partial x} (x^3y + y^3z + z^3x - x^2y^2z^2) i$$

$$+ \frac{\partial}{\partial y} (x^3y + y^3z + z^3x - x^2y^2z^2) j$$

$$+ \frac{\partial}{\partial z} (x^3y + y^3z + z^3x - x^2y^2z^2) k$$

$$\vec{F} = (3x^2y + z^3 - 2xy^2z^2) i + (x^3 + 3y^2z - 2x^2yz^2) j + (y^3 + 3z^2x - 2x^2y^2z) k$$

$$\therefore \operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$\begin{aligned} &= \frac{\partial}{\partial x} (3x^2y + z^3 - 2xy^2z^2) \\ &\quad + \frac{\partial}{\partial y} (x^3 + 3y^2z - 2x^2yz^2) \\ &\quad + \frac{\partial}{\partial z} (y^3 + 3z^2x - 2x^2y^2z) \end{aligned}$$

$$= 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6zx - 2x^2y^2$$

$$\operatorname{div} \vec{F} |_{(1,2,3)} = 12 - 36 + 54 - 72 + 18 - 8 = -32$$

$$\begin{aligned} \operatorname{Curl}(\vec{F}) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2y + z^3 - 2xy^2z^2) & (x^3 + 3y^2z - 2x^2yz^2) & (y^3 + 3z^2x - 2x^2y^2z) \end{vmatrix} \\ &= i [3y^2 - 4x^2yz - 3y^2 + 4x^2yz] \\ &\quad - j [3z^2 - 4xy^2z - 3z^2 + 4xy^2z] \\ &\quad + k [3x^2 - 4xyz^2 - 3x^2 + 4xyz^2] \\ &= 0i + 0j + 0k \\ &= \vec{0} \end{aligned}$$

Problem 2.5.6. Prove that $\nabla \cdot \vec{r} = 3$ and $\nabla \times \vec{r} = \vec{0}$

Solution: (i)

$$\begin{aligned} \nabla \cdot \vec{r} &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (xi + yj + zk) \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\ &= 1 + 1 + 1 = 3 \end{aligned}$$

$$\begin{aligned}
\nabla \times \vec{r} &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times (xi + yj + zk) \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= i \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y) \right] - j \left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x) \right] \\
&\quad + k \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right] \\
&= 0i + 0j + 0k = \vec{0}
\end{aligned}$$

Problem 2.5.7. If $\vec{V} = \vec{w} \times \vec{r}$, P.T. $\text{curl } \vec{V} = \frac{1}{2}(\nabla \times \vec{V})$ or $\text{curl } \vec{V} = 2\vec{w}$ where \vec{w} is a constant vector.

Solution: Given $\vec{v} = \vec{w} \times \vec{r}$

$\vec{r} = xi + yj + zk$ [$\because \vec{r}$ is the position vector of (x, y, z)]

and let $\vec{w} = w_1i + w_2j + w_3k$, w_1, w_2, w_3 are constants.

$$\begin{aligned}
\vec{w} \times \vec{r} &= \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} \\
&= i [w_2z - w_3y] - j [w_1z - w_3x] + k [w_1y - w_2x]
\end{aligned}$$

$$\begin{aligned}
\text{curl } \vec{v} &= \nabla \times (\vec{w} \times \vec{r}) \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2z - w_3y & w_3x - w_1z & w_1y - w_2x \end{vmatrix} \\
&= i \left[\frac{\partial}{\partial y} (w_1y - w_2x) - \frac{\partial}{\partial z} (w_3x - w_1z) \right] \\
&\quad - j \left[\frac{\partial}{\partial x} (w_1y - w_2x) - \frac{\partial}{\partial z} (w_2z - w_3y) \right] \\
&\quad + k \left[\frac{\partial}{\partial x} (w_3x - w_1z) - \frac{\partial}{\partial y} (w_2z - w_3y) \right] \\
&= i [w_1 + w_1] - j [-w_2 - w_2] + k [w_3 + w_3] \\
&= 2 [w_1i + w_2j + w_3k] \\
&= 2\vec{w} \Rightarrow \vec{w} = \frac{1}{2} \text{curl } \vec{v}
\end{aligned}$$

Problem 2.5.8. Find divergence and curl of the vector $\vec{V} = xyz\mathbf{i} + 3x^2y\mathbf{j} + (xz^2 - y^2z)\mathbf{k}$ at $(2, -1, 1)$.

Solution:

$$\vec{v} = (xyz)\mathbf{i} + (3x^2y)\mathbf{j} + (xz^2 - y^2z)\mathbf{k}$$

$$\text{Div. } \vec{v} = \nabla \cdot \vec{v}$$

$$= \left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right] \cdot \vec{v}$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

$$\text{Div. } \vec{v} |_{(2,-1,1)} = -1 + 12 + 4 - 1 = 14$$

$$\begin{aligned}
\text{Curl } \vec{v} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} \\
&= -2yzi - (z^2 - xy)j + (6xy - xz)k \\
&= -2yzi + (xy - z^2)j + (6xy - xz)k \\
\text{Curl at } (2, -1, 1) &= -2(-1)(1)i + \{(2)(-1) - 1\}j \\
&\quad + 6(2)(-1) - 2(1)k \\
&= 2i - 3j - 14k
\end{aligned}$$

Problem 2.5.9. Find curl (curl \vec{A}) where $\vec{A} = xyi + y^2zj + z^2yk$

$$\begin{aligned}
\vec{A} &= xyi + y^2zj + z^2yk \\
\text{curl } \vec{A} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2y \end{vmatrix} \\
&= i(z^2 - y^2) - j(0 - 0) + k(0 - x) \\
\text{curl } \vec{A} &= (z^2 - y^2)i - 0j - xk \\
\therefore \text{curl}(\text{curl } \vec{A}) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & 0 & -x \end{vmatrix} \\
&= i(0 - 0) - j(-1 - 2z) + k(0 + 2y) \\
\text{curl}(\text{curl } \vec{A}) &= 0i + (2z + 1)j + 2yk
\end{aligned}$$

Problem 2.5.10. If $F = (x + y + 1)i + j - (x + y)k$, then show that $\vec{F} \cdot \text{curl } \vec{F} = 0$

Solution:

$$\vec{F} = (x + y + 1)\mathbf{i} + j - (x + y)\mathbf{k} \text{ and}$$

$$\begin{aligned} \text{curl } \vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + 1 & 1 & -(x + y) \end{vmatrix} \\ &= \mathbf{i}[-1 - 0] - \mathbf{j}[-1 - 0] + \mathbf{k}[0 - 1] \end{aligned}$$

$$\text{curl } \vec{F} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\begin{aligned} \therefore \vec{F} \cdot \text{curl } \vec{F} &= ((x + y + 1)\mathbf{i} + \mathbf{j} - (x + y)\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &= -(x + y + 1) + 1 + (x + y) \\ &= 0 \end{aligned}$$

Problem 2.5.11. If $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\vec{r}|$ then prove that $\nabla r^n = nr^{n-2}\vec{r}$

Solution: We have $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned} \nabla r^n &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) (r^n) \\ &= \mathbf{i} \frac{\partial}{\partial x} (r^n) + \mathbf{j} \frac{\partial}{\partial y} (r^n) + \mathbf{k} \frac{\partial}{\partial z} (r^n) \\ &= \mathbf{i} \left(nr^{n-1} \frac{\partial r}{\partial x} \right) + \mathbf{j} \left(nr^{n-1} \frac{\partial r}{\partial y} \right) + \mathbf{k} \left(nr^{n-1} \frac{\partial r}{\partial z} \right) \end{aligned}$$

$$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{r}$$

$$\text{Similarly } \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned} \therefore \nabla r^n &= nr^{n-1} \left[\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k} \right] \\ &= \frac{nr^{n-1}}{r} [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] \\ &= nr^{n-2}\vec{r} \end{aligned}$$

Problem 2.5.12. Prove that $\nabla \left(\frac{1}{r} \right) = \frac{-1}{r^3}\vec{r} = \frac{-1}{r^2}\hat{r}$

Solution: We have $\vec{r} = xi + yj + zk$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned}\nabla \left(\frac{1}{r} \right) &= i \frac{\partial}{\partial x} \left(\frac{1}{r} \right) + j \frac{\partial}{\partial y} \left(\frac{1}{r} \right) + k \frac{\partial}{\partial z} \left(\frac{1}{r} \right) \\ &= i \left(-\frac{1}{r^2} \frac{\partial r}{\partial x} \right) + j \left(-\frac{1}{r^2} \frac{\partial r}{\partial y} \right) + k \left(-\frac{1}{r^2} \frac{\partial r}{\partial z} \right)\end{aligned}$$

$$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{r}$$

$$\text{Similarly } \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned}\therefore \nabla \left(\frac{1}{r} \right) &= -\frac{1}{r^2} \left[\frac{x}{r} i + \frac{y}{r} j + \frac{z}{r} k \right] \\ &= -\frac{1}{r^3} (xi + yj + zk) \\ &= -\frac{\vec{r}}{r^3} = -\frac{1}{r^2} \frac{\vec{r}}{r} = \frac{-1}{r^2} \hat{r}\end{aligned}$$

Problem 2.5.13. Compute curl of $V(x, y, z) = xyz^2 i + xy^2z j + x^2yz k$

$$\begin{aligned}\text{Curl } V &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & xy^2z & x^2yz \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y} (x^2yz) - \frac{\partial}{\partial z} (xy^2z) \right) i + \left(\frac{\partial}{\partial z} (xyz^2) - \frac{\partial}{\partial x} (x^2yz) \right) j \\ &\quad + \left(\frac{\partial}{\partial x} (xy^2z) - \frac{\partial}{\partial y} (xyz^2) \right) k \\ &= (x^2z - xy^2) i + (2xyz - 2xyz) j + (y^2z - xz^2) k \\ &= (x^2z - xy^2) i + (y^2z - xz^2) k\end{aligned}$$

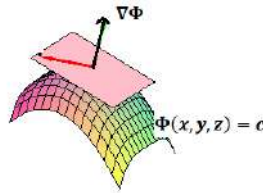
Normal vector :

If $\Phi(x, y, z)$ represents equation of a surface, then $\nabla \Phi$ is a vector normal to the surface.

Thus

$$\boxed{\text{normal vector} = \nabla \Phi}$$

$$\text{unit normal vector, } \hat{n} = \frac{\nabla\Phi}{|\nabla\Phi|}$$



Problem 2.5.14. Find a unit normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.

Solution: Given surface is

$$xy^3z^2 = 4$$

$$\Rightarrow \phi(x, y, z) = xy^3z^2$$

Hence the normal vector is given by

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = y^3z^2,$$

At the point $(-1, -1, 2)$,

$$\frac{\partial\phi}{\partial x} = (-1)^3 \cdot 2^2 = -4$$

$$\frac{\partial\phi}{\partial y} = 3(-1)(-1)^2 \cdot 2^2 = -12$$

$$\frac{\partial\phi}{\partial z} = 2(-1)(-1)^3 \cdot 2 = 4$$

\therefore at the point $(-1, -1, 2)$,

$$\nabla\phi = -4i - 12j + 4k = -4(i + 3j - k)$$

\therefore unit normal to the given surface at the point $(-1, -1, 2)$ is

$$\begin{aligned} \hat{n} &= \frac{\nabla\phi}{|\nabla\phi|} \\ &= \frac{-4(i + 3j - k)}{4\sqrt{1 + 9 + 1}} = -\frac{(i + 3j - k)}{\sqrt{11}} \end{aligned}$$

2.6 Angle between two surfaces :

If $\Phi_1(x, y, z) = c_1$ and $\Phi_2(x, y, z) = c_2$ are two surfaces, then the angle between these two surfaces is equal to the angle between their normals $\nabla\Phi_1$ and $\nabla\Phi_2$.

Hence the angle is given by

$$\cos\theta = \frac{\nabla\Phi_1 \cdot \nabla\Phi_2}{|\nabla\Phi_1| |\nabla\Phi_2|}$$

Problem 2.6.1. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2, -1, 2)$.

Solution: The given surfaces are

$$x^2 + y^2 + z^2 = 9 \dots (1) \quad \text{and} \quad x^2 + y^2 - z = 3 \dots (2)$$

Let

$$\Phi_1 = x^2 + y^2 + z^2 \quad \text{and} \quad \Phi_2 = x^2 + y^2 - z$$

Now, vector normal to the first surface is given by

$$\begin{aligned} \nabla\Phi_1 &= i \frac{\partial\Phi_1}{\partial x} + j \frac{\partial\Phi_1}{\partial y} + k \frac{\partial\Phi_1}{\partial z} \\ &= 2xi + 2yj + 2zk \end{aligned}$$

$$\nabla\Phi_1 |_{(2,-1,2)} = 4i - 2j + 4k$$

Now, vector normal to the second surface is given by

$$\begin{aligned} \nabla\Phi_2 &= i \frac{\partial\Phi_2}{\partial x} + j \frac{\partial\Phi_2}{\partial y} + k \frac{\partial\Phi_2}{\partial z} \\ &= 2xi + 2yj - k \end{aligned}$$

$$\nabla\Phi_2 |_{(2,-1,2)} = 4i - 2j - k$$

If θ is the angle between the surfaces (1) and (2) at $(2, -1, 2)$, then

$$\begin{aligned}\cos \theta &= \frac{\nabla \Phi_1 \cdot \nabla \Phi_2}{|\nabla \Phi_1| |\nabla \Phi_2|} \\ &= \frac{(4i - 2j + 4k) \cdot (4i - 2j - k)}{\sqrt{16 + 4 + 16} \sqrt{16 + 4 + 1}} \\ &= \frac{16 + 4 - 4}{\sqrt{36} \sqrt{21}} \\ &= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}} \\ \Rightarrow \theta &= \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right)\end{aligned}$$

Problem 2.6.2. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1, 1, 1)$

Recall that angle between the surfaces at a point of their intersection is defined to be the angle between the normals to the surfaces at the point of intersection. Thus we need to find the normals to both the surfaces at $(1, 1, 1)$.

Let $\phi_1(x, y, z) = x \log z - y^2 + 1$ and $\phi_2(x, y, z) = x^2 y - 2 + z$

Normal to the first surface is given by

$$\begin{aligned}\text{grad} \phi_1 &= \nabla \phi_1 \\ &= \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \\ &= \frac{\partial}{\partial x} (x \log z - y^2 + 1) i \\ &\quad + \frac{\partial}{\partial y} (x \log z - y^2 + 1) j \\ &\quad + \frac{\partial}{\partial z} (x \log z - y^2 + 1) k \\ &= (\log z) i + (-2y) j + \left(\frac{x}{z} \right) k \\ \therefore \text{grad} \phi_1 |_{(1,1,1)} &= -2j + k\end{aligned}$$

Normal to the second surface is given by

$$\begin{aligned}
 \text{grad}\phi_2 &= \nabla\phi_2 \\
 &= \frac{\partial\phi_2}{\partial x}i + \frac{\partial\phi_2}{\partial y}j + \frac{\partial\phi_2}{\partial z}k \\
 &= \frac{\partial}{\partial x}(x^2y - 2 + z)i + \frac{\partial}{\partial y}(x^2y - 2 + z)j \\
 &\quad + \frac{\partial}{\partial z}(x^2y - 2 + z)k \\
 &= 2xyi + (x^2)j + (1)k
 \end{aligned}$$

$$\nabla\phi_2|_{(1,1,1)} = 2i + j + k$$

Angle between the surfaces is given by

$$\begin{aligned}
 \cos\theta &= \frac{\nabla\Phi_1 \cdot \nabla\Phi_2}{|\nabla\Phi_1||\nabla\Phi_2|} \\
 &= \frac{[0i - 2j + k] \cdot [2i + j + k]}{\sqrt{0^2 + (-2)^2 + 1^2} \sqrt{2^2 + 1^2 + 1^2}} \\
 &= \frac{0 - 2 + 1}{\sqrt{5} \sqrt{6}} = -\frac{1}{\sqrt{30}} \\
 \Rightarrow \theta &= \cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)
 \end{aligned}$$

Problem 2.6.3. Find the angle between the normal to the surface $xy = z^2$ at $(1,4,2)$ and $(-3,-3,3)$

Solution: The given surface is $xy - z^2 = 0$ ∴

$$\phi = xy - z^2$$

We know $\nabla\phi$ is normal to the surface at the point (x, y, z)

Let \vec{n}_1, \vec{n}_2 , be the normals to the surface at the points $(1, 4, 2)$ and $(-3, -3, 3)$ respectively.

Now

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \quad \dots (1)$$

$$\frac{\partial\phi}{\partial x} = y, \quad \frac{\partial\phi}{\partial y} = x, \quad \text{and} \quad \frac{\partial\phi}{\partial z} = -2z$$

Substituting in (1) we get

$$\nabla\phi = yi + xj - 2zk$$

At the point $(1, 4, 2)$, $\nabla\phi = 4i + j - 4k$

and at the point $(-3, -3, 3)$, $\nabla\phi = -3i - 3j - 6k$

$$\therefore \vec{n}_1 = 4i + j - 4k, \quad \vec{n}_2 = -3i - 3j - 6k$$

If θ is the angle between the normals, then

$$\begin{aligned} \cos\theta &= \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} \\ &= \frac{(i + j - 4k) \cdot (-3i - 3j - 6k)}{\sqrt{16 + 1 + 16} \cdot \sqrt{9 + 9 + 36}} \\ &= \frac{-12 - 3 + 24}{\sqrt{33}\sqrt{54}} \\ &= \frac{9}{\sqrt{3} \sqrt{11} 3\sqrt{3} \sqrt{2}} \\ &= \frac{1}{\sqrt{22}} \end{aligned}$$

$$\text{Hence } \theta = \cos^{-1} \left(\frac{1}{\sqrt{22}} \right)$$

Problem 2.6.4. Find the constants a & b so that the surface $ax^2 - byz = (a+2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$. **Ans :**

Solution: The given surfaces are

$$ax^2 - byz - (a+2)x = 0$$

$$4x^2y + z^3 - 4 = 0$$

$$\text{Let } \phi_1 = ax^2 - byz - (a+2)x \quad \dots (1)$$

and

$$\phi_2 = 4x^2y + z^3 - 4 \quad \dots (2)$$

Given that surfaces (1) and (2) cut orthogonally at the point $(1, -1, 2)$.

$$\therefore \nabla\phi_1 \cdot \nabla\phi_2 = 0$$

we have

$$\nabla\phi_1 = i \frac{\partial\phi_1}{\partial x} + j \frac{\partial\phi_1}{\partial y} + k \frac{\partial\phi_1}{\partial z}$$

$$\text{where } \frac{\partial \phi_1}{\partial x} = 2ax - a - 2,$$

$$\frac{\partial \phi_1}{\partial y} = -bz \text{ and}$$

$$\frac{\partial \phi_1}{\partial z} = -by$$

$$\therefore \nabla \phi_1 = (2ax - a - 2)i - bzj - byk$$

Similarly,

$$\nabla \phi_2 = i \frac{\partial \phi_2}{\partial x} + j \frac{\partial \phi_2}{\partial y} + k \frac{\partial \phi_2}{\partial z}$$

$$\text{where } \frac{\partial \phi_2}{\partial x} = 8xy,$$

$$\frac{\partial \phi_2}{\partial y} = 4x^2$$

$$\text{and } \frac{\partial \phi_2}{\partial z} = 3z^2$$

$$\therefore \nabla \phi_2 = 8xyi + 4x^2j + 3z^2k$$

At the point (1,-1,2),

$$\begin{aligned} \nabla \phi_1 &= (2a - a - 2)i - b(2)j - b(-1)k \\ &= (a - 2)i - 2bj + bk \text{ and} \end{aligned}$$

$$\nabla \phi_2 = -8i + 4j + 12k$$

$$\nabla \phi_1 \cdot \nabla \phi_2 = 0$$

$$\Rightarrow ((a - 2)i - 2bj + bk) \cdot (-8i + 4j + 12k) = 0$$

$$\Rightarrow -8(a - 2) - 8b + 12b = 0$$

$$\Rightarrow -8a + 4b + 16 = 0$$

$$\Rightarrow 2a - b = 4 \quad \dots (3)$$

Since the point of intersection $(1, -1, 2)$ is a common point on the surfaces (1) and (2), it should satisfy (1). Hence we get

$$a + 2b - (a + 2) = 0$$

$$\Rightarrow 2b = 2$$

$$\Rightarrow b = 1$$

$$\therefore \Rightarrow 2a = 4 + b = 4 + 1 = 5 \quad (\text{from (3)})$$

$$\Rightarrow a = \frac{5}{2}$$

$$\therefore a = \frac{5}{2}, \quad b = 1$$

2.7 Directional Derivative :

The directional derivative of a scalar point function ϕ in a given direction \vec{a} is the rate of change of ϕ in that direction. It is given by the component of $\nabla\phi$ in the direction of \vec{a}

$$\therefore \text{the directional derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

We can write

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{|\nabla\phi| |\vec{a}|}{|\vec{a}|} \cos\theta = |\nabla\phi| \cos\theta$$

, where θ is the angle between $\nabla\phi$ and \vec{a} .

So, the directional derivative at a given point is maximum if $\cos\theta$ is maximum. i.e., $\cos\theta = 1 \Rightarrow \theta = 0$

\therefore The maximum directional derivative at a point is in the direction of $\nabla\phi$ and the maximum directional derivative is $|\nabla\phi|$.

Problem 2.7.1. Find the directional derivative of $\Phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of the vector $2i - j - 2k$

Solution: Given

$$\phi(x, y, z) = x^2yz + 4xz^2$$

We know

$$\begin{aligned}\text{grad } \phi &= \nabla \phi \\ &= i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}\end{aligned}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial \phi}{\partial y} = x^2z, \quad \frac{\partial \phi}{\partial z} = x^2y + 8xz$$

At the point $(1, -2, -1)$,

$$\frac{\partial \phi}{\partial x} = 2 \cdot 1(-2)(-1) + 4(-1)^2 = 8$$

$$\frac{\partial \phi}{\partial y} = 1^2 \cdot (-1) = -1$$

$$\frac{\partial \phi}{\partial z} = 1^2(-2) + 8 \cdot 1(-1) = -2 - 8 = -10$$

\therefore at the point $(1, -2, -1)$, $\nabla \phi = 8i - j - 10k$

Given direction is $\vec{a} = 2i - j - 2k$

\therefore the directional derivative of ϕ at the point $(1, -2, -1)$ in the direction of \vec{a} is given by

$$\begin{aligned}\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} &= (8i - j - 10k) \cdot \frac{(2i - j - 2k)}{\sqrt{4 + 1 + 4}} \\ &= \frac{16 + 1 + 20}{\sqrt{9}} \\ &= \frac{37}{3}\end{aligned}$$

Problem 2.7.2. Find the maximum value of the directional derivative of $\phi = x^3yz$ at the point $(1, 4, 1)$

Solution: Given

$$\begin{aligned}\phi &= x^3yz \\ \nabla \phi &= i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}\end{aligned}$$

We know, The directional derivative is maximum in the direction of $\nabla \phi$ and the maximum value = $|\nabla \phi|$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial \phi}{\partial x} = 3x^2yz, \quad \frac{\partial \phi}{\partial y} = x^3z, \quad \frac{\partial \phi}{\partial z} = x^3y$$

At the point $(1, 4, 1)$,

$$\frac{\partial \phi}{\partial x} = 3(1)(4)(1) = 12,$$

$$\frac{\partial \phi}{\partial y} = 1^3(1) = 1,$$

$$\frac{\partial \phi}{\partial z} = 1^3(4) = 4$$

\therefore at the point $(1, 4, 1)$, $\nabla \phi = 12i + j + 4k$

Maximum value of the directional derivative is

$$\begin{aligned} |\nabla \phi| &= |12i + j + 4k| \\ &= \sqrt{144 + 1 + 16} \\ &= \sqrt{161} \end{aligned}$$

Problem 2.7.3. Find the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at the point $(-1, 2, 1)$.

Solution:: Given

$$\begin{aligned} f &= xy^2 + yz^3 \\ \therefore \nabla f &= i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \\ &= y^2 i + (2xy + z^3) j + 3yz^2 k \end{aligned}$$

At the point $(2, -1, 1)$,

$$\begin{aligned} \nabla f &= (-1)^2 i + (-4 + 1) j + 3(-1)1^2 k \\ \Rightarrow \nabla f &= i - 3j - 3k \end{aligned}$$

The directional derivative of f in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point $(-1, 2, 1)$ is required.

Let $g = x \log z - y^2 + 4$.

Normal to this surface is,

$$\begin{aligned} \nabla g &= i \frac{\partial g}{\partial x} + j \frac{\partial g}{\partial y} + k \frac{\partial g}{\partial z} \\ &= \log z i - 2y j + \frac{x}{z} k \end{aligned}$$

At the point $(-1, 2, 1)$,

$$\begin{aligned}\vec{a} &= \nabla g \\ &= \log 1i - 4j + \left(\frac{-1}{1}\right)k \\ &= 0i - 4j - k = -4j - k\end{aligned}$$

$$\therefore \vec{a} = -4j - k$$

Required directional derivative is

$$\begin{aligned}\nabla f \cdot \frac{\vec{a}}{|\vec{a}|} &= (i - 3j - 3k) \cdot \frac{(-4j - k)}{\sqrt{16 + 1}} \\ &= \frac{12 + 3}{\sqrt{17}} = \frac{15}{\sqrt{17}}\end{aligned}$$

Problem 2.7.4. Find the directional derivative of $\phi(x, y, z) = 2xy + 3y^2 - z^3$ at the point $P(1, -1, 2)$ in the direction of the vector $\vec{v} = i - j + k$.

Solution: First we find that

$$\begin{aligned}\text{grad } \phi &= \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k \\ &= 2yi + (2x + 6y)j + (-3z^2)k\end{aligned}$$

$$\nabla \phi|_{(1, -1, 2)} = -2i - 4j - 12k$$

The directional derivative of ϕ in the direction of v is given by

$$\begin{aligned}\nabla \phi \cdot \frac{\vec{v}}{|\vec{v}|} &= (-2i - 4j - 12k) \cdot \left(\frac{i - j + k}{\sqrt{3}}\right) \\ &= \frac{-2 + 4 - 12}{\sqrt{3}} \\ &= -\frac{10}{\sqrt{3}}\end{aligned}$$

12. Find the directional derivative of the function $\phi = xy + yz + zx$ in the direction of the vector $2i + 3j + 6k$ at the point $(3, 1, 2)$. Ans : $45/7$
13. Find the directional derivative of the function $2yz + z^2$ in the direction of the vector $i + 2j + 2k$ at the point $(1, -1, 3)$ Ans : $\frac{20}{3}$
14. Find the directional derivative of $x^3 + y^3 + z^3$ at the point $(1, -1, 2)$ in the direction of $i + 2j + k$ Ans : $\frac{7\sqrt{6}}{2}$

2.8 Solenoidal and irrotational vector fields

A vector \vec{F} said to be **solenoidal** if $div\vec{F} = 0$. i. e.

$$\nabla \cdot \vec{F} = 0$$

A vector \vec{F} said to be **irrotational** if $curl\vec{F} = \vec{0}$. i. e.

$$\nabla \times \vec{F} = \vec{0}$$

Note : (1) An irrotational vector field is also called as conservative field or potential field.

(2) If \vec{F} is irrotational, then there always exists a scalar function Φ such that $\nabla\phi = \vec{F}$

Problem 2.8.1. Show that $V(x, y, z) = (4xy - z^3)i + 2x^2j - 3xz^2k$ is irrotational

$$\begin{aligned} \text{Curl } \vec{V} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y} (-3xz^2) - \frac{\partial}{\partial z} (2x^2) \right) i \\ &\quad + \left(\frac{\partial}{\partial z} (4xy - z^3) - \frac{\partial}{\partial x} (-3xz^2) \right) j \\ &\quad + \left(\frac{\partial}{\partial x} (2x^2) - \frac{\partial}{\partial y} (4xy - z^3) \right) k \\ &= (0 - 0)i + (-3z^2 - (-3z^2))j + (4x - 4x)k = 0 \end{aligned}$$

Hence \vec{V} is irrotational.

Problem 2.8.2. Show that $V(x, y, z) = 3y^4z^2i + 4x^3z^2j + 3x^2y^2k$ is solenoidal.

Solution:

$$\begin{aligned} \text{div } \vec{V} &= \nabla \cdot \vec{V} \\ &= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \right) \cdot (3y^4z^2i + 4x^3z^2j + 3x^2y^2k) \\ &= \frac{\partial}{\partial x} (3y^4z^2) + \frac{\partial}{\partial y} (4x^3z^2) + \frac{\partial}{\partial z} 3(x^2y^2) \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

Hence, V is solenoidal.

Problem 2.8.3. Find the value of a so that $V(x, y, z) = (x + 3y)i + (y - 2z)j + (x - az)k$ is solenoidal.

Solution: Given that \vec{V} is solenoidal.

Hence $\text{div } V = 0$

$$\Rightarrow \nabla \cdot V = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \right) \cdot ((x + 3y)i + (y - 2z)j + (x - az)k) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x - az) = 0$$

$$\Rightarrow 1 + 1 - a = 0$$

$$\Rightarrow a = 2$$

Problem 2.8.4. Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational.

Solution: For solenoidal, we have to prove $\nabla \cdot \vec{F} = 0$.

Now,

$$\nabla \cdot \vec{F}$$

$$= \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot \left[\begin{array}{l} (y^2 - z^2 + 3yz - 2x)i \\ +(3xz + 2xy)j \\ +(3xy - 2xz + 2z)k \end{array} \right]$$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= (-2) + (2x) + (-2x + 2)$$

$$= 0$$

Thus, \vec{F} is solenoidal.

For irrotational, we have to prove $\text{Curl } \vec{F} = 0$.

$$\begin{aligned} & \text{Curl } \vec{F} \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix} \\ &= (3x - 3x)i - (-2z + 3y - 3y + 2z)j \\ &\quad + (3z + 2y - 2y - 3z)k \\ &= 0i + 0j + 0k = 0 \end{aligned}$$

Hence \vec{F} is irrotational.

Problem 2.8.5. Find a, b, c if $(x + y + az)i + (bx + 2y - z)j + (-x + cy + 2z)k$ is irrotational.

Solution: Let $\vec{F} = (x + y + az)i + (bx + 2y - z)j + (-x + cy + 2z)k$

Given \vec{F} is irrotational.

$$\therefore \nabla \times \vec{F} = \vec{0}$$

$$\begin{aligned} & \Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & -x + cy + 2z \end{vmatrix} = \vec{0} \\ & \Rightarrow i \left[\frac{\partial}{\partial y}(-x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z) \right] \\ & \quad - j \left[\frac{\partial}{\partial x}(-x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az) \right] \\ & \quad + k \left[\frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az) \right] = 0 \\ & \Rightarrow i(c + 1) - j(-1 - a) + k(b - 1) = \vec{0} \\ & \Rightarrow (c + 1)i + (1 + a)j + (b - 1)k = \vec{0} \\ & \Rightarrow c + 1 = 0, 1 + a = 0, b - 1 = 0 \\ & \Rightarrow a = -1, b = 1 \text{ and } c = -1 \end{aligned}$$

Problem 2.8.6. Prove that $\frac{x^i + y^j}{x^2 + y^2}$ is both solenoidal & irrotational.

Solution::

$$\begin{aligned}\vec{F} &= \frac{xi + yj}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2}i + \frac{y}{x^2 + y^2}j\end{aligned}$$

$$\begin{aligned}\operatorname{div} \vec{F} &= \nabla \cdot \vec{F} \\ &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) \\ &= \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} \\ &= \frac{(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \\ &= 0 \\ &\Rightarrow \vec{F} \text{ is solenoidal.}\end{aligned}$$

$$\begin{aligned}\operatorname{curl} \vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix} \\ &= i[0 - 0] - j[0 - 0] + k \left(\frac{-2xy}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2} \right) \\ &= (0)i + (0)j + (0)k = \vec{0}\end{aligned}$$

$\operatorname{curl} \vec{F} = \vec{0} \Rightarrow \vec{F}$ is irrotational.

Problem 2.8.7. A vector field is given by $\vec{A} = (x^2 + xy^2)i + (y^2 + x^2y)j$. Show that the field is irrotational and find the scalar potential.

Solution:

$$\begin{aligned}\operatorname{Curl} \vec{A} &= \nabla \times \vec{A} \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix} \\ &= i(0 - 0) - j(0 - 0) + k(2xy - 2xy) = 0\end{aligned}$$

Hence, \vec{A} is irrotational. If ϕ is the scalar potential, then

$$\vec{A} = \text{grad } \phi$$

$$i.e. \vec{A} = \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k$$

$$(x^2 + xy^2)i + (y^2 + x^2y)j = \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right)$$

Equating the components on both sides

$$\frac{\partial \phi}{\partial x} = (x^2 + xy^2) \quad \dots (1)$$

$$\frac{\partial \phi}{\partial y} = (y^2 + x^2y) \quad \dots (2)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \dots (3)$$

Integrating (1), only w.r.to x , treating y and z as constants,

$$\phi = \frac{x^3}{3} + \frac{x^2}{2}y^2 + f(y, z) \quad \dots (4)$$

Integrating (1), only w.r.to y , treating x and z as constants,

$$\phi = \frac{y^3}{3} + \frac{y^2}{2}x^2 + g(x, z) \quad \dots (5)$$

Integrating (1), only w.r.to z , treating x and y as constants,

$$\phi = 0 + h(x, y) \quad \dots (6)$$

Equating all the expressions for ϕ from equations (4), (5) and (6), we get

$$\frac{x^3}{3} + \frac{x^2}{2}y^2 + f(y, z) = \frac{y^3}{3} + \frac{y^2}{2}x^2 + g(x, z) = 0 + h(x, y)$$

Comparing we get

$$f(y, z) = \frac{y^3}{3} + 0$$

$$g(x, z) = \frac{x^3}{3} + 0$$

$$h(x, y) = \frac{y^3}{3} + \frac{y^2}{2}x^2 + \frac{x^3}{3}$$

Substituting these values in (4), (5) and (6), we get a unique expression for ϕ as

$$\phi = \frac{y^3}{3} + \frac{y^2}{2}x^2 + \frac{x^3}{3}$$

Problem 2.8.8. Show that $\vec{F} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ is irrotational and hence find its scalar potential.

Solution: let

$$\vec{f} = (x^2 - yz) i + (y^2 - zx) j + (z^2 - xy) k$$

$$\begin{aligned} \text{Then curl } \vec{f} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} \\ &= \sum i(-x + x) = \vec{0} \end{aligned}$$

$\therefore \vec{f}$ is Irrotational. Then there exists ϕ such that $\vec{f} = \nabla \phi$

$$\Rightarrow i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} = (x^2 - yz) i + (y^2 - zx) j + (z^2 - xy) k$$

Comparing components, we get

$$\frac{\partial \phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int (x^2 - yz) dx = \frac{x^3}{3} - xyz + f_1(y, z) \quad \dots (1)$$

$$\frac{\partial \phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z, x) \quad \dots (2)$$

$$\frac{\partial \phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x, y) \quad \dots (3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = \frac{x^3 + y^3 + z^3}{3} - xyz + \text{Constant}$$

Which is the required scalar potential.

Problem 2.8.9. Find the constants a and b such that $F = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational. Find Φ such that $F = \nabla \Phi$

Solution: Given that F is irrotational. Hence

$$\nabla \times \vec{F} = 0$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = 0$$

$$\Rightarrow i(-1 + 1) - j(bz^2 - 3z^2) + k(6x - ax) = 0$$

$$\Rightarrow 0i - z^2(b - 3)j + x(6 - a)k = 0$$

$$\Rightarrow b = 3 \text{ and } a = 6$$

$$\therefore \vec{F} = (6xy + z^3) i + (3x^2 - z) j + (3xz^2 - y) k$$

Given, $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy + z^3 \Rightarrow \phi = 3x^2y + xz^3 + f_1(y, z) \quad \dots (1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \Rightarrow \phi = 3x^2y - yz + f_2(x, z) \quad \dots (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \Rightarrow \phi = xz^3 - yz + f_3(x, y) \quad \dots (3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = 3x^2y + xz^3 - yz + c$$

Problem 2.8.10. Find the constants a, b, c such that the vector field $\vec{f} = (x + y + az)i + (x + cy + 2z)k + (bx + 2y - z)j$ is irrotational. Also find Φ such that $\vec{F} = \nabla\Phi$

Solution: Given that the vector field is irrotational. Therefore $\nabla \times \vec{F} = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & x + cy + 2z \end{vmatrix} = 0$$

$$i(c + 1) - j(1 - a) + k(b - 1) = 0$$

$$c + 1 = 0, \quad 1 - a = 0, \quad b - 1 = 0 \quad \text{Hence } \vec{f} = (x + y + z)i + (x + 2y -$$

$$c = -1, \quad a = 1, \quad b = 1$$

$$z)j + (x - y + 2z)k$$

Then there exists ϕ such that $\vec{f} = \nabla\phi$

$$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= (x + y + z)i + (x + 2y - z)j + (x - y + 2z)k$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x + y + z \Rightarrow \phi = \frac{x^2}{2} + xy + xz + f_1(y, z) \quad \dots (1)$$

$$\frac{\partial\phi}{\partial y} = x + 2y - z \Rightarrow \phi = xy + 2\frac{y^2}{2} - zy + f_2(x, z) \quad \dots (2)$$

$$\frac{\partial\phi}{\partial z} = x - y + 2z \Rightarrow \phi = xz - yz + 2\frac{z^2}{2} + f_3(x, y) \quad \dots (3)$$

Equating all the expressions for ϕ from equations (1), (2) and (3), we get

$$\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2 + \text{constant}$$

Problem 2.8.11. Find constants a, b and c if the vector

$\vec{f} = (2x + 3y + az)i + (bx + 2y + 3z)j + (2x + cy + 3z)k$ is Irrotational.

Solution: Given

$$\vec{f} = (2x + 3y + az)i + (bx + 2y + 3z)j + (2x + cy + 3z)k$$

$$\begin{aligned} \text{Curl } \vec{f} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 3y + az & bx + 2y + 3z & 2x + cy + 3z \end{vmatrix} \\ &= (c - 3)i - (2 - a)j + (b - 3)k \end{aligned}$$

If the vector is irrotational then $\text{curl } \vec{f} = \vec{0}$

$$\therefore 2 - a = 0 \Rightarrow a = 2, b - 3 = 0 \Rightarrow b = 3, c - 3 = 0 \Rightarrow c = 3$$

Problem 2.8.12. Find the constants a, b, c such that the vector field $\vec{f} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational. Hence find its scalar potential.

Solution:: Given vector is $\vec{A} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is Irrotational.

$$\Rightarrow \text{curl } \vec{A} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow (c + 1)i + (a - 4)j + (b - 2)k = \vec{0}$$

$$\Rightarrow (c + 1)i + (a - 4)j + (b - 2)k = 0i + 0j + 0k$$

Comparing both sides,

$$c + 1 = 0, a - 4 = 0, b - 2 = 0$$

$$c = -1, a = 4, b = 2$$

$$\text{Now } \vec{A} = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k,$$

on substituting the values of a, b, c

we have $\vec{A} = \nabla\phi$.

$$\begin{aligned} \Rightarrow \vec{A} &= (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k \\ &= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \end{aligned}$$

Comparing both sides, we have

$$\frac{\partial\phi}{\partial x} = x + 2y + 4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4zx + f_1(y, z)$$

$$\frac{\partial\phi}{\partial y} = 2x - 3y - z \Rightarrow \phi = 2xy - \frac{3y^2}{2} - yz + f_2(x, z)$$

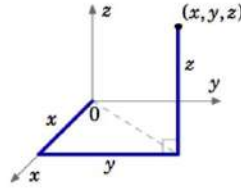
$$\frac{\partial\phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2 + f_3(y, x)$$

$$\text{Hence } \phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - yz + c$$

2.9 Curvilinear coordinates

Curvilinear coordinates are alternative coordinate systems used to describe points in space.

The Cartesian coordinates of a point (x, y, z) are determined by straight paths starting from the origin: first along the x -axis, then parallel to the y -axis, and finally parallel to the z -axis, as shown in the following figure.



In curvilinear coordinate systems, these paths can be curved.

Let the rectangular coordinates (x, y, z) of any point P be expressed as functions of (u, v, w) i.e.,

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$$

If these equations are solved for u, v, w in terms of x, y, z , we get inverse transformation i.e.,

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$

Here, (u, v, w) are called curvilinear coordinates of (x, y, z) .

The surfaces $u = c_1, v = c_2, w = c_3$ are called coordinate surfaces through $P(c_1, c_2, c_3)$. The curve of intersection of $u = c_1$ and $v = c_2$ will be called w -curve. Similarly u -curve and v -curve are defined.

Let $\text{vec} \mathbf{R} = xi + yj + zk$ be the position vector of the point P, where x, y and z are in terms of u, v, w

A tangent vector to the u -curve at P (for which v and w are constants) is $\frac{\partial \vec{R}}{\partial u}$.

The magnitude of this tangent vector is denoted by the scale factor h_1 .

$$i.e. h_1 = \left| \frac{\partial \vec{R}}{\partial u} \right|$$

\therefore The unit vector along the tangent vector to the u -curve is

$$e_1 = \frac{\partial \vec{R} / \partial u}{h_1} = \frac{\partial \vec{R} / \partial u}{|\partial \vec{R} / \partial u|}$$

Similarly, A tangent vectors to the v -curve and w -curves at P are respectively given by $\frac{\partial \vec{R}}{\partial v}$ and $\frac{\partial \vec{R}}{\partial w}$.

The magnitudes of these tangent vectors are denoted by scale factors h_2 and h_3 .

$$i.e. h_2 = \left| \frac{\partial \vec{R}}{\partial v} \right|$$

and

$$h_3 = \left| \frac{\partial \vec{R}}{\partial w} \right|$$

∴ The unit vectors along the tangent vectors to the v -curve and w -curve are respectively given by

$$e_2 = \frac{\partial \vec{R} / \partial v}{h_2} = \frac{\partial \vec{R} / \partial v}{|\partial \vec{R} / \partial v|}$$

$$e_3 = \frac{\partial \vec{R} / \partial w}{h_3} = \frac{\partial \vec{R} / \partial w}{|\partial \vec{R} / \partial w|}$$

The two types of curvilinear coordinates which we will consider are cylindrical and spherical coordinates.

2.10 Cylindrical coordinates

Cylindrical coordinates specify a point in space using three quantities: radial distance (ρ), angle (θ), and height (z).

$$x = \rho \cos(\theta), \quad y = \rho \sin(\theta), \quad z = z$$

Here, Radial distance (ρ) is the distance from the z -axis to the point projected onto the xy -plane.

Angle (θ) is the angle measured counterclockwise from the positive x -axis in the xy -plane.

Height (z) is the Vertical distance along the z -axis from the xy -plane to the point.

The inverse transformation is given by

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad z = z$$

Cylindrical coordinates are often used when there is symmetry around the z -axis.

2.11 Spherical Coordinates

Spherical coordinates specify a point in space using three quantities: radial distance (ρ), polar angle (θ), and an angle (ϕ).

$$x = r \sin(\theta) \cos(\phi), \quad y = r \sin(\theta) \sin(\phi), \quad z = r \cos(\theta)$$

Here, Radial distance r is the distance from the origin to the point.

Angle (θ) is the angle measured counterclockwise from the positive x -axis in the xy -plane to the projection of the point onto the xy -plane.

Polar angle (ϕ) is the angle measured from the positive z -axis to the line segment connecting the origin to the point. The inverse transformation is given by

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \quad \phi = \arctan\left(\frac{y}{x}\right)$$

Spherical coordinates are useful when there is symmetry about the origin.

Problem 2.11.1. *Prove that the cylindrical coordinate system is orthogonal.*

Solution : In cylindrical coordinate, we have

$$x = \rho \cos(\theta), \quad y = \rho \sin(\theta), \quad z = z$$

The position vector of any point in cylindrical coordinates is

$$\vec{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \rho \cos \phi \mathbf{i} + \rho \sin \phi \mathbf{j} + z\mathbf{k}$$

The tangent vectors to the ρ , ϕ and z curves are given respectively by $\frac{\partial \vec{R}}{\partial \rho}$, $\frac{\partial \vec{R}}{\partial \phi}$ and $\frac{\partial \vec{R}}{\partial z}$ where

$$\frac{\partial \vec{R}}{\partial \rho} = \cos \phi \mathbf{i} + \sin \phi \mathbf{j} + 0\mathbf{k},$$

$$\frac{\partial \vec{R}}{\partial \phi} = -\rho \sin \phi \mathbf{i} + \rho \cos \phi \mathbf{j} + 0\mathbf{k}$$

$$\frac{\partial \vec{R}}{\partial z} = \mathbf{k}$$

The unit vectors in these directions are

$$\begin{aligned}
 e_1 &= \frac{\partial \vec{R} / \partial \rho}{|\partial / \partial \rho|} \\
 &= \frac{\cos \phi i + \sin \phi j + 0k}{\sqrt{\cos^2 \phi + \sin^2 \phi}} \\
 &= \cos \phi i + \sin \phi j \\
 e_2 &= \frac{\partial \vec{R} / \partial \phi}{|\partial \vec{R} / \partial \phi|} \\
 &= \frac{-\rho \sin \phi i + \rho \cos \phi j + 0k}{\sqrt{\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi}} \\
 &= -\sin \phi i + \cos \phi j \\
 e_3 &= \frac{\partial \vec{R} / \partial z}{|\partial \vec{R} / \partial z|} = k
 \end{aligned}$$

Then

$$e_1 \cdot e_2 = (\cos \phi i + \sin \phi j + 0k) \cdot (-\sin \phi i + \cos \phi j + 0k) = 0$$

$$e_1 \cdot e_3 = (\cos \phi i + \sin \phi j) \cdot (0i + 0j + k) = 0$$

$$e_2 \cdot e_3 = (-\sin \phi i + \cos \phi j + 0k) \cdot (0i + 0j + k) = 0$$

and so e_1 , e_2 and e_3 are mutually perpendicular and the coordinate system is orthogonal.

Problem 2.11.2. Prove that the spherical coordinate system is orthogonal.

Solution : At any point P , we have

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

$$\vec{R} = xi + yj + zk$$

$$= r \sin \theta \cos \phi i + r \sin \theta \sin \phi j + r \cos \theta k$$

and If $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ be the unit vectors at P in the directions of the tangents to the r, θ, ϕ curves respectively, then

$$\begin{aligned}
 \hat{e}_1 &= \frac{\partial \vec{R} / \partial r}{h_1} \\
 &= \frac{\partial \vec{R} / \partial r}{|\partial \vec{R} / \partial r|} \\
 &= \frac{\sin \theta \cos \phi i + \sin \theta \sin \phi j + \cos \theta k}{\sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta}} \\
 &= \sin \theta \cos \phi i + \sin \theta \sin \phi j + \cos \theta k \\
 \hat{e}_2 &= \frac{\partial \vec{R} / \partial \theta}{h_2} \\
 &= \frac{\partial \vec{R} / \partial \theta}{|\partial \vec{R} / \partial \theta|} \frac{r \cos \theta \cos \phi i + r \cos \theta \sin \phi j - r \sin \theta k}{\sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta}} \\
 &= \cos \theta \cos \phi i + \cos \theta \sin \phi j - \sin \theta k \\
 \hat{e}_3 &= \frac{\partial \vec{R} / \partial \phi}{h_3} \\
 &= \frac{\partial \vec{R} / \partial \phi}{|\partial \vec{R} / \partial \phi|} \\
 &= \frac{-r \sin \theta \sin \phi i + r \sin \theta \cos \phi j}{\sqrt{r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi}} \\
 &= -\sin \phi i + \cos \phi j
 \end{aligned}$$

Now,

$$\hat{e}_1 \cdot \hat{e}_2$$

$$\begin{aligned} &= (\sin \theta \cos \phi i + \sin \theta \sin \phi j + \cos \theta k) \cdot (\cos \theta \cos \phi i + \cos \theta \sin \phi j - \sin \theta k) \\ &= \sin \theta \cos \theta \cos^2 \phi + \sin \theta \cos \theta \sin^2 \phi - \sin \theta \cos \theta \\ &= \sin \theta \cos \theta - \sin \theta \cos \theta = 0, \end{aligned}$$

$$\hat{e}_2 \cdot \hat{e}_3$$

$$\begin{aligned} &= (\cos \theta \cos \phi i + \cos \theta \sin \phi j - \sin \theta k) \cdot (-\sin \phi i + \cos \theta j) \\ &= -\cos \theta \sin \phi \cos \phi + \cos \theta \sin \phi \cos \phi = 0 \end{aligned}$$

$$\hat{e}_3 \cdot \hat{e}_1$$

$$\begin{aligned} &= (-\sin \phi i + \cos \theta j) \cdot (\sin \theta \cos \phi i + \sin \theta \sin \phi j + \cos \theta k) \\ &= -\sin \theta \sin \phi \cos \phi + \sin \theta \sin \phi \cos \phi = 0 \end{aligned}$$

Hence the spherical coordinate system is orthogonal.

Problem 2.11.3. Express the vector field $\vec{A} = zi - 2xj + yk$ in cylindrical polar coordinates.

Solution: $x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z$

$$\vec{R} = xi + yj + zk$$

$$= \rho \cos \phi i + \rho \sin \phi j + zk$$

Then in cylindrical polar coordinates,

$$\vec{A} = A_1 e_1 + A_2 e_2 + A_3 e_3$$

where

$$\hat{e}_1 = \frac{\frac{\partial \vec{R}}{\partial \rho}}{\left| \frac{\partial \vec{R}}{\partial \rho} \right|} = \cos \phi i + \sin \phi j$$

$$\hat{e}_2 = \frac{\frac{\partial \vec{R}}{\partial \phi}}{\left| \frac{\partial \vec{R}}{\partial \phi} \right|} = -\sin \phi i + \cos \phi j$$

$$\hat{e}_3 = \frac{\frac{\partial \vec{R}}{\partial z}}{\left| \frac{\partial \vec{R}}{\partial z} \right|} = k$$

$$\begin{aligned} A_1 &= \vec{A} \cdot e_1 \\ &= (zi - 2xj + yk) \cdot (\cos \phi i + \sin \phi j + 0k) \\ &= z \cos \phi - 2x \sin \phi \\ &= z \cos \phi - 2\rho \cos \phi \sin \phi \end{aligned}$$

$$\begin{aligned} A_2 &= \vec{A} \cdot e_2 \\ &= (zi - 2xj + yk) \cdot (-\sin \phi i + \cos \phi j + 0k) \\ &= -z \sin \phi - 2x \cos \phi \\ &= -z \sin \phi - 2\rho \cos \phi \cos \phi \\ &= -z \sin \phi - 2\rho \cos^2 \phi \end{aligned}$$

$$\begin{aligned} A_3 &= \vec{A} \cdot e_3 \\ &= (zi - 2xj + yk) \cdot (0i + 0j + k) \\ &= y \\ &= \rho \sin \phi \end{aligned}$$

Hence

$$\begin{aligned} \vec{A} &= A_1 e_1 + A_2 e_2 + A_3 e_3 \\ &= (z \cos \phi - 2\rho \cos \phi \sin \phi) e_1 + (-z \sin \phi - 2\rho \cos^2 \phi) e_2 + \rho \sin \phi \hat{e}_3 \end{aligned}$$

Problem 2.11.4. Express the vector field $2yi - zj + 3xk$ in cylindrical polar coordinate system.

Solution : We have, so that

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z$$

$$\vec{R} = \rho \cos \phi i + \rho \sin \phi j + zk$$

If e_1, e_2, e_3 be the unit vectors along the tangents to ρ, ϕ, z -curves respectively, then

$$e_1 = \frac{\partial \vec{R} / \partial \rho}{|\partial \vec{R} / \partial \rho|} = \cos \phi i + \sin \phi j$$

$$e_2 = \frac{\partial \vec{R} / \partial \phi}{|\partial \vec{R} / \partial \phi|} = -\sin \phi i + \cos \phi j$$

$$e_3 = \frac{\partial \vec{R} / \partial z}{|\partial \vec{R} / \partial z|} = \frac{k}{|1|} = k$$

Let the expression for $\vec{F} = 2yi - zj + 3xk$ in cylindrical coordinates be

$$\vec{F} = f_1 e_1 + f_2 e_2 + f_3 e_3$$

$$f_1 = \vec{F} \cdot e_1 = (2yi - zj + 3xk) \cdot (\cos \phi i + \sin \phi j)$$

$$= (2\rho \sin \phi i - zj + 3\rho \cos \phi k) \cdot (\cos \phi i + \sin \phi j)$$

$$= 2\rho \sin \phi \cos \phi - z \sin \phi$$

Then

$$f_2 = \vec{F} \cdot e_2 = (2\rho \sin \phi i - zj + 3\rho \cos \phi k) \cdot (-\sin \phi i + \cos \phi j)$$

$$= -2\rho \sin^2 \phi - z \cos \phi$$

$$f_3 = \vec{F} \cdot e_3 = (2\rho \sin \phi i - zj + 3\rho \cos \phi k) \cdot k = 3\rho \cos \phi$$

$$\therefore \vec{F} = (2\rho \sin \phi \cos \phi - z \sin \phi) e_1 - (2\rho \sin^2 \phi + z \cos \phi) e_2 + 3\rho \cos \phi e_3$$

which is the required expression.

Problem 2.11.5. Express the vector field $A = yzi - yj + xz^2k$ in cylindrical polar coordinates.

Sol :

$$\text{Solution: } x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z$$

$$\vec{R} = \rho \cos \phi i + \rho \sin \phi j + zk$$

If e_1, e_2, e_3 be the unit vectors along the tangents to ρ, ϕ, z -curves respectively, then

$$e_1 = \frac{\partial \vec{R}/\partial \rho}{|\partial \vec{R}/\partial \rho|} = \cos \phi i + \sin \phi j + 0k$$

$$e_2 = \frac{\partial \vec{R}/\partial \phi}{|\partial \vec{R}/\partial \phi|} = -\sin \phi i + \cos \phi j + 0k$$

$$e_3 = \frac{\partial \vec{R}/\partial z}{|\partial \vec{R}/\partial z|} = \frac{k}{|1|} = k$$

Let the expression for $\vec{A} = yzi - yj + xz^2k$ in cylindrical coordinates be

$$\vec{A} = A_1 e_1 + A_2 e_2 + A_3 e_3$$

$$\begin{aligned} A_1 &= \vec{A} \cdot e_1 = (yzi - yj + xz^2k) \cdot (\cos \phi i + \sin \phi j) \\ &= (\rho \sin \phi zi - \rho \sin \phi j + \rho \cos \phi z^2k) \cdot (\cos \phi i + \sin \phi j + 0k) \\ &= z\rho \sin \phi \cos \phi - \rho \sin^2 \phi \end{aligned}$$

Then

$$\begin{aligned} A_2 &= \vec{A} \cdot e_2 = (\rho \sin \phi zi - \rho \sin \phi j + \rho \cos \phi z^2k) \cdot (-\sin \phi i + \cos \phi j + 0k) \\ &= -z\rho \sin^2 \phi - \rho \sin \phi \cos \phi \end{aligned}$$

$$\begin{aligned} A_3 &= \vec{A} \cdot e_3 = (\rho \sin \phi zi - \rho \sin \phi j + \rho \cos \phi z^2k) \cdot k \\ &= \rho \cos \phi z^2 \end{aligned}$$

$$\begin{aligned} \therefore \vec{A} &= A_1 e_1 + A_2 e_2 + A_3 e_3 \\ &= (z\rho \sin \phi \cos \phi - \rho \sin^2 \phi) \hat{e}_1 - (z\rho \sin^2 \phi + \rho \sin \phi \cos \phi) \hat{e}_2 + \rho \cos \phi z^2 \hat{e}_3 \end{aligned}$$

Problem 2.11.6. Express the vector field $\mathbf{A} = xi + 2yj + yzk$ in spherical polar coordinates

Solution: $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

$$\vec{R} = r \sin \theta \cos \phi \mathbf{i} + r \sin \theta \sin \phi \mathbf{j} + r \cos \theta \mathbf{k}$$

$$\hat{e}_1 = \frac{\frac{\partial \vec{R}}{\partial r}}{\left| \frac{\partial \vec{R}}{\partial r} \right|} = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k}$$

$$\hat{e}_2 = \frac{\frac{\partial \vec{R}}{\partial \theta}}{\left| \frac{\partial \vec{R}}{\partial \theta} \right|} = \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k}$$

$$\hat{e}_3 = \frac{\frac{\partial \vec{R}}{\partial \phi}}{\left| \frac{\partial \vec{R}}{\partial \phi} \right|} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}$$

Let the expression for $\vec{A} = x\mathbf{i} + 2y\mathbf{j} + yz\mathbf{k}$ in cylindrical coordinates be

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

$$A_1 = \vec{A} \cdot \hat{e}_1$$

$$= (x\mathbf{i} + 2y\mathbf{j} + yz\mathbf{k}) \cdot (\sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k})$$

$$= x \sin \theta \cos \phi + 2y \sin \theta \sin \phi + yz \cos \theta$$

$$= (r \sin \theta \cos \phi) \sin \theta \cos \phi + 2(r \sin \theta \sin \phi) \sin \theta \sin \phi + ((r \sin \theta \sin \phi) \cos \theta)$$

$$= r \sin^2 \theta \cos^2 \phi + 2r \sin^2 \theta \sin^2 \phi + r^2 \sin \theta \sin \phi \cos^2 \theta A_2$$

$$= (x\mathbf{i} + 2y\mathbf{j} + yz\mathbf{k}) \cdot (\cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k})$$

$$= x \cos \theta \cos \phi + 2y \cos \theta \sin \phi - yz \sin \theta$$

Then

$$= (r \sin \theta \cos \phi) \cos \theta \cos \phi + 2(r \sin \theta \sin \phi) \cos \theta \sin \phi - ((r \sin \theta \sin \phi) \cos \theta)$$

$$= r \sin \theta \cos \theta \cos^2 \phi + 2r \sin \theta \cos \theta \sin^2 \phi - r^2 \sin^2 \theta \sin \phi \cos \theta$$

$$A_3 = \vec{A} \cdot \hat{e}_3$$

$$= (x\mathbf{i} + 2y\mathbf{j} + yz\mathbf{k}) \cdot (-\sin \phi \mathbf{i} + \cos \phi \mathbf{j} + 0\mathbf{k})$$

$$= -x \sin \phi + 2y \cos \phi + 0$$

$$= -(r \sin \theta \cos \phi) \sin \phi + 2(r \sin \theta \sin \phi) \cos \phi$$

$$= -r \sin \theta \cos \phi \sin \phi + 2r \sin \theta \sin \phi \cos \phi$$

$$= r \sin \theta \sin \phi \cos \phi$$

Thus the expression for \vec{A} in cylindrical coordinates is,

$$\begin{aligned}\therefore \vec{A} &= A_1 e_1 + A_2 e_2 + A_3 e_3 \\ &= [(r \sin^2 \theta \cos^2 \phi + 2r \sin^2 \theta \sin^2 \phi + r^2 \sin \theta \sin \phi \cos^2 \theta)] e_1 \\ &+ [r \sin \theta \cos \theta \cos^2 \phi + 2r \sin \theta \cos \theta \sin^2 \phi - r^2 \sin^2 \theta \sin \phi \cos \theta] e_2 \\ &+ r \sin \theta \sin \phi \cos \phi e_3\end{aligned}$$

2.12 Question Bank : Module 2

The Gradient, Divergence and Curl :

- Find $grad\phi$ where $\phi = x^3 + y^3 + z^3 - 3xyz$ at the point $(1, -1, 2)$ (VTU Model 2023) Ans: $-12i - 9j - 16k$
- If $\vec{F} = \nabla(xy^3z^2)$, find $div\vec{F}$ and $curl\vec{F}$ at the point $(1, -1, 1)$ (VTU June 2023, Model 2023) Ans: $-8, 0$
- Find $grad\phi$ where $\phi = 3x^2y - y^3z^2$ at the point $(1, -2, -1)$ (VTU Jan 2017)
- If $\vec{F} = \nabla(xy^3z^2)$ find $divF$ and $curlF$ at $(1,-1,1)$ (VTU Model 2018)
- Find $divF$ and $curlF$ where $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$.
(VTU Nov 2003, Model 2023, Model 2022, Jan 2020, June 2019, July 2017, Jul 2015, Jul 2013, Jan 2008, Aug 2002) Ans: $6(x + y + z), 0$
- If $F = (x + y + 1)i + j - (x + y)k$, then show that $\vec{F} \cdot curl\vec{F} = 0$ (VTU Jan 2014, July 2013)
- If $\Phi = 2x^3y^2z^4$, find $div(grad\Phi)$ at $(1,1,1)$ (VTU Jan 2018) Ans: 40
- If $\vec{V} = \vec{w} \times \vec{r}$, P.T. $w = \frac{1}{2}(\nabla \times \vec{V})$ or $curl\vec{V} = 2\vec{w}$ where \vec{w} is a constant vector.
(VTU Jan 2015)
- If $\vec{r} = xi + yj + zk$, and $|\vec{r}| = r$, find $graddiv\left(\frac{\vec{r}}{r}\right)$ (VTU Jan 2015)

10. If $\phi = x^2 + y^2 + z^2$, and $\vec{F} = x^2i + y^2j + z^2k$, then find $grad\phi$, $div\vec{F}$ and $curl\vec{F}$.

(VTU July 2014) **Ans:** $2xi + 2yj + 2zk, 2x + 2y + 2z, \vec{0}$.

11. If $\vec{F} = 2xy^3z^4i + 3x^2y^2z^4j + 4x^2y^3z^3k$, find (i) $\nabla \cdot \vec{F}$ (ii) $\nabla \times \vec{F}$ (VTU Jan 2014)

12. $div\vec{F}$ and $curlF$ at the point (1,2,3) if $\vec{F} = grad(x^3y + y^3z + z^3x - x^2y^2z^2)$ (VTU 2007) **Ans :** -32, 0

13. Find the divergence and curl of the vector $\vec{V} = xyz i + 3x^2y j + (xz^2 - y^2z)k$ at the point (2,-1,1)

(VTU Jan 2017) **Ans :** 14, $2i - 3j - 14k$

14. If $\vec{F} = x^2y i + y^2z j + z^2xk$, find $Curl(Curl\vec{F})$ (VTU Model 2022)

15. If $F = xy^2i + 2x^2yzj + 3yz^2k$, then find $div(curlF)$

Ans : 0

16. If $F = (3x^2y - z)i + (xz^3 + y^4)j - 2x^3z^2k$, then find $grad(divF)$ at (2, -1, 0)

Ans : $-6i + 24j - 32k$

17. If $\vec{F} = 3xyz^2i - 4x^2yj - xy^2zk$, then find $\nabla(\nabla \cdot F)$ at (-1, 2, 1)

Ans : $-16i + 7j + 12k$

18. If $A = x^2yi - 2xzj + 2yzk$, find $curl(curlA)$.

Ans : $2(x + 2)j$

19. If $A = x^2yi + xzj + 2yzk$, find $curl(curlA)$ at (1,1,1)

Ans : $4j$

20. If $\phi = x^2 + y^2 + z^2$, and $\vec{F} = x^2i + y^2j + z^2k$, then find $grad\phi$, $div\vec{F}$ and $curl\vec{F}$. (VTU July 2014)

Ans: $2xi + 2yj + 2zk, 2x + 2y + 2z, \vec{0}$.

21. Find $\text{div}\vec{F}$ and $\text{curl}F$ at the point (1,2,3) if $\vec{F} = x^2yzi + xy^2zj + xyz^2k$
Ans : $12, 5i - 16j + 9k$

22. $\text{div}\vec{F}$ and $\text{curl}F$ at the point (1,2,3) if $\vec{F} = 3x^2i + 5xy^2j + 5xyz^3k$
Ans : $278, 5(27i - 54j + 8k)$

23. Find $\text{curl}(\text{grad } f)$ given $f(x, y, z) = x^2 + y^2 - z$
Ans : 0

24. Find $\text{curl}(\text{curl}\vec{A})$, given $\vec{A} = x^2yi + y^2zj + z^2yk$ (VTU 2003)
Ans : $2(x + z)j + 2yk$

25. Find $\text{div}F$ and $\text{curl}F$ where $\vec{F} = \nabla(xy^3z^2)$ (VTU Model 2018)

Angle between two surfaces :

1. Find the angle between two surfaces $x^2 + y^2 + z^2 = 9$ & $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$

(VTU Nov 2023, Model 2022, June 2019, July 2017, Dec 2010, Jul 2009) Ans: $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

2. Find the angle between two surfaces $x^2 + y^2 - z^2 = 4$ & $z = x^2 + y^2 - 13$ at the point $(2, 1, 2)$ (VTU Model 2023) Ans: $\cos^{-1}\left(\frac{-4}{\sqrt{21}}\right)$

3. Show that two surfaces $xz + y + z^2 = 9$ and $z = 4 - 4xy$ at $(1, -1, 2)$ are orthogonal. (VTU June 2023)

4. Find the angle between two surfaces $x^2 + y^2 + z^2 = 4$ & $z = x^2 + y^2 - 13$ at the point $(2, 1, 2)$

(VTU Model 2018)

5. Find the constants a & b so that the surface $ax^2 - byz = (a + 2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$. (VTU Model 2018)

6. Find the angle between two surfaces $x \log z = y^2 - 1$, $x^2 y = 2 - z$ at the point $(1,1,1)$
Ans : $\cos^{-1} \left(\frac{-1}{\sqrt{30}} \right)$
7. Find the angle between the directions of the normal to the surface $x^2 y z = 1$ at the points $(-1, 1, 1)$ and $(1, -1, -1)$.
Ans : $\theta = \pi$
8. Show that the surfaces $5x^2 - 2yz - 9x = 0$ and $4x^2 y + z^3 - 4 = 0$ are orthogonal at the point $(1, -1, 2)$.
9. Find the angle between two surfaces $x \log z = y^2 - 1$, $x^2 y = 2 - z$ at the point $(1,1,1)$
Ans : $\cos^{-1} \left(\frac{-1}{\sqrt{30}} \right)$
10. Find the constants a & b so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$
Ans : $a = -6, b = -10$

Directional Derivative :

1. Find the directional derivative of $4xz^3 - 3x^2 y^2 z$ at $(2, -1, 2)$ along $2i - 3j + 6k$.
 (VTU Jan 2020, Model 2018, Jan 2015) **Ans :** $\frac{376}{7}$
2. Find the directional derivative of $xy^3 + yz^3$ at $(2, -1, 1)$ in the direction of the vector $i + 2j + 2k$
 (VTU July 2017)
3. Find the directional derivative of $\Phi = x^2 y z + 4x z^2$ at $(1, -2, -1)$ in the direction of the vector $2i - j - 2k$ (VTU Model 2023) **Ans:** 7
4. Find the directional derivative of $f = xy^2 + yz^3$ at the point $(1, 2, -1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at the point $(-1, 2, 1)$.
 (VTU Feb 2004)
5. Find the directional derivative of $x \log z - y^2 = -4$ at the point $(-1, 2, 1)$ in the direction of the vector $2i - j - 2k$.
 (VTU June 2023)

6. In which direction the directional derivative of x^2yz^3 is maximum at $(2, 1, -1)$ and find the magnitude of this maximum. (VTU Jan 2009, Dec 2008) **Ans :**
 $-4i - 4j + 12k, 4\sqrt{11}$
7. Find the directional derivative of $\Phi = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of the vector $2i - j - 2k$ (VTU 2007) **Ans :** $\frac{37}{3}$
8. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2i - j - 2k$ (VTU Nov 2023, Model 2022)
9. Find the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of $i + 2j + 2k$ **Ans :** $\frac{-11}{3}$
10. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.
Ans : $96(i + 3j - 3k), 96\sqrt{19}$
11. Find the directional derivative of $\nabla \cdot \nabla \Phi$ at the point $(1, -2, 1)$ in the direction of normal to the surface $x^2yz = 3x + z^2$, where $\Phi = 2x^2y^2z^4$ **Ans :** $\frac{1724}{21}$

Solenoidal and irrotational vector fields

- Find 'a' for which $\vec{F} = (x + 3y)i + (y - 2z)j + (x + az)k$ is solenoidal. (VTU Jan 2017)
- Prove that $\frac{xi+yj}{x^2+y^2}$ is both solenoidal & irrotational. (VTU Model 2023, Model 2018, Model 2014)
- Find the constants a, b, c such that the vector field $\vec{f} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational. (VTU July 2014)
- Find the value of the constant a such that the vector field $\vec{F} = (axy - z^3)i + (bx^2 + z)j + (bxz^3 + cy)k$ is irrotational.

5. Find the value of the constant a such that the vector field $\vec{F} = (axy + bz^3)i + (3x^2 - cz)j + (3xz^2 - y)k$ is a conservative force field. Hence find a scalar function Φ such that $\vec{F} = \nabla\Phi$ (VTU June 2019, Model 2018)
6. Find the constants a, b, c such that the vector field $\vec{f} = (x + y + az)i + (x + cy + 2z)k + (bx + 2y - z)j$ is irrotational. (VTU July 2017). Also find Φ such that $\vec{F} = \nabla\Phi$ (VTU Jul 2015)
7. Find the value of the constant a such that the vector field $\vec{F} = (axy - z^3)i + (a - 2)x^2j + (1 - a)xz^2k$ is irrotational. Hence find a scalar function Φ such that $\vec{F} = \nabla\Phi$ (VTU Jul 2014)
8. Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational. (VTU Nov 2023, Jan 2016)
9. Find the constants a and b such that $F = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational. Find Φ such that $F = \nabla\Phi$ (VTU Jan 2020, July 2014, Jan 2014, Jun 2012, Feb 2005)
10. Show that $\vec{F} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ is irrotational and hence find its scalar potential. (VTU July 2015, Jan 2013)
11. If $\vec{u} = x^2i + y^2j + z^2k$ and $\vec{v} = yzi + zxj + xyk$ Show that $\vec{u} \times \vec{v}$ is a solenoidal vector. (VTU July 2017)
12. A vector field is given by $\vec{A} = (x^2 + xy^2)i + (y^2 + x^2y)j$, show that the field is irrotational and find the scalar potential. (VTU July 2017)
13. Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational. (VTU Model 2022)
14. Define an irrotational vector. Find the constants a, b and c such that $\vec{F} = (axy - z^3)i + (bx^2 - yz)j + (3xz^2 - y)k$ is irrotational. (VTU Model 2022)

15. Find a for which $\vec{F} = (x + 3y)i + (y - 2z)j + (x + az)k$ is solenoidal.
(VTU Jan 2017)
16. Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$
is both solenoidal and irrotational. (VTU Jan 2016)
17. Show that $\vec{F} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational.
Find Φ such that $\vec{F} = \nabla\Phi$ (VTU Aug 2003) Ans $\Phi = 3x^2y + xz^3 - yz$
18. Show that $\vec{F} = (2x + yz)i + (4y + zx)j - (6z - xy)k$ is both solenoidal
and irrotational and also find its scalar potential Φ Ans :
 $\Phi = x^2 + 2y^2 - 3z^2 + xyz$
19. Determine the constants a and b such that the curl of vector
 $\vec{A} = (2xy + 3yz)i + (x^2 + axz - 4z^2)j - (3xy + byz)k$ is zero.
Ans : $a = 3, b = 8$

Curvilinear coordinates

1. Prove that the cylindrical coordinate system is orthogonal. (VTU June 2023)
2. Show that the spherical coordinate system is orthogonal. (VTU Nov 2023, Model 2023)
3. Express the vector field $\vec{A} = zi - 2xj + yk$ in cylindrical polar coordinates.
(VTU Model 2023)
4. Express the vector field $2yi - zj + 3xk$ in cylindrical polar coordinate system.
5. Express the vector field $\mathbf{A} = xi + 2yj + yzk$ in spherical polar coordinates
6. Express the vector field $\mathbf{A} = yzi - yj + xz^2k$ in cylindrical polar coordinates.



Mathematics II for CSE Stream - Lecture Notes

Subject Code : BMATS201

Module 3- Vector Spaces and Linear Transformation

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Mathematics II for CSE Stream (BMATS201)

Module 3 - Vector Spaces and Linear

Transformations

3.1 Vector Spaces

Let F be a set. A vector space over F , is a non-empty set V together with two operations (called addition and scalar multiplication) such that for each $u, v \in V$ there is a unique element $u + v \in V$ and for each $\alpha \in F$ and $u \in V$ there is a unique element $\alpha u \in V$, and it satisfies the following conditions:

I. (i) $u + (v + w) = (u + v) + w$, for all $u, v, w \in V$ (Associativity)

(ii) $u + v = v + u$, for all $u, v \in V$ (commutativity)

(iii) There exists an element $0 \in V$ such that $u + 0 = u = 0 + u$, for all $u \in V$ (Existence of identity element)

(iv) For each $u \in V$, there exists a unique element $-u \in V$ such that $u + (-u) = 0 = (-u) + u$ (Existence of additive inverse)

II. There is an external composition in V over F called scalar multiplication. i.e $\forall \alpha \in F$ and $u \in V \Rightarrow \alpha \cdot u \in V$ In other words V is closed with respect to scalar multiplication. III. The two compositions, i.e. vector addition and scalar multiplication satisfy the following postulates,

$\forall a, b \in F$ and $v, w \in V$

(i) $(a + b) \cdot v = a \cdot v + b \cdot v$

(ii) $a \cdot (b \cdot v) = (ab) \cdot v$

(iii) $a \cdot (v + w) = a \cdot v + a \cdot w$

(iv) $1 \cdot (v) = v$.

3.1.1 Vectors in R^n : –

The set of all ordered triples (a, b, c) of real numbers is called Euclidean 3-space and is denoted by R^3 & the set of all n -tuples of real numbers, denoted by R^n , is called Euclidean n -space.

Eg:- The ordered pair $(2, -3)$ belongs to R^2 ; it is a 2-tuple of dimension two.

The ordered triple $(7, 3, 6)$ belongs to R^3 ; it is a 3-tuple of dimension three.

3.1.2 Examples or Problems

1) Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Sol:- Let V denotes the set of all coplanar vectors and R be the field of real numbers.

∴ The elements of V are the ordered pairs (x, y) where $x, y \in R$.

$$V = \{(x, y) : x, y \in R\}$$

I) $(V, +)$ is an abelian group. i) Associativity:- We know that for all $u, v, w \in V$

$$(u + v) + w = u + (v + w).$$

ii) Commutativity:- We know that for all $u, v \in V$

$$u + v = v + u$$

iii) Existence of additive identity for every vector $u \in V$ then exists a zero vector $0 \in V$ such that $u + 0 = 0 + u = u$.

iv) Existence of additive inverse: For every vector $u \in V$ there exists a vector $-u \in V$ such that $u + (-u) = (-u) + u = 0$.

II) Scalar multiplication in V . i) For $u, v \in V$ and $\alpha \in R$ we have

$$\alpha(u + v) = \alpha u + \alpha v$$

ii) For $u \in V$ and $a, b \in R$, we have

$$(a + b)u = au + bu.$$

ii) For $u \in V$ and $a, b \in R$, we have

$$a(bu) = (ab)u$$

iv) For $u \in V$ and $1 \in R$, we have

$$1u = u.$$

The set V of coplanar vectors satisfies all the properties of vector addition and scalar multiplication.

$\therefore V$ is a vector space.

2) Prove that the set C of all complex numbers (the set of all ordered pairs of real numbers) is a vector space over the field R of all real numbers where vector addition is defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \text{ for all } (x_1, x_2), (y_1, y_2)$$

and scalar multiplication is defined by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2), \text{ for all } \alpha \in R.$$

Soln:- We observe that C is closed under vector addition and under scalar multiplication.

I. (i) Associativity: For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in C$, we have

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2)$

$$\begin{aligned} (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2) \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\ &= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) \end{aligned}$$

(ii) Commutativity: For all $(x_1, x_2), (y_1, y_2) \in C$, we have

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2) \end{aligned}$$

(iii) Existence of identity: For $(x_1, x_2) \in C$, there exists $(0, 0) \in C$ such that

$$\begin{aligned}(x_1 + x_2) + (0, 0) &= (x_1 + 0, x_2 + 0) \\ &= (x_1, x_2) \\ (0 + 0) + (x_1, x_2) &= (0 + x_1, 0 + x_2) = (x_1, x_2) \\ \therefore (x_1, x_2) + (0, 0) &= (x_1, x_2) = (0, 0) + (x_1, x_2)\end{aligned}$$

(iv) Existence of inverse: For any $(x_1, x_2) \in C$. there exists $(-x_1, -x_2) \in C$ such that

$$(x_1, x_2) + (-x_1, -x_2) = (0, 0) = (-x_1, -x_2) + (x_1, x_2).$$

Thus C is an abelian group with respect to vector addition.

II. Properties of Scalar multiplication in C . (i)

$$\begin{aligned}\alpha [(x_1, x_2) + (y_1, y_2)] &= \alpha (x_1 + y_1, x_2 + y_2) \\ &= (\alpha (x_1 + y_1), \alpha (x_2 + y_2)) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \\ &= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2) \\ &= \alpha (x_1, x_2) + \alpha (y_1, y_2)\end{aligned}$$

Thus $\alpha [(x_1, x_2) + (y_1, y_2)] = \alpha (x_1, x_2) + \alpha (y_1, y_2)$ for all $(x_1, x_2), (y_1, y_2) \in C, \alpha \in R$.

(ii)

$$\begin{aligned}(a + b) (x_1, x_2) &= ((a + b)x_1, (a + b)x_2) \\ &= (ax_1 + bx_1, ax_2 + bx_2) \\ &= (ax_1, ax_2) + (bx_1, bx_2) \\ &= a (x_1, x_2) + b (x_1, x_2)\end{aligned}$$

Thus $(a + b) (x_1, x_2) = a (x_1, x_2) + b (x_1, x_2)$ for all

$$(x_1, x_2), (y_1, y_2) \in C \text{ \& } a, b \in R.$$

(iii)

$$\begin{aligned}a (b (x_1, x_2)) &= a (bx_1, bx_2) = (abx_1, abx_2) \\ a (b (x_1, x_2)) &= (ab) (x_1, x_2)\end{aligned}$$

(iv) 1. $(x_1, x_2) = (x_1, x_2)$ for all $(x_1, x_2) \in C$.

Thus the set C satisfies all the properties of vector space.

Hence C is a vector space over R .

3) Show that the set V of all real valued continuous functions of x defined on interval $[0, 1]$ is a vector space over the field R of real numbers with respect to vector addition and scalar multiplication defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \text{ for all } f_1, f_2 \in V$$

$$(\alpha f_1)(x) = \alpha f_1(x), \text{ for all } \alpha \in R, f_1 \in V.$$

Soln: - I. (i) Associativity: Let $f_1, f_2, f_3 \in V$ be arbitrary

$$\begin{aligned} [(f_1 + f_2) + f_3](x) &= (f_1 + f_2)(x) + f_3(x) \\ &= [f_1(x) + f_2(x)] + f_3(x) \\ &= f_1(x) + [f_2(x) + f_3(x)] \\ &= f_1(x) + (f_2 + f_3)(x) \\ &= [f_1 + (f_2 + f_3)](x) \end{aligned}$$

$$(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3).$$

(ii) Commutativity: Let $f_1, f_2 \in V$ be arbitrary.

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ &= f_2(x) + f_1(x) \\ &= (f_2 + f_1)(x), \text{ for all } x. \end{aligned}$$

$$f_1 + f_2 = f_2 + f_1.$$

(iii) Existence of Identity: Define a function 0 such that $0(x) = 0$ (real number), for all $x \in [0, 1]$.

Also, 0 is a continuous function and belongs to V

$$(0 + f_1)(x) = 0(x) + f_1(x) = 0 + f_1(x) = f_1(x)$$

$$(f_1 + 0)(x) = f_1(x) + 0(x) = f_1(x)$$

$$0 + f_1 = f_1 + 0 = f_1$$

Thus, the function 0 defined above is an additive identity element of V .

(iv) Existence of Inverse: For a function f , the function $-f$ defined by $(-f)(x) = -f(x)$, is called additive inverse, as $[f + (-f)](x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x)$

Similarly, $[-f + f](x) = 0(x)$. Thus the set V is an abelian group under addition.

II. Properties of Scalar Multiplication in V .(i) For $f_1, f_2 \in Y$ and $\alpha \in R$, we have

$$\begin{aligned}
 [\alpha (f_1 + f_2)] x &= \alpha [(f_1 + f_2) (x)] \\
 &= \alpha [f_1(x) + f_2(x)] \\
 &= \alpha f_1(x) + \alpha f_2(x) \\
 &= (\alpha f_1) x + (\alpha f_2) x \\
 &= (\alpha f_1 + \alpha f_2) x
 \end{aligned}$$

$$\alpha (f_1 + f_2) = \alpha f_1 + \alpha f_2$$

(ii) For $f_1 \in V$ and $a, b \in R$.

$$\begin{aligned}
 [(a + b) f_1] (x) &= (a + b) f_1(x) \\
 &= a f_1(x) + b f_1(x) \\
 &= (a f_1) x + (b f_1) x \\
 &= (a f_1 + b f_1) (x)
 \end{aligned}$$

$$(a + b) f_1 = a f_1 + b f_1$$

(iii) For $f_1 \in V$ and $a, b \in R$,

$$\begin{aligned}
 [a (b f_1)] x &= a [(b f_1) x] \\
 &= a [b f_1(x)] \\
 &= (ab) f_1(x) \\
 &= [(ab) f_1] x
 \end{aligned}$$

$$a (b f_1) = (ab) f_1$$

(iv) For $f_1 \in V$ and $1 \in R$, we have

$$(1, f_1) x = 1 \cdot f_1(x) = f_1(x)$$

$$1 f_1 = f_1.$$

Thus V satisfies all the properties of a vector space and hence V is a vector space.

4) Show that the set of all matrices of the type $m \times n$ where m and n are fixed positive integers is a vector space over R with respect to matrix addition and multiplication of matrix by a scalar. (α , a real number).

Soln: Let M denotes the set of all matrices of type $m \times n$.

I. $(M, +)$ is an abelian group.

(i) Associativity: Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, $C = [c_{ij}]_{m \times n}$ be three matrices belonging to set M .

$$\begin{aligned}(A + B) + C &= [[a_{ij}] + [b_{ij}]] + [C_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + (c_{ij})] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \\ &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij}] + ([b_{ij}] + [C_{ij}]) \\ &= A + (B + C).\end{aligned}$$

(ii) Commutativity: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices belonging to set M .

$$\begin{aligned}A + B &= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \\ &= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} \\ &= B + A.\end{aligned}$$

(iii) Existence of Identity: Let $A = [a_{ij}]_{m \times n} \in M$ be arbitrary. We know that a Zero matrix (null matrix) of type $m \times n$ also belongs to the set M and is denoted by 0 . Now, $A + 0 = [a_{ij}] + [0] = [a_{ij} + 0] = [a_{ij}] = A$.

Similarly, $0 + A = A$.

(iv) Existence of Inverse: If $A = [a_{ij}]_{m \times n}$ belongs to the set M , then $-A = [-a_{ij}]_{m \times n}$ also belongs to the set M . Now,

$$\begin{aligned}A + (-A) &= [a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}] \\ &= [0] = 0\end{aligned}$$

Similarly, $(-A) + (A) = 0$ Thus the set M is an abelian group under addition.

II. Properties of Scalar multiplication in M . (i) Let $A = [a_{ij}]_{m \times n}$ and $B : [b_{ij}]_{m \times n} \in M$ and

$\alpha \in R$, then

$$\begin{aligned}
 \alpha(A + B) &= \alpha([a_{ij}] + [b_{ij}]) \\
 &= \alpha[a_{ij} + b_{ij}] \\
 &= [\alpha[a_{ij} + b_{ij}]] \\
 &= [\alpha \cdot a_{ij} + \alpha \cdot b_{ij}] \quad [\because \text{Multiplication is distributive in } R] \\
 &= [\alpha \cdot a_{ij}] + [\alpha \cdot b_{ij}] \quad \alpha A + \alpha B
 \end{aligned}$$

(ii) Let $A = [a_{ij}]_{m \times n} \in M$ and $a, b \in R$, then

$$\begin{aligned}
 (a + b)A &= (a + b)[a_{ij}] \\
 &= [(a + b)a_{ij}] \\
 &= [a \cdot a_{ij} + b \cdot a_{ij}] \\
 &= a[a_{ij}] + b[a_{ij}] \\
 &= aA + bA.
 \end{aligned}$$

(iii) Let $A = [a_{ij}]_{m \times n} \in M$ and $a, b \in R$ then

$$\begin{aligned}
 a(bA) &= a(b[a_{ij}]) \\
 &= a([ba_{ij}]) \\
 &= [(ab)a_{ij}] \\
 &= (ab)[a_{ij}] \\
 &= (ab)A \\
 a(bA) &= (ab)A.
 \end{aligned}$$

(iv) Let $A = [a_{ij}]_{m \times n} \in M$, then

$$1. A = 1 \cdot [a_{ij}] = [1, a_{ij}] = [a_{ij}] = A.$$

Thus M satisfies all the properties of vector space and hence M is a vector space over R .

3.2 Subspaces

A non-empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V .

Eg:- Let

$$W_1 = \{(a, 0, 0) : a \in \mathbb{R}\}$$

$$W_2 = \{(a, b, 0) : a, b \in \mathbb{R}\}$$

Here W_1 is a subspace of W_2 . Also W_1 and W_2 are subspace of \mathbb{R}^3 .

3.2.1 Necessary and Sufficient conditions for a subspace

Theorem 1: W is a subspace of $V(F)$ iff

i) W is non empty. ii) W is closed under vector addition.

$$\text{i.e. } \forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W.$$

iii) W is closed under scalar multiplication. i.e. $\forall a \in F$ and $w \in W \Rightarrow a \cdot w \in W$.

Theorem 2 : W is a subspace of $V(F)$ iff

i) W is non empty.

ii) $\forall a, b \in F$ and $v, w \in W \Rightarrow a \cdot v + b \cdot w \in W$.

3.2.2 Problems

1) Show that W is a subspace of $V(\mathbb{R})$ where $W = \{f : f(9) = 0\}$.

Soln:- Since $0 \in W$ as $0(9) = 0$.

So W is non empty set.

Let $f, g \in W$.

i.e. $f(9) = 0$ and $g(9) = 0$

then $\forall a, b \in \mathbb{R}$

$$(a \cdot f + b \cdot g)(9) = a \cdot f(9) + b \cdot g(9) = a \cdot 0 + b \cdot 0 = 0$$

Hence $a \cdot f + b \cdot g \in W$. By theorem (2), W is a subspace of $V(\mathbb{R})$.

2) Show that W is a subspace of $V(\mathbb{R})$ where $W = \{f : f(2) = f(1)\}$

Soln:- $0 \in W$ since $0(2) = 0 = 0(1)$. Hence W is a non empty set.

Let $f, g \in W$ then $f(2) = f(1)$ and $g(2) = g(1)$

then $\forall a, b \in R$.

$$\begin{aligned}(a \cdot f + b \cdot g)(2) &= a \cdot f(2) + b \cdot g(2) = a \cdot f(1) + b \cdot g(1) \\ &= (a \cdot f + b \cdot g)(1)\end{aligned}$$

Hence $a \cdot f + b \cdot g \in W$.

So by theorem 2, W is a subspace of $V(R)$.

3) Let $V = R^3$ be the Euclidean 3 space. Let $W = \{(x, y, z) : ax + by + cz = 0; x, y, z \in R\}$, a, b, c being real numbers. Show that W is a subspace of V .

(OR)

Show that any plane passing through the origin is a subspace of R^3 .

Soln:- Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any two elements of W where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$.

Then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$.

For $\alpha \in R$,

$$\begin{aligned}\alpha u + v &= \alpha(x_1, y_1, z_1) + (x_2, y_2, z_2) \\ \alpha u + v &= (\alpha x_1, \alpha y_1, \alpha z_1) + (x_2, y_2, z_2) \\ &= (\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2) \rightarrow (1)\end{aligned}$$

where $\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2 \in R$.

Now

$$\begin{aligned}a(\alpha x_1 + x_2) + b(\alpha y_1 + y_2) + c(\alpha z_1 + z_2) &= \\ \alpha(ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) &= \\ \alpha(0) + 0 = 0 &\rightarrow (2)\end{aligned}$$

From (1) and (2), we have

$\alpha u + v \in W$.

Thus, W is a subspace of V .

4). Let V be the vector space of all square matrices over R . Determine which of the following are subspaces of V .

$$(i) W = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} ; x, y, z \in R \right\}$$

$$(ii) W = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in R \right\}$$

(iii) $W = \{A : A \in V \text{ and } A \text{ is singular} \}$

(iv) $W : \{A : A \in V, A^2 = A\}$

Soln :- (i) Let $A = \begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$ be any two elements of W .

If $a, b \in R$ then

$$\begin{aligned} aA + bB &= a \begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix} + b \begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 & ay_1 \\ az_1 & 0 \end{bmatrix} + \begin{bmatrix} bx_2 & by_2 \\ bz_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ az_1 + bz_2 & 0 \end{bmatrix} \end{aligned}$$

which is a matrix of the type $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2 \in R.$$

$$\therefore aA + bB \in W.$$

Thus, W is a sub-space of V .

ii) Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$ be any two elements of W .

If $a, b \in R$, then

$$\begin{aligned} aA + bB &= a \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + b \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + bx_2 & 0 \\ 0 & ay_1 + by_2 \end{bmatrix} \end{aligned}$$

which is a matrix of the type $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2 \in R.$$

$$aA + bB \in W.$$

(iii) Here W is the set of singular matrices.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two square matrices.

Since $|A| = 0$ and $|B| = 0$, therefore $A, B \in W$.

If $a, b \in \mathbb{R}$ are non zero, then

$$\begin{aligned} aA + bB &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \end{aligned}$$

$$\text{Also, } |aA_1 + bA_2| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \neq 0, \text{ as none of } a \text{ and } b \text{ is zero.}$$

$$\therefore aA_1 + bA_2 \notin W.$$

Thus, W is not a subspace of V .

$$(iv) \text{ Let } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ so that } A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A \therefore A \in W.$$

Now

$$\begin{aligned} A + A &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ (A + A)^2 &= \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \\ &= A + A \end{aligned}$$

$\therefore (A + A)$ is not a member of the set W .

$\Rightarrow W$ not closed under addition.

Hence W is not a sub-space of V .

5) Let V be the vector space of all real valued continuous functions over \mathbb{R} . Show that the set

W of solutions of differential equations $5 \frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 2y = 0$ is a subspace of V .

Soln: - Let $y_1, y_2 \in W$.

Then y_1, y_2 are solutions of differential equations.

$$5 \frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 2y = 0$$

$$\text{i.e. } 5 \frac{d^2 y_1}{dx^2} - 7 \frac{dy_1}{dx} + 2y_1 = 0; 5 \frac{d^2 y_2}{dx^2} - 7 \frac{dy_2}{dx} + 2y_2 = 0$$

Let $a, b \in \mathbb{R}$ be arbitrary.

$$\begin{aligned}
 & 5 \frac{d^2}{dx^2} (ay_1 + by_2) - 7 \frac{d}{dx} (ay_1 + by_2) + 2 (ay_1 + by_2) \\
 &= 5 \frac{d^2}{dx^2} (ay_1) - 7 \frac{d}{dx} (ay_1) + 2 (ay_1) + 5 \frac{d^2}{dx^2} (by_2) - 7 \frac{d}{dx} (by_2) + 2 (by_2) \\
 &= a \left[5 \frac{d^2 y_1}{dx^2} - 7 \frac{dy_1}{dx} + 2y_1 \right] + b \left[5 \frac{d^2 y_2}{dx^2} - 7 \frac{dy_2}{dx} + 2y_2 \right] \\
 &= a \cdot 0 + b \cdot 0 = 0.
 \end{aligned}$$

Thus $ay_1 + by_2$ is a solution of (1) & so $ay_1 + by_2 \in W$. $\therefore W$ is a subspace of V .

Linear Combination

Let V be a vector space over F and let $v_1, v_2 \dots v_n \in V$. Any vector of the form $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n$ in V , where $a_i \in F$ is called a linear combination of $v_1, v_2 \dots v_n$.

Linear Dependence

Let V be a vector space over F . The vectors $v_1, v_2 \dots v_n$ are said to be linearly dependent over F , if \exists scalars $a_1, a_2 \dots a_n \in F$ not all zero but linear combination is zero.

$$\text{ie } a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0.$$

but all $a_i \neq 0$, where $i \in \mathbb{N}$.

Linear Independence

Let V be a vector space over F . The vectors $v_1, v_2 \dots v_n$ are said to be linearly independent over F , if \exists scalars $a_1, a_2 \dots a_n \in F$ such that

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0 \Rightarrow \text{all } a_i = 0 \text{ where } i \in \mathbb{N}$$

Linear Span

Let S be a subset of the vector space V over F . The set of all linear combinations of vectors in S is called a linear span of S and is denoted by $L(S)$.

Basis or Base of Vector space V

Let V be a vector space over F . The set of vectors $\{v_1, v_2, \dots, v_n\}$ is called a basis of V , if

- (i) v_1, v_2, \dots, v_n are linearly independent,
- (ii) v_1, v_2, \dots, v_n span V . i.e. each vector of V can be Uniquely expressed as linear combination of v_1, v_2, \dots, v_n .

Dimension of a vector space V : –

Number of elements in a basis of vector space V is called the dimension of V and is denoted by $\dim V$.

If V contains a basis with n elements then the $\dim V = n$.

Note: 1) The vector space $\{0\}$ is defined to have $\dim 0$, since empty set ϕ is independent and generates $\{0\}$.

$\therefore \dim\{0\} =$ Number of elements in $\phi = 0$ [Since no element is in ϕ]

2) When a vector space is not of finite dimension, it is said to be of infinite dimension.

Problems

1) Express the vector $V = (1, -2, 5)$ as a linear combination of the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, $v_3 = (2, -1, 1)$ in the vector space $R^3(R)$.

Soln:- Let $v = a_1v_1 + a_2v_2 + a_3v_3$; $a_1, a_2, a_3 \in R$

$$(1, -2, 5) = a_1(1, 1, 1) + a_2(1, 2, 3) + a_3(2, -1, 1)$$

$$(1, -2, 5) = (a_1 + a_2 + 2a_3, a_1 + 2a_2 - a_3, a_1 + 3a_2 + a_3)$$

Equating the corresponding elements, we get

$$a_1 + a_2 + 2a_3 = 1; a_1 + 2a_2 - a_3 = -2; a_1 + 3a_2 + a_3 = 5$$

Solving above equations, we get

$$a_1 = -6, \quad a_2 = 3, \quad a_3 = 2$$

Hence $(1, -2, 5) = -6(1, 1, 1) + 3(1, 2, 3) + 2(2, -1, 1)$.

2) Write the vector $v = (1, 3, 9)$ as a linear combination of the vectors $u_1 = (2, 1, 3)$, $u_2 = (1, -1, 1)$, $u_3 = (3, 1, 5)$ (Another method of solving)

Soln:- Let $y = xu_1 + yu_2 + zu_3$

$$(1, 3, 9) = x(2, 1, 3) + y(1, -1, 1) + z(3, 1, 5)$$

$$\Rightarrow 2x + y + 3z = 1; \quad x - y + z = 3; \quad 3y + y + 5z = 9$$

Consider $Ax = B$.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

Augmented matrix.

$$[A : B] = \begin{bmatrix} 2 & 1 & 3 & 1 & 1 \\ 1 & -1 & 1 & : & 3 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 1 & 3 & : & 1 \\ 3 & 1 & 5 & 9 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 4 & 2 & : & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & -5 \\ 0 & 1 & 1 & 5 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & +5 \\ 0 & 3 & 1 & : & -5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & : & -20 \end{bmatrix}$$

$\therefore \text{Rank}[A] = \text{Rank}[A : B] = 3 = \text{Number of unknowns.}$

∴ The system of linear equations is consistent and possesses a unique solution.

$$x + y + z = 3; \quad y + z = 5; \quad -2z = -20$$

$$\therefore z = 10, \quad y = -5, \quad x = -12$$

$$\therefore (1, 3, 9) = -12(2, 1, 3) - 5(1, -1, 1) + 10(3, 1, 5).$$

3) Write the vector $v = (4, 2, 1)$ as a linear combination of the vectors $u_1 = (1, -3, 1), u_2 = (0, 1, 2), u_3 = (5, 1, 37)$.

Soln:- Let $v = xu_1 + yu_2 + zu_3$

$$(4, 2, 1) = x(1, -3, 1) + y(0, 1, 2) + z(5, 1, 3)$$

$$\Rightarrow x + 0y + 5z = 4$$

$$-3x + y + z = 2$$

$$x + 2y + 37z = 1$$

$$Ax = B \text{ where } A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$[A : B] = \begin{bmatrix} 1 & 0 & 5 & : & 4 \\ 0 & 1 & 16 & : & 14 \\ 0 & 2 & 32 & : & -3 \end{bmatrix}$$

$$[A : B] = \begin{bmatrix} 1 & 0 & 5 & : & 4 \\ 0 & 1 & 16 & : & 14 \\ 0 & 0 & 0 & : & -31 \end{bmatrix}$$

$$\text{Rank}[A] = 2 \neq \text{Rank}[A : B] = 3.$$

Since $\text{Rank}[A] \neq \text{Rank}[A : B]$.

Hence the system of linear equations is inconsistent.

i.e solution does not exist.

∴ v cannot be expressed as linear combination of the vectors u_1, u_2, u_3 .

4) For what value of k (if any) the vector $v = (1, -2, k)$ can be expressed as a linear combination of vectors $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$ in $R^3(R)$.

Soln: - Since vector $v = (1, -2, k)$ is a linear combination of $v_1 = (3, 0, -2)$ and $v_2 =$

$(2, -1, -5)$; therefore there exist scalars a and b such that

$$v = av_1 + bv_2$$

$$(1, -2, k) = a(3, 0, -2) + b(2, -1, -5)$$

$$(1, -2, k) = (3a + 2b, -b, -2a - 5b)$$

Equating the corresponding elements,

$$3a + 2b = 1, \quad -b = -2, \quad -2a - 5b = k$$

$$3a + 4 = 1 \quad b = 2 \quad -2(-1) - 5(2) = k$$

$$a = -1 \quad 2 - 10 = k$$

$$\therefore k = -8.$$

5) Find a condition on a, b, c so that $w = (a, b, c)$ is a linear combination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$ in \mathbb{R}^3 so that $w \in \text{span}(u, v)$.

Soln:-

$$\text{Let } w = xu + yv$$

$$(a, b, c) = x(1, -3, 2) + y(2, -1, 1)$$

$$1 + 2y = a; \quad -3x - y = b; \quad 2x + y = c$$

Let $AX = B$.

$$\text{Consider } [A : B] = \left[\begin{array}{cc|c} 1 & 2 & a \\ -3 & -1 & b \\ 2 & 1 & c \end{array} \right]$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$R_2 \rightarrow \frac{R_2}{5}$$

$$[A : B] : \begin{bmatrix} 1 & 2 & : & a \\ 0 & 1 & : & \frac{b+3a}{5} \\ 0 & -3 & : & c - 2a \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$[A : B] = \begin{bmatrix} 1 & 2 & : & a \\ 0 & 1 & : & \frac{b+3a}{5} \\ 0 & 0 & : & \frac{5c-a+3b}{5} \end{bmatrix}$$

$$\text{Rank } [A] = 2$$

The system of linear equations will be consistent if $\text{rank}[A : B] = 2$

$$\text{So, } \frac{5c-a+3b}{5} = 0 \Rightarrow a - 3b - 5c = 0.$$

i.e. w is linear combination of u and v if

$$a - 3b - 5c = 0.$$

6) Express the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of

$$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Soln: Let $A = a_1B + a_2C + a_3D$; $a_1, a_2, a_3 \in R \rightarrow (1)$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$$

Equating the corresponding elements.

$$a_1 + a_2 + a_3 = 3; \quad a_1 + a_2 - a_3 = -1; \quad -a_2 = 1; \quad -a_1 = -2$$

Solving the above equations,

$$a_1 = 2, a_2 = -1, a_3 = 2.$$

$$\therefore (1) \Rightarrow$$

$$A = 2B - C - 2D.$$

(or)

$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is the required linear combination of A .

7) Express the matrix $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$, $B =$

$$\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

Soln:-

$$\text{Let } \begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = a_1 \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix};$$

$$a_1, a_2, a_3 \in \mathbb{R}.$$

$$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 0 + 0 + 2a_3 & -3a_1 + 3a_3 \\ 2a_1 + 2a_2 & a_2 + 5a_3 \end{bmatrix}$$

$$2 = 2a_3 \Rightarrow a_3 = 1$$

$$0 = -3a_1 + 3a_3$$

$$0 = -3a_1 + 3a_3 \rightarrow 2$$

$$0 = -3a_1 + 3a_3 \Rightarrow a_1 = 1$$

$$4 = 2a_1 + 2a_2 \rightarrow (3)$$

$$4 = 2a_1 + 2a_2 \Rightarrow a_2 = 1$$

$$-5 = a_2 + 5a_3.$$

Equating the corresponding elements

$$2 = 2a_3 \Rightarrow a_3 = 1$$

$$2 = 2a_3 \Rightarrow a_3 = 1$$

But $a_1 = 1, a_2 = 1, a_3 = 1$ do not satisfy eq (4) Thus equations (1). (2): (3) have no solution.

Hence, the given matrix cannot be expressed as a linear combination of A, B, C.

Note: * Two vectors v_1 and v_2 are linearly dependent if one of them is a multiple of the other.

Eg:- Determine whether or not the vectors v_1, v_2 are linearly dependent.

i) $v_1 = (1, 3, 9)$ $v_2 = (2, 4, 1)$. \rightarrow No vector is the multiple of the other hence v_1 & v_2 are not linearly dependent.

ii) $v_1 = (3, 4)$ $v_2 = (6, 8)$. $\rightarrow v_2 = 2v_1$ hence v_1 and v_2 are linearly dependent vectors.

8) Determine whether the vectors $v_1 = (1, 2, 3), v_2 = (3, 1, 7)$ and $v_3 = (2, 5, 8)$ are linearly dependent or linearly independent.

Soln: $xv_1 + yv_2 + zv_3 = 0$

$$x(1, 2, 3) + y(3, 1, 7) + z(2, 5, 8) = (0, 0, 0)$$

$$x + 3y + 2z = 0; 2x + y + 5z = 0; 3x + 7y + 8z = 0.$$

Let $Ax = 0$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$R_2 \rightarrow -R_2$$

$$R_3 \leftrightarrow -\frac{R_3}{2}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 5R_2}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Rank $[A] = 3 =$ Number of unknowns. \Rightarrow Homogeneous system of linear equations possesses unique solution. $x = 0, y = 0, z = 0$. So vectors v_1, v_2, v_3 are linearly independent.

9) Determine whether the vectors $v_1 = (1, 9, 3), v_2 = (2, 5, 4)$ and $v_3 = (0, 0, 0)$ are linearly dependent or linearly independent.

Soln:- Vectors v_1, v_2, v_3 are linearly dependent as linearly independent set of vectors can not contain zero vector.

10) Determine whether the vectors $v_1 = (1, 4, 9), v_2 = (3, 1, 4)$ and $v_3 = (9, 3, 12)$ are linearly dependent or linearly independent.

Soln:-

$$xv_1 + yv_2 + zv_3 = 0$$

$$x(1, 4, 9) + y(3, 1, 4) + z(9, 3, 12) = (0, 0, 0)$$

$$x + 3y + 9z = 0; \quad 4x + y + 3z = 0; \quad 9x + 4y + 12z = 0$$

Consider $Ax = 0$. Where

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 4R_1, R_3 \rightarrow R_3 - 9R_1}$$

$$\begin{aligned}
 &= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix} R_2 \rightarrow -\frac{R_2}{11}, R_3 \rightarrow -\frac{R_3}{23} \\
 &= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} R_3 \rightarrow R_3 - R_2 \\
 &= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

$\text{Rank}[A] = 2 < 3$ (Number of unknowns).

So one unknown is constant and hence not always zero.

So the vectors are linearly dependent.

[OR]

$$v_3 = 3v_2$$

vectors v_1, v_2, v_3 are linearly dependent.

11) Determine whether or not each of the following forms a basis, $x_1 = (2, 2, 1), x_2 = (1, 3, 7), x_3 = (1, 2, 2)$ in \mathbb{R}^3 .

Soln: - Three vectors in \mathbb{R}^3 form a basis iff they are linearly independent.

$$x(2, 2, 1) + y(1, 3, 7) + z(1, 2, 2) = (0, 0, 0)$$

$$2x + y + z = 0; \quad 2x + 3y + 2z = 0; \quad x + 7y + 2z = 0.$$

Let $AX = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -13 & -3 \end{bmatrix}$$

$$R_2 \rightarrow -R_2$$

$$R_3 \rightarrow -R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 13 & 3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 1 & 1/2 \end{bmatrix} \quad R_3 \leftrightarrow R_2 = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 11 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -7/2 \end{bmatrix}$$

Rank $[A] = 3 =$ Number of unknowns. So unique solution and solution is

$$x = 0, y = 0, z = 0.$$

Hence vectors x_1, x_2, x_3 are linearly independent and hence form a basis.

12) Let W be the subspace of R^5 , spanned by

$$x_1 = (1, 2, -1, 3, 4), x_2 = (2, 4, -2, 6, 8), x_3 = (1, 3, 2, 2, 6)$$

$$x_4 = (1, 4, 5, 1, 8), x_5 = (2, 7, 3, 3, 9).$$

Find a subset of vectors which forms a basis of W .

Soln:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$R_5 \rightarrow R_5 - 2R_1$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - 2R_3, R_5 \rightarrow R_5 - 3R_3$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

Number of nonzero rows = 3.

$\therefore \dim W = 3$ and $\{x_1, x_3, x_5\}$ forms a basis in W .

13) Let V is a vector space of polynomials over R . Find a basis and dimension of the subspace W of V , spanned by the polynomials.

$$x_1 = t^3 - 2t^2 + 4t + 1, x_2 = 2t^3 - 3t^2 + 9t - 1$$

$$x_3 = t^3 + 6t - 5, x_4 = 2t^3 - 5t^2 + 7t + 5.$$

Soln:

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \quad R_3 \rightarrow R_3 - 2R_2$$

$$R_3 \rightarrow R_3 - R_1 \quad R_4 \rightarrow R_4 + R_2$$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - 2R_1$$

Nonzero rows of Echelon matrix forms basis.

Number of non Zero rows = 2

$\therefore \dim W = 2$ and $\{x_1, x_2\}$ forms a basis in W .

3.3 Linear Transformations

Let V and W be any two subspaces over R . A mapping T from V to W is called a linear transformation if,

i) $v_1, v_2 \in V$

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

ii) $\forall a \in F$ and $\forall v \in V$

$$T(a \cdot v) = a \cdot T(v)$$

Problems

1) Show that the function $T : R^2 \rightarrow R^3$ given by $T(x, y) = (x + y, x - y, y)$ is a linear transformation.

Soln :- Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be two vectors belonging to R^2 .

$$\begin{aligned} \therefore T(u + v) &= T(x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2, y_1 + y_2) \\ &= \{(x_1 + y_1) + (x_2 + y_2), (x_1 - y_1) + (x_2 - y_2), (y_1 + y_2)\} \\ &= (x_1 + y_1, x_1 - y_1, y_1) + (x_2 + y_2, x_2 - y_2, y_2) \\ &= T(u) + T(v) \end{aligned}$$

For $a \in R$ and $u \in R^2$, we have

$$\begin{aligned} T(au) &= T(ax_1, ay_1) \\ &= (ax_1 + ay_1, ax_1 - ay_1, ay_1) \\ &= a(x_1 + y_1, x_1 - y_1, y_1) \\ &= aT(u) \end{aligned}$$

$\therefore T$ is a linear transformation.

2) Which of the following functions are linear transformation?

i) $T : R^3 \rightarrow R^3$ defined by $T(x, y, z) = (y, -x, -z)$

ii) $T : R^3 \rightarrow R^2$ defined by $T(x, y, z) = (2x - 3y, 7y + 2z)$.

Sol:- i) Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in R^3$ be arbitrary.

$$\begin{aligned} T(u) &= T(x_1, y_1, z_1) = (y_1, -x_1, -z_1) \\ T(v) &= T(x_2, y_2, z_2) = (y_2, -x_2, -z_2) \\ T(u + v) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (y_1 + y_2, -x_1 - x_2, -z_1 - z_2) \\ &= (y_1, -x_1, -z_1) + (y_2, -x_2, -z_2) \\ &= T(u) + T(v) \end{aligned}$$

Also, for any scalar $a \in R$,

$$\begin{aligned} T(au) &= T(ax_1, ay_1, az_1) \\ &= (ay_1, -ax_1, -az_1) \\ &= a(y_1, -x_1, -z_1) \\ &= aT(u) \end{aligned}$$

Hence, T is a linear transformation.

ii)

Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2) \in \mathbb{R}^3$ be arbitrary.

$$T(u) = T(x_1, y_1, z_1) = (2x_1 - 3y_1, 7y_1 + 2z_1)$$

$$T(v) = T(x_2, y_2, z_2) = (2x_2 - 3y_2, 7y_2 + 2z_2)$$

$$\therefore T(u + v) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= (2x_1 + 2x_2 - 3y_1 - 3y_2, 7y_1 + 7y_2 + 2z_1 + 2z_2)$$

$$= \{(2x_1 - 3y_1) + (2x_2 - 3y_2), (7y_1 + 2z_1) + (7y_2 + 2z_2)\}$$

$$= (2x_1 - 3y_1, 7y_1 + 2z_1) + (2x_2 - 3y_2, 7y_2 + 2z_2)$$

$$= T(u) + T(v)$$

Also, for any scalar $a \in \mathbb{R}$

$$T(au) = T(ax_1, ay_1, az_1)$$

$$= (2ax_1 - 3ay_1, 7ay_1 + 2az_1) = a(2x_1 - 3y_1, 7y_1 + 2z_1)$$

$$= aT(u).$$

Hence, T is a linear transformation.

3) Let $I : V(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ be a mapping $f(x) = (3x, 5x)$. Show that f is a linear transformation.

Soln:- i)

$$f(x_1 + x_2) = (3(x_1 + x_2), 5(x_1 + x_2))$$

$$= (3x_1 + 3x_2, 5x_1 + 5x_2)$$

$$= (3x_1 + 5x_1) + (3x_2, 5x_2)$$

$$= f(x_1) + f(x_2)$$

ii)

$$f(a \cdot x) = (3(a \cdot x), 5(a \cdot x))$$

$$= (3ax, 5ax)$$

$$= a(3x, 5x)$$

$$= a \cdot f(x)$$

Hence f is a linear transformation.

4) Show that the translation mapping $f : V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by $f(x, y) = (x + 6, y + 2)$

is not linear.

Soln:

$$f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + 6, y_1 + y_2 + 2) \rightarrow (1)$$

$$\begin{aligned} f(x_1, y_1) + f(x_2, y_2) &= (x_1 + 6, y_1 + 2) + (x_2 + 6, y_2 + 2) \\ &= (x_1 + x_2 + 12, y_1 + y_2 + 4) \rightarrow (2) \end{aligned}$$

Equation (1) \neq Equation (2).

Hence f is not a linear transformation.

3.4 Kernel of T and Image of T

Let T be a linear transformation from V to W .

Kernel of T or $\ker(T) = \{v \in V : T(v) = 0, 0 \in W\}$.

Remark:- Kernel of T is also known as null space of T .

Image of T or $\text{Im}(T) = \{w \in W; w = T(v) \text{ for some } v \in V\}$

3.4.1 Algebra of Linear Transformations

Sum of linear transformations

Let $T_1 : U \rightarrow V$ and $T_2 : U \rightarrow V$ be two transformations where U and V are two vector spaces over R . We define the sum of T_1 and T_2 by $T_1 + T_2 : U \rightarrow V$, such that $(T_1 + T_2)u = T_1(u) + T_2(u)$, $u \in U$.

Scalar multiplication of Linear Transformations

If $a \in R$, the function aT , ie product of a linear transformation $T : U \rightarrow V$ with a , and defined by $(aT) : U \rightarrow V$, such that $(aT)u = a(T(u))$ for all $u \in U$ is a linear transformation from U into V .

Composition (Product) of two Linear transformations:

Let U, V, W be three vector spaces over R . We define T_2T_1 , called the composition or product of T_2 and T_1 , by $(T_2T_1)(u) = T_2(T_1(u))$.

T_2T_1 is also denoted by $(T_2 \circ T_1)$.

Problems:

1) Let the linear transformations $T_1 : R^3 \rightarrow R^2$ such that $T_1(x, y, z) = (4x, 3y - 2z)$ and $T_2 : R^2 \rightarrow R^2$ such that $T_2(x, y) = (-2x, y)$. Compute T_1T_2 and T_2T_1 .

Soln: - As the range of T_2 is not contained in the domain of T_1 , $\therefore T_1T_2$ is not defined.

Now, T_2T_1 is defined as the range of T_1 is contained in the domain of T_2 .

$$\begin{aligned} T_2T_1(x, y, z) &= T_2(T_1(x, y, z)) \\ &= T_2(4x, 3y - 2z) \\ &= (-8x, 3y - 2z). \end{aligned}$$

2) Illustrate with the help of an example that there exist linear transformations $T_1 : R^2 \rightarrow R^2$ and $T_2 : R^2 \rightarrow R^2$ such that $T_2T_1 = 0$ but $T_1T_2 \neq 0$.

Soln: - Let us define $T_1 : R^2 \rightarrow R^2$ by

$$T_1(x, y) = (0, 4x)$$

& $T_2 : R^2 \rightarrow R^2$ by $T_2(x, y) = (x, 0)$

$\therefore T_1T_2$ and T_2T_1 both are defined.

$$\begin{aligned} (T_2T_1)(x, y) &= T_2(T_1(x, y)) = T_2(0, 4x) \\ &= (0, 0) = 0(x, y) \end{aligned}$$

$$T_2T_1 = 0.$$

Also

$$\begin{aligned} (T_1T_2)(x, y) &= T_1(T_2(x, y)) = T_1(x, 0) \\ &= (0, 4x) \neq 0(x, y) \end{aligned}$$

$$T_1T_2 \neq 0.$$

3.4.2 Matrix of a Linear transformation

Let $T : U \rightarrow V$ be the linear transformation, where U and V are vector spaces over R . Let $B = \{u_1, u_2 \dots u_n\}$ and $B' = \{v_1, v_2 \dots v_m\}$ be ordered basis for the finite dimensional vector spaces U and V respectively. Since $T(u_1), T(u_2) \dots T(u_n) \in V$ and $\{v_1, v_2 \dots v_m\}$ spans V , each $T(u_i)$ can be expressed at a linear combination of the vectors $v_1, v_2 \dots, v_m$.

$$T(u_1) = a_{11}v_1 + a_{21}v_2 + \dots + a_{m1}v_m$$

$$T(u_2) = a_{12}v_1 + a_{22}v_2 + \dots + a_{m2}v_m$$

$$T(u_n) = a_{1n}v_1 + a_{2n}v_2 + \dots + a_{mn}v_m \text{ where } a_{ij} \in R.$$

The coefficient matrix of this system of equations is

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ - & - & - & \dots & - \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{mn} \end{bmatrix}$$

The transpose of this matrix is a matrix representation of T , called matrix of T with respect to ordered basis B and B' . It is denoted by $[T : B, B']$.

Problems

1) Find the matrix representing the transformation $T : R^2 \rightarrow R^3$ given by

$$T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2) \text{ relative to the standard basis of } R^2 \text{ and } R^3.$$

Soln: - The ordered standard basis of R^2 is

$$B = \{(1, 0), (0, 1)\} \text{ and that of } R^3 \text{ is}$$

$$B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

$$\therefore T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2)$$

$$T(1, 0) = (3, 2, 5) \text{ \& } T(0, 1) = (-1, 4, -6)$$

$$T(1, 0) = (3, 2, 5) = 3(1, 0, 0) + 2(0, 1, 0) + 5(0, 0, 1)$$

$$T(0, 1) = (-1, 4, -6) = -1(1, 0, 0) + 4(0, 1, 0) + (-6)(0, 0, 1)$$

Hence, matrix of the transformation is
$$\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$$

2) Find the matrix representing the transformation $T : R^3 \rightarrow R^4$ defined by $T(x, y, z) =$

$(x + y + z, 2x + z, 2y - z, 6y)$ relative to the standard basis of R^3 and R^4 .

Sol: - We know that ordered standard basis of R^3 is $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and R^4 is $B' = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

Since $T(x, y, z) = (x + y + z, 2x + z, 2y - z, 6y)$

$$T(1, 0, 0) = (1, 2, 0, 0)$$

$$T(0, 1, 0) = (1, 0, 2, 6)$$

$$T(0, 0, 1) = (1, 1, -1, 0)$$

Let $e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)$

$$\therefore T(1, 0, 0) = (1, 2, 0, 0) = 1e_1 + 2e_2 + 0e_3 + 0e_4$$

$$T(0, 1, 0) = (1, 0, 2, 6) = 1e_1 + 0e_2 + 2e_3 + 6e_4$$

$$T(0, 0, 1) = (1, 1, -1, 0) = 1e_1 + 1e_2 - 1e_3 + 0e_4$$

Matrix of T relative to B and B' is transpose of matrix of coefficients in above system of equations

$$\text{i.e., } [T : B, B'] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}.$$

3.4.3 Change of coordinates or change of Basis Matrix or Transition Matrix

Let $B_1 = \{u_1, u_2, \dots, u_n\}$ be a basis of n -dimensional vector space V and $B_2 = \{v_1, v_2, \dots, v_n\}$ be another basis of V . Then, each element in B_2 can be expressed as a linear combination of the vectors in basis B_1 .

Let

$$v_1 = a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n$$

$$v_2 = a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n$$

⋮

$$v_n = a_{n1}u_1 + a_{n2}u_2 + \dots + a_{nn}u_n$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & \dots a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Then the transpose of the matrix of coefficients in above system of equations is said to be the transition matrix or change of basis matrix from the old basis B_1 to the new basis B_2 . It is denoted by P and written as

$$P = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

Since the vectors in B_2 are linearly independent, so the matrix P is invertible. Its inverse P^{-1} is the transition matrix from the basis B_2 to the basis B_1 .

Problems:- 1) Consider the following basis of R^2 :

$$E = \{e_1, e_2\} = \{(1, 0), (0, 1)\} \text{ and } S = \{u_1, u_2\} = \{(1, 3), (1, 4)\}$$

(a) Find the change-of-basis matrix P from the usual basis E to S .

(b) Find the change - of -basis matrix Q from S back to E .

(c) Find the coordinate vector $[v]$ of $v = (5, -3)$ relative to S .

Soln: (a) Since E is the usual basis of R^2 ,

$$\therefore P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

(b) Find P^{-1}

$$P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

(c) Let $[v]_S = P^{-1}[v]_E$ we have $[v]_E = [5, -3]^T$

$$[v]_S = P^{-1}[v]_E = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -18 \end{bmatrix}.$$

2) The vectors $u_1 = (1, 2, 0), u_2 = (1, 3, 2), u_3 = (0, 1, 3)$ form a basis S of R^3 . Find:

a) The change of basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of R^3 to the basis S .

b) The change of basis matrix Q from the above basis S back to the usual basis E of R^3 .

Soln: - (a) Since E is a usual basis of R^3 .

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

(b) Find P^{-1}

$$P^{-1} = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

3) Find the co-ordinates of vector $(1, 1, 1)$ relative to basis $(1, 1, 2), (2, 2, 1), (1, 2, 2)$.

Sol:-

$$\text{Let } (1, 1, 1) = a(1, 1, 2) + b(2, 2, 1) + c(1, 2, 2)$$

$$a + 2b + c = 1 \rightarrow (1)$$

$$a + 2b + 2c = 1 \rightarrow (2)$$

$$2a + b + 2c = 1 \rightarrow (3)$$

Subtracting (1) from (2), we have $c = 0$.

putting $c = 0$ in (2) and (3), we get

$$a + 2b = 1 \rightarrow (4)$$

$$2a + b = 1 \rightarrow (5)$$

Solving (4) and (5). we get $a = b = \frac{1}{3}$.

\therefore Coordinates of $(1, 1, 1)$ relative to given basis are $(\frac{1}{3}, \frac{1}{3}, 0)$.

4) Find the co-ordinates of the following vectors relative to the basis $y_1 = (1, 1, 2), y_2 = (2, 2, 1), y_3 = (1, 2, 2)$.

(i) $(1, 0, 1)$ soln : $(\frac{4}{3}, \frac{1}{3}, -1)$

(ii) $(1, 1, 0)$ soln : $(-\frac{1}{3}, \frac{2}{3}, 0)$

3.5 Rank and nullity of a linear Operator

Rank of T : – Dimension of the $\text{Im}(T)$ is known as rank of T .

Nullity of T : - Dimension of the $\text{Ker}(T)$ is known as nullity of T .

3.5.1 Rank-nullity theorem

Let V and W be vector spaces over R and let T be a linear transformation from V to W . If V is a finite dimensional then $\text{rank}(T) + \text{nullity of } (T) = \dim(V)$.

Problems:

1) For the linear transformation $T : R^2 \rightarrow R^3$ such that $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$, find a basis and dimension of its range space and its null space. Also verify that $\text{rank}(T) + \text{nullity}(T) = \dim R^2$.

Sol: - i) To find null space of T and its dimension:

By definition, null space = $N(T) = \{u \in R^2 : T(u) = 0 \in R^3\}$

Let $u = (x_1, x_2) \in R^2$ be an arbitrary element of null space.

$$T(u) = 0$$

$$T(x_1, x_2) = 0$$

$$(x_1 - x_2, x_2 - x_1, -x_1) = 0$$

$$x_1 - x_2 = 0$$

$$\Rightarrow x_2 - x_1 = 0$$

$$-x_1 = 0$$

On solving the above equations, We get

$$x_1 = 0, x_2 = 0.$$

Thus, the only member of null space is a zero vector $\in R$

$$\text{i.e } N(T) = \{0\}.$$

\therefore Nullity of $T = \dim(N(T)) = 0$.

ii) To find range space of T and its dimension:

The range space consists of ordered triples $(x_1 - x_2, x_2 - x_1, -x_1)$ for $(x_1, x_2) \in \mathbb{R}^2$ such that

$$T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$$

Let v be any element of $R(T)$.

Thus there exist $(x_1, x_2) \in \mathbb{R}^2$ such that $v = T(x_1, x_2)$

$$\begin{aligned} \Rightarrow (x_1, x_2) &= x_1(1, 0) + x_2(0, 1) \\ &= x_1e_1 + x_2e_2 \text{ where } e_1 = (1, 0), e_2 = (0, 1) \end{aligned}$$

$$\begin{aligned} T(x_1, x_2) &= T(x_1e_1 + x_2e_2) \\ &= x_1T(e_1) + x_2T(e_2) \rightarrow (2) \end{aligned}$$

$$T(e_1) = T(1, 0) = (1 - 0, 0 - 1, -1) = (1, -1, -1)$$

$$T(e_2) = T(0, 1) = (0 - 1, 1 - 0, 0) = (-1, 1, 0)$$

Putting the values of $T(e_1), T(e_2)$ in (2)

$$\begin{aligned} T(x_1, x_2) &= x_1(1, -1, -1) + x_2(-1, 1, 0) \\ v &= x_1(1, -1, -1) + x_2(-1, 1, 0) \end{aligned}$$

Since $v \in R(T)$ is arbitrary.

(iii) Let $a(1, -1, -1) + b(-1, 1, 0) = 0$ for $a, b \in \mathbb{R}$.

$$\begin{aligned} \Rightarrow (a - b, -a + b, -a) &= (0, 0, 0) \\ \Rightarrow a - b = 0, b - a = 0, -a = 0 \\ \Rightarrow a = b = 0. \end{aligned}$$

Thus $\{(1, -1, -1), (-1, 1, 0)\}$ is linearly independent.

\therefore It is a basis set of $R(T)$.

\therefore Dimension of $R(T) = 2$

Also dimension of $\mathbb{R}^2 = 2$

\therefore Rank $N(T) + \text{nullity}(T) = 2 + 0 = 2 = \dim \mathbb{R}^2$.

2) Find a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ whose range is spanned by the vectors $(1, 2, 3), (4, 5, 6)$.

Soln:- We know that $\{e_1, e_2, e_3\}$ is a standard basis of \mathbb{R}^3 , where $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$

Since \mathbb{R}^3 is a three dimensional vector space and $\{e_1, e_2, e_3\}$ is a basis of \mathbb{R}^3 , there exist a unique

linear transformation T such that

$$T(e_1) = (1, 2, 3)$$

$$T(e_2) = (4, 5, 6)$$

$$T(e_3) = (0, 0, 0)$$

Also $R(T)$ is spanned by $\{(1, 2, 3), (4, 5, 6)\}$

i.e by $\{(1, 2, 3), (4, 5, 6), (0, 0, 0)\}$ or by $\{T(e_1), T(e_2), T(e_3)\}$

Now, for each $(x_1, x_2, x_3) \in R^3$.

$$\begin{aligned}(x_1, x_2, x_3) &= x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1) \\ &= x_1e_1 + x_2e_2 + x_3e_3\end{aligned}$$

$$\begin{aligned}T(x_1, x_2, x_3) &= x_1T(e_1) + x_2T(e_2) + x_3T(e_3) \\ &= x_1(1, 2, 3) + x_2(4, 5, 6) + x_3(0, 0, 0) \\ &= (x_1 + 4x_2, 2x_1 + 5x_2, 3x_1 + 6x_2)\end{aligned}$$

which is the required linear transformation.

3) Verify the Rank-nullity theorem for the $T : R^3 \rightarrow R^3$ defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

Soln:- To find the basis of nullity of T

Let $v = (x, y, z) \in R^3$ such that Nullity of $T = \{v \in V \mid T(v) = 0\}$

$$T(x, y, z) = 0.$$

$$(x + 2y - z, y + z, x + y - 2z) = (0, 0, 0).$$

$$x + 2y - z = 0 \Rightarrow x = 3z$$

$$y + z = 0 \Rightarrow y = -z.$$

$$x + y - 2z = 0$$

On solving above equations we get

$$x = 3z, \quad y = -z$$

$$\therefore \{(x, y, z)\} = \{3z, -z, z\} = (z\{3, -1, 1\})$$

Thus $\{(3, -1, 1)\}$ is a basis of nullity of T & Nullity $(T) = 1$.

To find the basis of range of T .

As $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ generates R^3

$\Rightarrow \{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\}$ generates range

$\Rightarrow \{(1, 0, 1), (2, 1, 1), (-1, 1, -2)\}$ generates range of T

To find the basis of range, consider a matrix.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_3 \rightarrow R_3 + R_2$$

Thus $\{(1, 0, 1), (0, 1, -1)\}$ form a basis of Range of T & $\dim(\text{Range of } T) = 2$.

$$\therefore \text{Rank}(T) + \text{Nullity}(T) = 2 + 1 = 3 = \dim(\mathbb{R}^3).$$

Hence rank-Nullity theorem verified.

4) Verify the Rank-Nullity theorem for the $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + 2y, y - z, x + 2z)$.

3.6 Inner Product Spaces and Orthogonality

3.6.1 Inner Product Space

Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number, denoted by $\langle u, v \rangle$. This function is called inner product on V if it satisfies the following axioms:

(i) Linear Property: $\langle au_1 + bu_2, v \rangle = a \langle u_1, v \rangle + b \langle u_2, v \rangle$

(ii) Symmetric Property: $\langle u, v \rangle = \langle v, u \rangle$

(iii) Positive Definite Property: $\langle u, u \rangle \geq 0$; and $\langle u, u \rangle = 0$ if and only $u = 0$.

Norm of a vector

An inner product, $\langle u, u \rangle$ is nonnegative for any vector u .

$$\|u\| = \sqrt{\langle u, u \rangle} \text{ or } \|u\|^2 = \langle u, u \rangle$$

This non negative number is called the norm or length of u .

* Every non- zero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector $\hat{v} = \frac{1}{\|v\|}v$, which is a positive multiple. This process is called normalizing v .

Orthogonality

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if $\langle u, v \rangle = 0$.

Problems

1) Consider vectors $u = (1, 2, 4), v = (2, -3, 5), w = (4, 2, -3)$ in R^3 . Find

- $\langle u \cdot v \rangle$
- $\langle u \cdot w \rangle$
- $\langle v \cdot w \rangle$
- $\langle (u + v) \cdot w \rangle$
- $\|u\|$
- $\|v\|$

Soln: a) $\langle u \cdot v \rangle = 2 - 6 + 20 = 16$

b) $\langle u \cdot w \rangle = 4 + 4 - 12 = -4$

c) $\langle v \cdot w \rangle = 8 - 6 - 15 = -13$

d) $\langle (u + v) \cdot w \rangle = (3, -1, 9) \cdot (4, 2, -3) = 12 - 2 - 27 = -17.$

e) $\|u\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21}.$

f) $\|v\| = \sqrt{4 + 9 + 25} = \sqrt{38}.$

2) Consider the vectors $u = (1, 5)$ and $v = (3, 4)$ in R^2 , Find:

a) $\langle u, v \rangle$ with respect to the usual inner product in R^2 .

b) $\|v\|$ using the inner product in R^2 .

3) Consider the following polynomials in $P(t)$ and inner product:

$$f(t) = t + 2, g(t) = 3t - 2, h(t) = t^2 - 2t - 3 \text{ and}$$

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

(a) find $\langle f, g \rangle$ and $\langle f, h \rangle$

(b) find $\|f\|$ and $\|g\|$

(c) normalize f and g .

Soln: : (a)

$$\langle f, g \rangle = \int_0^1 (t+2)(3t-2) dt = \int_0^1 (3t^2 + 4t - 4) dt$$

$$\langle f, g \rangle = t^3 + 2t^2 - 4t \Big|_0^1 = -1.$$

$$\langle f, h \rangle = \int_0^1 (t+2)(t^2 - 2t - 3) dt = \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t \right]_0^1 = -\frac{37}{4}$$

(b)

$$\langle f, f \rangle = \int_0^1 (t+2)(t+2) dt = \frac{19}{3}; \|f\| = \frac{\sqrt{19}}{\sqrt{3}} = \frac{\sqrt{57}}{3}.$$

$$\langle g, g \rangle = \int_0^1 (3t-2)(3t-2) dt = 1; \|g\| = \sqrt{1} = 1$$

(c) Since $\|f\| = \frac{\sqrt{57}}{3}$ and g is already a unit vector,

$$\hat{f} = \frac{1}{\|f\|} f = \frac{3}{\sqrt{57}}(t+2)$$

$$\hat{g} = \frac{1}{\|g\|} g = 3t - 2.$$

4) Let $M = M_{2,3}$ with inner product, $\langle A, B \rangle = \text{tr} \langle B^T A \rangle$ and let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$, $B =$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$ Find (a) $\langle A, B \rangle, \langle A, C \rangle, \langle B, C \rangle$ (b) $\langle 2A + 3B, 4C \rangle$

(c) $\|A\|$ and $\|B\|$

$$(a) \langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij}$$

$$\langle A, B \rangle = 9 + 16 + 21 + 24 + 25 + 24 = 119$$

$$\langle A, C \rangle = 27 - 40 + 14 + 6 + 0 - 16 = -9$$

$$\langle B, C \rangle = 3 - 10 + 6 + 4 + 0 - 24 = -21$$

(b)

$$2A + 3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix} \quad 4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$$

$$\langle 2A + 3B, 4C \rangle = 252 - 440 + 96 + 0 - 416 = -324$$

(c) $\|A\|^2 = \langle A, A \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2$, the sum of the squares of all the elements of A .

$$\|A\|^2 = \langle A, A \rangle = 9^2 + 8^2 + 7^2 + 6^2 + 5^2 + 4^2 = 271 \Rightarrow \|A\| = \sqrt{271}$$

$$\|B\|^2 = \langle B, B \rangle = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91 \Rightarrow \|B\| = \sqrt{91}$$

5) Verify the vectors $u = (1, 1, 1)$, $v = (1, 2, -3)$ and $w = (1, -4, 3)$ in R^3 are orthogonal or not.

Soln: $\langle u, v \rangle = 1 + 2 - 3 = 0$, $\langle u, w \rangle = 1 - 4 + 3 = 0$, $\langle v, w \rangle = 1 - 8 - 9 = -16$

Thus u is orthogonal to v and w , v and w are not orthogonal.

AJIEET, Mangaluru



Lecture Notes

BMATS201

Mathematics-II for CSE Stream

Module 4: Numerical Methods I

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Module 4

Numerical Methods I

Syllabus:

Importance of numerical methods for discrete data in the field of computer science & engineering.

Solution of algebraic and transcendental equations - Regula-Falsi and Newton-Raphson methods (only formulae). Problems.

Finite differences, Interpolation using Newton's forward and backward difference formulae, Newton's divided difference formula and Lagrange's interpolation formula (All formulae without proof). Problems.

Numerical integration: Trapezoidal, Simpson's (1/3)rd and (3/8)th rules(without proof). Problems.

Self-Study: Bisection method, Lagrange's inverse Interpolation.

4.1 Numerical Solution of algebraic and transcendental equations:

- Algebraic equation is an equation in the form of a polynomial having a finite number of terms.

Example: $x^3 - 4x - 9 = 0$, $x^4 + x^3 = 80$

- A transcendental equation is an equation containing a transcendental function of the variable(s) being solved for.

Example: $xe^x - 2 = 0$, $\tan x = 2x$, $x \log x - 1.2 = 0$

- Numerical method of finding approximate roots of the given function is a repetitive type of process known as **iteration process**.

4.2 Newton Raphson Method

Let the given equation be $f(x) = 0$

We first find an interval (a,b) such that f(a) and f(b) are of opposite signs.

Then we select a real number in (a,b) as the initial approximation x_0 to the required root.

Use the Newton Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots$$

repeatedly, until we obtain the answer with the desired accuracy.

Problem 4.2.1. Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.

Solution:: Let $f(x) = x^4 - x - 10$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$$f(1) = -10 \quad (\text{Here value is -ve})$$

$$f(2) = 16 - 2 - 10 = 4 \quad (\text{Here value is +ve})$$

\therefore a root of $f(x) = 0$ lies between $a = 1$ and $b = 2$.

Let us take $x_0 = 2$

Differentiating, $f(x) = x^4 - x - 10$ w.r.to x , we get $f'(x) = 4x^3 - 1$

Newton's Raphson formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (*)$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 2 - \frac{f(2)}{f'(2)} \\ &= 2 - \frac{2^4 - 4 - 10}{4 \times 2^3 - 1} \\ &= 2 - \frac{4}{31} \\ &= 1.871 \end{aligned}$$

Put $n = 1$, in (*), the second approximation x_2 is given by,

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1.871 - \frac{f(1.871)}{f'(1.871)} \\ &= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} \\ &= 1.871 - \frac{0.3835}{25.199} \\ &= 1.856 \end{aligned}$$

Put $n = 2$, in (*), the third approximation x_3 is given by,

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1} \\ &= 1.856 - \frac{0.010}{24.574} \\ &= 1.856 \end{aligned}$$

Since $x_2 = x_3 = 1.856$, the required root is $x = 1.856$

Problem 4.2.2. Using Newton-Raphson method, find the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

Solution: : Let $f(x) = 3x - \cos x - 1$ (1)

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

Note : Make sure your calculator is in **radian mode**

$$f(0) = 3(0) - \cos(0) - 1 = -2, \text{ (Here value is -ve)}$$

$$f(1) = 3(1) - \cos(1) - 1 = 1.4597, \text{ (Here value is +ve)}$$

Hence interval $(a,b) = (0,1)$

\therefore a root of $f(x) = 0$ lies between 0 and 1.

Let us take $x_0 = 0.6$

Also differentiating (1) we get,

$$f'(x) = 3 + \sin x$$

Newton's Raphson formula gives,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \\ &= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad (*) \end{aligned}$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$\begin{aligned} x_1 &= \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} \\ &= \frac{0.6 \sin 0.6 + \cos 0.6 + 1}{3 + \sin 0.6} \\ &= 0.6071 \end{aligned}$$

Put $n = 1$ in (*), then second approximation is

$$\begin{aligned} x_2 &= \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} \\ &= \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} \\ &= 0.6071 \end{aligned}$$

Clearly $x_1 = x_2$.

Hence, desired root is 0.6071 correct to 4 decimal places.

Problem 4.2.3. Using Newton Raphson method, find a real root of $x \sin x + \cos x$ near $x = \pi$ correct to 3 decimal places.

Solution : Given $x_0 = \pi = 3.1416$

$$f(x) = x \sin x + \cos x$$

$$\begin{aligned} f'(x) &= x \cos x + \sin x - \sin x \\ &= x \cos x \end{aligned}$$

The iteration formula is,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n} \quad (*) \end{aligned}$$

Note : Make sure your calculator is in **radian mode**

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$\begin{aligned} x_1 &= x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0} \\ &= \pi - \frac{\pi \sin \pi + \cos \pi}{\pi \cos \pi} \quad (\because x_0 = \pi) \\ &= 2.8233 \end{aligned}$$

Put $n = 1$, in (*),

$$\begin{aligned} x_2 &= x_1 - \frac{x_1 \sin x_1 + \cos x_1}{x_1 \cos x_1} \\ &= 2.8233 - \frac{2.8233 \sin(2.8233) + \cos(2.8233)}{2.8233 \cos(2.8233)} \\ &= 2.7986 \end{aligned}$$

similarly Putting $n = 2$, in (*), and calculating we get

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 \sin x_2 + \cos x_2}{x_2 \cos x_2} \\ &= 2.7986 - \frac{2.7986 \sin(2.7986) + \cos(2.7986)}{2.7986 \cos(2.7986)} \\ &= 2.7984 \end{aligned}$$

since x_2 and x_3 are same upto three decimal places, we stop the procedure and required root is $x = 2.798$

Problem 4.2.4. Using Newton Raphson method, find a real root of $x \log_{10} x - 1.2$ correct to four decimal places.

Solution: : Here initial approximation is not given.

$$f(x) = x \log_{10} x - 1.2 \quad (1)$$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$$f(1) = -1.2 \text{ (Here value is -ve)}$$

$$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979 \text{ (Here value is -ve)}$$

$$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136 \text{ (Here value is +ve)}$$

Hence a root of $f(x) = 0$ lies between a=2 and b=3

Let us take $x_0 = 2$

Also,

$$\begin{aligned} f'(x) &= \frac{d}{dx} (x \log_{10} x - 1.2) \\ &= \log_{10} x + x \frac{d}{dx} (\log_{10} x) \\ &= \log_{10} x + x \frac{d}{dx} \left(\frac{\log_e x}{\log_e 10} \right) \\ &= \log_{10} x + x \frac{1}{x} \log_{10} e \\ &= \log_{10} x + 0.43429 \end{aligned}$$

Newton's formula gives,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.43429} \\ &= \frac{0.43429x_n + 1.2}{\log_{10} x_n + 0.43429} \quad (*) \end{aligned}$$

Put $n = 0$, the first approximation is

$$\begin{aligned} x_1 &= \frac{0.43429x_0 + 1.2}{\log_{10} x_0 + 0.43429} \\ &= \frac{0.43429(2) + 1.2}{\log_{10} 2 + 0.43429} \\ &= 2.8132 \end{aligned}$$

Similarly, Putting $n = 1$ in (*), we get

$$\begin{aligned} x_2 &= \frac{0.43429x_1 + 1.2}{\log_{10} x_1 + 0.43429} \\ &= \frac{0.43429(2.8132) + 1.2}{\log_{10}(2.8132) + 0.43429} \\ &= 2.7411 \end{aligned}$$

Similarly, Putting $n = 2$ in (*), we get

$$\begin{aligned} x_3 &= \frac{0.43429x_2 + 1.2}{\log_{10} x_2 + 0.43429} \\ &= \frac{0.43429(2.7411) + 1.2}{\log_{10}(2.7411) + 0.43429} \\ &= 2.7406 \end{aligned}$$

Similarly, Putting $n = 3$ in (*), we get

$$\begin{aligned} x_4 &= \frac{0.43429x_3 + 1.2}{\log_{10} x_3 + 0.43429} \\ &= \frac{0.43429(2.7406) + 1.2}{\log_{10}(2.7406) + 0.43429} \\ &= 2.7406 \end{aligned}$$

Clearly, $x_3 = x_4$. Hence, the required root is 2.7406 correct to four decimal places.

Problem 4.2.5. Find a real root of $\cos x = xe^x$ which is nearer to $x = 0.5$ correct to three decimal places.

Solution::

Given $x_0 = 0.5$

$$f(x) = \cos x - xe^x$$

$$f'(x) = -\sin x - (xe^x + e^x)$$

$$= -\sin x - e^x(x + 1)$$

The iteration formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (*)$$

Note : Make sure your calculator is in **radian mode**

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 0.5 - \frac{f(0.5)}{f'(0.5)} \\ &= 0.5 - \frac{\cos(0.5) - (0.5)e^{0.5}}{\sin(0.5) - e^{0.5}(0.5 + 1)} \\ &= 0.5 - \frac{0.0532}{-2.9525} \\ &= 0.51802 \end{aligned}$$

Put $n = 1$, in (*), the second approximation x_2 is given by,

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 0.51802 - \frac{f(0.51802)}{f'(0.51802)} \\ &= 0.51802 - \frac{\cos(0.51802) - (0.51802)e^{0.51802}}{\sin(0.51802) - e^{0.51802}(0.51802 + 1)} \\ &= 0.51802 - \frac{-0.0008}{-3.0435} \\ &= 0.518 \end{aligned}$$

Since $x_1 = x_2 = 0.518$, the required root is $x = 0.518$

Problem 4.2.6. Using Newton-Raphson method, find the real root of the equation $x^3 - 2x - 5 = 0$ correct to 5 decimal places.

Solution::

$$f(x) = x^3 - 2x - 5, \quad f'(x) = 3x^2 - 2$$

Let us find an interval (a,b) such that f(a) and f(b) are of opposite signs.

$$f(1) = -6 \text{ (Here value is -ve)}$$

$$f(2) = -1 \text{ (Here value is -ve)}$$

$$f(3) = 16 \text{ (Here value is +ve)}$$

Hence root lies between $(a, b) = (2, 3)$.

Let us take initial point $x_0 = 2.5$

Newton's Raphson formula gives,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} \quad (*) \end{aligned}$$

Put $n = 0$, in (*), the first approximation x_1 is given by,

$$\begin{aligned} x_1 &= x_0 - \frac{x_0^3 - 2x_0 - 5}{3x_0^2 - 2} \\ &= 2.5 - \frac{2.5^3 - 2(2.5) - 5}{3(2.5)^2 - 2} \\ &= 2.16418 \end{aligned}$$

Put $n = 1$, in (*), the first approximation x_2 is given by,

$$\begin{aligned} x_2 &= x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} \\ &= 2.16418 - \frac{(2.16418)^3 - 2(2.16418) - 5}{3(2.16418)^2 - 2} \\ &= 2.09714 \end{aligned}$$

Similarly, with $n = 2, 3, 4$ in (*) we get

$$x_2 = 2.09714$$

$$x_3 = 2.09455$$

$$x_4 = 2.09455$$

Since $x_3 = x_4 = 2.09455$, the required root is $x = 2.09455$

4.3 Practice Problems :

- Find the real root of the equation $xe^x - 2 = 0$ correct to three decimal places using Newton- Raphson method. [VTU- Jan17] Ans: 0.8526

- Using Newton-Raphson method, find the root that lies near $x = 4.5$ of the equation $\tan x = x$ correct to four decimal places. [VTU- Jan 2018, Jan17]

Ans: 4.4934

- Compute one positive root of $2x - \log_{10} x = 7$ by Newton-Raphson method correct to four decimal places.

Ans : 3.7892

4.4 Regula Falsi Method or Method of False Position

- Let the given equation be $f(x) = 0$
- We first find an interval (x_0, x_1) such that $f(x_0)$ and $f(x_1)$ are of opposite signs.
- Regula-Falsi formula to find the kth approximation x_{k+2} is given by,

$$x_{k+2} = x_k - \frac{f(x_k)(x_{k+1} - x_k)}{f(x_{k+1}) - f(x_k)}$$

In particular, the first approximation x_2 is given by

$$x_2 = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)} \quad (*)$$

- Using equation * find the first approximation x_2 and then find $f(x_2)$.
- Determine the new interval: We come across two possibilities.
 - (i) If $f(x_2)$ and $f(x_1)$ are of opposite signs, then set the new interval as (x_2, x_1) .
 - (ii) If $f(x_0)$ and $f(x_2)$ are of opposite signs, then set the new interval as (x_0, x_2) .
- The second approximation x_3 is given by using the new interval in the Regula Falsi formula *.

– If the new interval is (x_0, x_2) :

$$x_3 = x_0 - \frac{f(x_0)(x_2 - x_0)}{f(x_2) - f(x_0)}.$$

– If the new interval is (x_2, x_1) :

$$x_3 = x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)}.$$

- Evaluate $f(x_3)$, decide the new interval and repeat the process until the root is obtained with desired accuracy.
- Each step reduces the interval size, converging towards the root.

Problem 4.4.1. Use the Regula-falsi method to find a real root of the equation, $x^3 - 2x - 5 = 0$ correct to 2 decimal places.

Solution: Let $f(x) = x^3 - 2x - 5$

$$f(0) = -5$$

$$f(1) = -6,$$

$$f(2) = 2^3 - 2(2) - 5 = -1 \quad (\text{negative})$$

$$f(3) = 3^3 - 2(3) - 5 = 16 \quad (\text{positive})$$

Since $f(2)$ and $f(3)$ are of opposite signs, a real root lies in $(2, 3)$.

Let us take $x_0 = 2$ and $x_1 = 3$.

The first approximation to root x_2 is given by

$$\begin{aligned} x_2 &= x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)} \\ &= 2 - \frac{3 - 2}{f(3) - f(2)} \\ &= 2 - \frac{3 - 2}{16 - (-1)} \\ &= 2.058 \end{aligned}$$

Now $f(x_2) = f(2.058) = (2.058)^3 - 2(2.058) - 5 = -0.4$ (negative)

Now, $f(x_2)$ and $f(x_1)$ are of opposite signs, and hence the new interval is $(x_2, x_1) = (2.058, 3)$.

The second approximation to the root x_3 is given by

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)} \\ &= 2.058 - \frac{f(2.058)(3 - 2.058)}{f(3) - f(2.058)} \\ &= 2.058 - \frac{(-0.4)(3 - (-0.4))}{16 - (-0.4)} \\ &= 2.081 \end{aligned}$$

Now $f(x_3) = f(2.081) = (2.081)^3 - 2(2.081) - 5 = -0.15$ (negative)

Now, $f(x_3)$ and $f(x_1)$ are of opposite signs, and hence the new interval is $(x_3, x_1) = (2.081, 3)$.

The third approximation to the root x_4 is given by

$$\begin{aligned} x_4 &= x_3 - \frac{f(x_3)(x_1 - x_3)}{f(x_1) - f(x_3)} \\ &= 2.081 - \frac{f(2.081)(3 - 2.081)}{f(3) - f(2.081)} \\ &= 2.081 - \frac{(-0.15)(3 - (-0.15))}{16 - (-0.15)} \\ &= 2.089 \end{aligned}$$

Now $f(x_4) = f(2.089) = (2.089)^3 - 2(2.089) - 5 = -0.0617$ (negative)

Now, $f(x_4)$ and $f(x_1)$ are of opposite signs, and hence the new interval is $(x_4, x_1) = (2.089, 3)$.

The 4th approximation to the root x_5 is given by

$$\begin{aligned} x_5 &= x_4 - \frac{f(x_4)(x_1 - x_4)}{f(x_1) - f(x_4)} \\ &= 2.089 - \frac{f(2.089)(3 - 2.089)}{f(3) - f(2.089)} \\ &= 2.089 - \frac{(-0.0617)(3 - (-0.0617))}{16 - (-0.0617)} \\ &= 2.092 \end{aligned}$$

x_4 and x_5 are almost equal correct to two decimal places, the required approximate root is $x_5 = 2.09$

Problem 4.4.2. Use the Regula-falsi method to find a real root of the equation, $x \log_{10} x - 1.2 = 0$ which lies in $(2, 3)$.

Solution: Let $f(x) = x \log_{10} x - 1.2$

Here given interval is $(x_0, x_1) = (2, 3)$

$$f(x_0) = f(2) = 2 \log_{10} 2 - 1.2 = -0.59794 \quad (-ve)$$

$$f(x_1) = f(3) = 3 \log_{10} 3 - 1.2 = 0.23136 \quad (+ve)$$

By method of false position, we have

$$\begin{aligned} x_2 &= x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)} \quad \dots (*) \\ &= 2 - \frac{3 - 2}{0.23136 - (-0.59794)} \\ &= 2.72102 \end{aligned}$$

Now, $f(x_2) = f(2.72102) = (2.72102) \log_{10} 2.72102 - 1.2 = -0.01709$ ($-ve$)
since, $f(x_2)$ and $f(x_1)$ are of opposite sign, the new interval is $(x_2, x_1) = (2.72102, 3)$

The second approximation to the root x_3 is given by

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)} \\ &= 2.72102 - \frac{3 - 2.72102}{0.23136 - (-0.01709)} \\ &= 2.74021 \end{aligned}$$

Now, $f(x_3) = f(2.74021) = (2.74021) \log_{10} 2.74021 - 1.2 = -0.00038$ ($-ve$)
since $f(x_3)$ and $f(x_1)$ are of opposite sign, the new interval is $(x_3, x_1) = (2.74021, 3)$

The 3rd approximation to the root x_3 is given by

$$\begin{aligned} x_4 &= x_3 - \frac{f(x_3)(x_1 - x_3)}{f(x_1) - f(x_3)} \\ &= 2.74021 - \frac{3 - 2.74021}{0.23136 - (-0.00038)} \\ &= 2.74064 \end{aligned}$$

Now, $f(x_4) = f(2.74064) = (2.74064) \log_{10} 2.74064 - 1.2 = -0.000005$ ($-ve$)

since $f(x_4) = f(2.74064) \approx 0$, we conclude that $x = 2.7406$ is a root correct to 4 decimal places.

Another form of Regula Falsi Formula :

If the root lies in the interval (a, b) , then Regula Falsi Formula can also be written in the following form:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Problem 4.4.3. Use the Regula-falsi method to find a real root of the equation $\cos x = xe^x$, which lies in $(0, 1)$. Carryout 3 iterations. Write the answer correct to 5 decimal places.

Solution: Let $f(x) = \cos x - xe^x = 0$

Given $(a, b) = (0, 1)$

So that $f(a) = f(0) = 1$, and $f(b) = f(1) = \cos 1 - e = -2.17798$

The first approximation to root (i.e. x_1) is given by

$$\begin{aligned} x_1 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1} \\ &= 0.31467 \end{aligned}$$

Now $f(x_1) = f(0.31467) = 0.51987$

i.e., the root lies between $(a, b) = (0.31467, 1)$.

$f(a) = f(0.31467) = 0.51987$ and $f(b) = f(1) = -2.17798$

The second approximation to the root (i.e. x_2) is given by

$$\begin{aligned} x_2 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0.31467(-2.17798) - 1(0.51987)}{-2.17798 - 0.51987} \\ &= 0.44673 \end{aligned}$$

Now $f(x_2) = f(0.44673) = 0.20356$

the root lies between $(a, b) = (0.44673, 1)$ $f(a) = f(0.44673) = 0.20356$ and $f(b) = f(1) = -2.17798$

The third approximation to the root (i.e. x_3) is given by

$$\begin{aligned} x_3 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0.44673(-2.17798) - 1(0.20356)}{-2.17798 - 0.20356} \\ &= 0.49402 \end{aligned}$$

After 3 iterations, the root of the equation $\cos x = xe^x$ is $x = 0.49402$ correct to 5 decimal places.

Problem 4.4.4. Using Regula Falsi Method, compute the real root of the equation $xe^x - 2 = 0$ which lies in $(0.5, 1)$ correct up to three decimals places.

Solution: Given that root lies in the interval $(a, b) = (0.5, 1)$

$$\text{Now, } f(0.5) = 0.5 \times e^5 - 2 = -1.17 \text{ (-ve)}$$

$$f(1) = 1 \times e^1 - 2 = 0.718 \text{ (+ve)}$$

Since, $(a, b) = (0.5, 1)$, we have $f(a) = f(0.5) = -1.17$ and $f(b) = f(1) = 0.718$

By method of false position, we have

$$\begin{aligned} x_1 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0.5(0.718) - 1(-1.17)}{0.718 - (-1.17)} \\ &= 0.8098 \end{aligned}$$

$$\text{Now, } f(x_1) = f(0.8098) = 0.8098e^{0.8098} - 2 = -0.18 \text{ (negative)}$$

Root lies between $a = 0.8098$ and $b = 1$

$$f(a) = f(0.8098) = -0.18 \text{ and } f(b) = f(1) = 0.718$$

$$\begin{aligned} x_2 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0.8098(0.718) - 1(-0.18)}{0.718 - (-0.18)} \\ &= 0.8479 \end{aligned}$$

Now, $f(x_2) = f(0.8479) = 0.8479 - e^{0.8479} - 2 = -0.02$ (negative)

Root lies between $a = 0.8479$ and $b = 1$

$f(a) = f(0.8479) = -0.02$ and $f(b) = f(1) = 0.718$

$$\begin{aligned} x_3 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0.8479(0.718) - 1(-0.02)}{0.718 - (-0.02)} \\ &= 0.8519 \end{aligned}$$

Now, $f(x_3) = f(0.8519) = e^{0.8519} - 2 = -0.0031$ (negative)

Root lies between $a = 0.8519$ and $b = 1$

$f(a) = f(0.8519) = -0.0031$ and $f(b) = f(1) = 0.718$

$$\begin{aligned} x_4 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0.8519(0.718) - 1(-0.0031)}{0.718 - (-0.0031)} \\ &= 0.8525 \end{aligned}$$

Now, $f(x_4) = f(0.8525) = 0.8525 - e^{0.8525} - 2 = -0.0005$ (negative)

Root lies between $a = 0.8525$ and $b = 1$

$f(a) = f(0.8525) = -0.0005$ and $f(b) = f(1) = 0.718$

$$\begin{aligned} x_5 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{0.8525(0.718) - 1(-0.0005)}{0.718 - (-0.0005)} \\ &= 0.8525 \end{aligned}$$

Now, $x_4 = x_5 = 0.8525$. Hence approximate root correct to 4 decimal of place is
, $x = 0.8525$

4.5 Forward and Backward Differences

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \dots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \dots, \Delta y_{n-1}$ are respectively, called the first forward differences where Δ is the forward difference operator.

Thus the first forward differences are

$$\Delta y_r = y_{r+1} - y_r, \quad r = 0, 1, 2, 3, \dots$$

The difference of first forward differences is called second forward differences. i.e,

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r, \quad r = 0, 1, 2, 3, \dots$$

Similarly, the other higher order differences namely the third; fourth, etc. can be obtained using the following general formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r$$

defines the n^{th} forward differences.

The tabular arrangement of these differences, known as the forward difference table, illustrates how the forward differences of all orders can be systematically formed.

The Forward Difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0	Δy_0				
x_1	y_1	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$		
x_2	y_2	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_0$	
x_3	y_3	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_0$
x_4	y_4	Δy_4	$\Delta^2 y_3$			
x_5	y_5					

4.6 Backward differences :

The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ respectively are called first backward differences where ∇ is the backward difference operator.

Similarly, we define higher order backward differences as,

$$\nabla y_r = y_r - y_{r-1}$$

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

Backward Difference Table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
x_1	y_1	∇y_1				
x_2	y_2	∇y_2	$\nabla^2 y_2$			
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$		
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$	
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$

4.7 Interpolation:

If $y_0, y_1, y_2, \dots, y_n$ be the set of values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \dots, x_n$, the process of finding the values of y for any given value of x between x_0 and x_n is called interpolation. Similarly, the process of finding the values of y outside the given range of x is called extrapolation.

4.8 Newton's Forward Interpolation Formula [NFIF]

Let $y = f(x)$ be a function which takes values $y_0 = f(x_0), y_1 = f(x_0 + h), y_2 = f(x_0 + 2h), \dots, y_n$ corresponding to various equi-spaced values of x with spacing h , denoted as $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$.

Suppose we wish to evaluate the function $y = f(x)$ for a value $x_p = x_0 + ph$ where p is any real number. Then, for any real number p ,

$$y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

This formula is mainly used for interpolating the values of y near the beginning of a set of tabular values.

4.9 Newton's Backward Interpolation Formula [NBIF]

The value of $y = f(x)$ at $x_p = x_n + ph$ is approximately given by

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

This formula is mainly used for interpolating the values of y near the end of a set of tabular values (second half).

4.10 Step by step working procedure for the problems :

Step 1: We construct the difference table in accordance with the interpolation formula. **Step 2:** We compute the value of p , where:

- $p = \frac{x_p - x_0}{h}$ in the case of the forward interpolation formula, with x_0 being the first value of x and h being the step length.
- $p = \frac{x_p - x_n}{h}$ in the case of the backward interpolation formula, with x_n being the last value of x and h being the step length.

Step 3: The value of p along with the finite differences is substituted into the interpolation formula, resulting in the value of y at the desired value of x .

Problem 4.10.1. The population of a town in the decimal census was as given below.
Estimate the population for the year 1895

year	1891	1901	1911	1921	1931
Population(in thousands)	46	66	81	93	101

Solution:

Here $x_0 = 1891$, $h = 10$, $x_p = 1895$

$$\Rightarrow p = \frac{x_p - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

The difference table is given by

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46	20	-5	2	-3
1901	66	15	-3	-1	
1911	81	12	-4		
1921	93	8			
1931	101				

$$\begin{aligned}
 y(x_p) &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \\
 &\quad + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \\
 y(1895) &= 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2}(-5) \\
 &\quad + \frac{(0.4)(0.4-1)(0.4-2)}{6}(2) \\
 &\quad + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3) \\
 &= 54.8528 \text{ thousands}
 \end{aligned}$$

Problem 4.10.2. Given $f(40) = 184$, $f(50) = 204$, $f(60) = 226$, $f(70) = 250$, $f(80) = 276$, $f(90) = 304$, find $f(85)$ using suitable interpolation formula.

Solution:

Here, we need to find y at $x = 85$.

Since the value $x = 85$ is in the second half of the table near $x = 90$, Newton's Backward Interpolation Formula (NBIF) is appropriate, and the backward differences are tabulated as below.

The difference table is given by:

x	y				
40	184				
		20			
50	204		2		
		22		0	
60	226		2		0
		24		0	
70	250		2		0
		26		0	
80	276		2		0
		28		0	
90	304				

We have Newton's backward difference interpolation formula

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

Here, $x_n = 90$, $h = 10$, $\nabla y_n = 28$, $\nabla^2 y_n = 2$, $x_p = 85$

$$p = \frac{x_p - x_n}{h}$$

$$p = \frac{85 - 90}{10} = -0.5$$

$$y = 304 + (-0.5)28 + \frac{(-0.5)((-0.5) + 1)}{2!}2 = 289.75$$

Thus $f(85) = 289.75$

Problem 4.10.3. In the following table, values of y are consecutive terms of a series in which 23.6 is the 6th term.

$x :$	3	4	5	6	7	8	9
$y :$	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Find the first and tenth terms of the series.

Solution: To find the first term (i.e., y when $x = 1$), let us use Newton's forward interpolation formula.

Here, $x_0 = 3, h = 1, x_p = 1$

$$\therefore p = \frac{x_p - x_0}{h} = -2$$

The difference table is given by:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8	3.6	2.5	0.5	0
4	8.4	6.1	3	0.5	0
5	14.5	9.1	3.5	0.5	0
6	23.6	12.6	4	0.5	0
7	36.2	16.6	4.5	0.5	0
8	52.8	21.1			
9	73.9				

$$y(x_p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

$$Y(1) = 4.8 + (-2) \times 3.6 + \frac{(-2)(-3)}{2}(2.5) + \frac{(-2)(-3)(-4)}{6}(0.5) = 3.1$$

To obtain tenth term, we use Newton's Backward interpolation formula

$$x_n = 9, h = 1, x_p = x_n + ph = 10$$

$$10 = 9 + p \Rightarrow p = 1$$

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

$$y_{10} = 73.9 + 1(21.1) + \frac{1(2)}{2!}(4.5) + \frac{1(2)(3)}{3!}(0.5) = 73.9 + 21.1 + 4.5 + 0.5 = 100$$

Problem 4.10.4. From the following table find the number of students who have obtained :

1. Less than 45 marks

2. Between 40 and 45 marks

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	31	42	51	35	31

Solution: : Let us first prepare a new table with the following data :

x: Marks	40	50	60	70	80
y: No. of students with marks < x	31	31+42=73	73+51=124	124+35=159	190

Now we shall first find y_{45} , the number of students with marks less than 45.

$$x_0 = 40, \quad h = 10, \quad x_p = 45$$

$$x_0 + hp = 40 + 10p = 45 \Rightarrow p = 0.5$$

The difference table is:

Marks less than (x)	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42	9		
50	73	51	-16	-25	
60	124	35	-4	12	37
70	159	31			
80	190				

By Newton's forward difference formula,

$$\begin{aligned}
 y(x_p) &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 \\
 &+ \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \\
 &= 31 + (0.5)(42) + \frac{(0.5)(0.5-1)}{2}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25) \\
 &+ \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(37) \\
 &= 47.8672 \approx 48
 \end{aligned}$$

Hence no. of students getting marks less than 45 = 48

By given data, no. of students getting marks less than 40 = 31

Hence no. of students getting marks between 40 and 45 = 48 - 31 = 17

Problem 4.10.5. Find the cubic polynomial which takes the following values :

$$\begin{array}{cccc} x : & 0 & 1 & 2 & 3 \\ f(x) : & 1 & 2 & 1 & 10 \end{array}$$

Here, $h = 1$. Hence using the formula, $x_p = x_0 + ph$, and choosing $x_0 = 0$, we get $p = \frac{x_p - x_0}{h} = \frac{x - 0}{1} = x$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1	1		
1	2	-1	-2	
2	1	9	10	12
3	10			

\therefore By Newton's forward difference formula,

$$\begin{aligned} y &= y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 \\ &= 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12) \\ &= 2x^3 - 7x^2 + 6x + 1 \end{aligned}$$

Simplify and verify !

Problem 4.10.6. Using Newton's backward interpolation formula find the interpolating polynomial from the following table and hence find $f(12.5)$.

x	10	11	12	13
y	22	24	28	34

Solution::The Backward difference table is :

x	$f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
10	22			
		2		
11	24		2	
		4		0
12	28		2	
		6		
13	34			

$$P = \frac{x-x_n}{h} = \frac{x-13}{1} = x - 13$$

By Newton's backward interpolation formula,

$$y(x_p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$y = 34 + (x-13)6 + \frac{(x-13)(x-13+1)2}{2}$$

$$y = 34 + 6x - 78 + (x-13)(x-12)$$

$$y = 34 + 6x - 78 + x^2 - 25x + 156$$

$$y = x^2 - 19x + 112$$

is the interpolating polynomial for the data.

Now

$$\begin{aligned} f(12.5) &= (12.5)^2 - 19(12.5) + 112 \\ &= 268.25 - 237.5 \end{aligned}$$

$$\therefore f(12.5) = 30.75$$

Problem 4.10.7. Given, $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$. Find $\sin 48^\circ$ using Newton's forward interpolation formula

Hint : Difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

Final Answer : $y(48) = 0.7431$

4.11 Lagrange's interpolation formula

Let $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \dots, y_n = f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \dots, x_n$ at unequal intervals then

$$\begin{aligned}
 y(x) &= \frac{(x - x_1)(x - x_2)(x - x_3) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \dots (x_0 - x_n)} y_0 \\
 &+ \frac{(x - x_0)(x - x_2)(x - x_3) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n)} y_1 \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_3) \dots (x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \dots (x_2 - x_n)} y_2 + \dots \\
 &+ \dots \\
 &+ \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} y_n
 \end{aligned}$$

Problem 4.11.1. Use Lagrange's interpolation formula to find y at $x = 10$ given

x	5	6	9	11
y	12	13	14	16

Solution:

$$\text{Let } x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$$

$$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$$

Lagrange's interpolation formula :

$$\begin{aligned}
 y(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 \\
 &+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3
 \end{aligned}$$

When $x = 10$

$$\begin{aligned} y &= \frac{4(1)(-1)12}{(-1)(-4)(-6)} + \frac{5(1)(-1)13}{(1)(-3)(-5)} \\ &+ \frac{5(4)(-1)14}{(4)(3)(-2)} + \frac{5(4)(1)16}{(6)(5)(2)} \\ &= 14.666 \end{aligned}$$

Thus y at $x = 10$ is 14.67

Problem 4.11.2. Given the values

$x :$	5	7	11	13	17
$f(x) :$	150	392	1452	2366	5202

Evaluate $f(9)$, using Lagrange's formula.

Solution:

$$x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$$

$$y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202$$

Lagrange's interpolation formula :

$$\begin{aligned} y(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} y_0 \\ &+ \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} y_1 \\ &+ \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} y_2 \\ &+ \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} y_3 \\ &+ \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} y_4 \end{aligned}$$

Putting $x = 9$ and substituting the given values in Lagrange's formula, we get

$$\begin{aligned}
 f(9) &= \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\
 &+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\
 &+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\
 &+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\
 &+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 \\
 &= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810
 \end{aligned}$$

Problem 4.11.3. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

$x :$	0	1	2	5
$f(x) :$	2	3	12	147

Solution:

$$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$$

$$y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147$$

Lagrange's interpolation formula :

$$\begin{aligned}
 y(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 \\
 &+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3
 \end{aligned}$$

$$\begin{aligned}
 f(x) &= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) \\
 &+ \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) \\
 &+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) \\
 &+ \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)
 \end{aligned}$$

$$f(x) = x^3 + x^2 - x + 2 \text{ (on simplification)}$$

$$f(3) = 27 + 9 - 3 + 2 = 35$$

Problem 4.11.4. Using Lagrange formula, calculate $f(3)$ from the following table.

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solution:: Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$
 $y_0 = f(x_0) = 1, y_1 = f(x_1) = 14, y_2 = f(x_2) = 15, y_3 = f(x_3) = 5,$
 $y_4 = f(x_4) = 6, y_5 = f(x_5) = 19$

$$\begin{aligned}
 y(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0 \\
 &+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5
 \end{aligned}$$

Here $x = 3$ then

$$\begin{aligned}
 f(3) &= \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1 \\
 &+ \frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14 \\
 &+ \frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 \\
 &+ \frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5 \\
 &+ \frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 \\
 &+ \frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19 \\
 &= \frac{12}{240} - \frac{18}{60} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{40} \times 19 \\
 &= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95 = 10 \\
 f(3) &= 10
 \end{aligned}$$

Problem 4.11.5. Find y at $x = 5$ if $y(1) = -3, y(3) = 9, y(4) = 30, y(6) = 132$ using Lagrange's interpolation formula.

Solution : Here $x + 0 = 1, x_1 = 3, x_2 = 4, x_3 = 6$ and $y_0 = -3, y_1 = 9, y_2 = 30, y_3 = 132$

$$\begin{aligned}
 y = f(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 \\
 &+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3
 \end{aligned}$$

Substitute $x = 5$ and other values from the given data,

$$\begin{aligned}
 y(5) &= \frac{(5-3)(5-4)(5-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(5-1)(5-4)(5-6)}{(3-1)(3-4)(3-6)}9 + \\
 &\frac{(5-1)(5-3)(5-6)}{(4-1)(4-3)(4-6)}30 + \frac{(5-1)(5-3)(5-4)}{(6-1)(6-3)(6-4)}132. \\
 &= -0.2 + (-6) + 40 - 35.2 \\
 &= -1.4
 \end{aligned}$$

Problem 4.11.6. If $y(0) = -12, y(1) = 0, y(3) = 6$ and $y(4) = 12$, find the Lagrange's interpolation polynomial and estimate $y(2)$.

Solution :

$$\begin{aligned}
 y = f(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}y_0 \\
 &+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}y_1 \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}y_2 \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}y_3 \\
 &= \frac{(x - 1)(x - 3)(x - 4)(-12)}{(-1)(-3)(-4)} \\
 &+ \frac{(x)(x - 3)(x - 4)(0)}{(1)(-2)(-3)} \\
 &+ \frac{x(x - 1)(x - 4)6}{(3)(2)(-1)} \\
 &+ \frac{x(x - 1)(x - 3)12}{(4)(3)(1)} \\
 &= x^3 - 7x^2 + 18x - 12 \text{ (after simplification)}
 \end{aligned}$$

putting $x = 2$, we get

$$y(2) = 2^3 - 7(2^2) + 18(2) - 12 = 4$$

Problem 4.11.7. Using Lagrange's interpolation formula find a polynomial which passes through the points $(0, -12)$, $(1, 0)$, $(3, 6)$ and $(4, 12)$.

Solution : Here, $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$, $y_0 = -12$, $y_1 = 0$, $y_2 = 6$, $y_3 = 12$

Using Lagrange's formula,

$$\begin{aligned}
 y(x) &= \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \times (-12) \\
 &+ \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} \times 0 \\
 &+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \times 6 \\
 &+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \times 12 \\
 &= \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)} (-12) + 0 \\
 &+ \frac{(x)(x-1)(x-4)}{(3)(2)(-1)} \times 6 + \frac{(x)(x-1)(x-3)}{(4)(3)(1)} \times 12 \\
 &= \frac{(x-1)(x^2-7x+12)}{(-12)} \times (-12) \\
 &+ \frac{x(x^2-5x+4)}{-6} \times 6 \\
 &+ \frac{x(x^2-4x+3)}{12} \times 12 \\
 &= (x^3-7x^2+12x-x^2+7x-12) - (x^3-5x^2+4x) \\
 &+ (x^3-4x^2+3x) \\
 &= (x^3-8x^2+19x-12) - (x^3-5x^2+4x) \\
 &+ (x^3-4x^2+3x) \\
 &= x^3-8x^2+19x-12-x^3+5x^2-4x+x^3-4x^2+3x \\
 &= x^3-7x^2+18x-12
 \end{aligned}$$

4.12 Divided Differences :

Let $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \dots, x_n$ at unequal intervals.

The first order divided differences are defined as follows

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0};$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1};$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}; \dots$$

$$\text{In general } f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

The second order divided differences are defined as follows

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

⋮

$$\text{In general } f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$$

Similarly the other higher order divided differences are defined.

4.13 Newton's divided difference formula :

$$\begin{aligned} y(x) = f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) \\ &+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \dots \\ &\vdots \\ &+ (x - x_0)(x - x_1) \dots (x - x_{n-1}) f(x_0, x_1, x_2, \dots, x_n) \end{aligned}$$

Problem 4.13.1. Use Newton's divided difference formula to find $f(4)$ given the

data :

x	0	2	3	6
$f(x)$	-4	2	14	158

Solution: Here,

$$x_0 = 0, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 6$$

$$f(x_0) = -4, \quad f(x_1) = 2, \quad f(x_2) = 14, \quad f(x_3) = 158$$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2 + 4}{2 - 0} = 3$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{14 - 2}{3 - 2} = 12$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{158 - 14}{6 - 3} = 48$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{12 - 3}{3 - 0} = 3$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{48 - 12}{6 - 2} = 9$$

Third order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$= \frac{9 - 3}{6 - 0} = 1$$

Newton's divided difference formula is given by:

$$y(x) = f(x) = f(x_0) + (x - x_0) f(x_0, x_1)$$

$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$y(4) = -4 + (4 - 0)3 + (4 - 0)(4 - 2)3 + (4 - 0)(4 - 2)(4 - 3)1$$

$$= 40$$

Problem 4.13.2. Given the values

$x :$	5	7	11	13	17
$f(x) :$	150	392	1452	2366	5202

Calculuate $f(9)$, using Newton's divided difference formula.

Solution: : Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

$f(x_0) = 150, f(x_1) = 392, f(x_2) = 1452, f(x_3) = 2366, f(x_4) = 5202$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{392 - 150}{7 - 5} = 121$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1452 - 392}{11 - 7} = 265$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{2366 - 1452}{13 - 11} = 457$$

$$f(x_3, x_4) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{5202 - 2366}{17 - 13} = 709$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{265 - 121}{11 - 5} = 24$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{457 - 265}{13 - 7} = 32$$

$$f(x_2, x_3, x_4) = \frac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2} = \frac{709 - 457}{17 - 11} = 42$$

Third order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3) = \frac{32 - 24}{13 - 5} = 1$$

$$f(x_1, x_2, x_3, x_4) = \frac{42 - 32}{17 - 7} = 1$$

Newton's divided difference formula is given by:

$$\begin{aligned} y(x) = f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) \\ &+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \\ &+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

Taking $x = 9$ in the above divided difference formula, we obtain

$$\begin{aligned} f(9) &= 150 + (9 - 5) \times 121 + (9 - 5)(9 - 7) \times 24 \\ &+ (9 - 5)(9 - 7)(9 - 11) \times 1 = 150 + 484 + 192 - 16 = 810 \end{aligned}$$

Problem 4.13.3. Determine $f(x)$ as a polynomial in x for the following data:

$$x : \quad -4 \quad -1 \quad 0 \quad 2 \quad 5$$

$$f(x) : \quad 1245 \quad 33 \quad 5 \quad 9 \quad 1335$$

Solution: Here

$$x_0 = -4, \quad f(x_0) = 1245$$

$$x_1 = -1, \quad f(x_1) = 33$$

$$x_2 = 0, \quad f(x_2) = 5$$

$$x_3 = 2, \quad f(x_3) = 9$$

$$x_4 = 5, \quad f(x_4) = 1335$$

First order Divided Differences are give by,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{33 - 1245}{-1 + 4} = -404$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{5 - 33}{0 + 1} = -28$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{9 - 5}{2 - 0} = 2$$

$$f(x_3, x_4) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{1335 - 9}{5 - 2} = 442$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{-28 - (-404)}{0 - (-4)} = 94$$

$$f(x_1, x_2, x_3) = \frac{2 - (-28)}{2 - (-1)} = 10$$

$$f(x_2, x_3, x_4) = \frac{442 - 2}{5 - 0} = 88$$

Third order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3) = \frac{10 - 94}{2 - (-4)} = -14$$

$$f(x_1, x_2, x_3, x_4) = \frac{88 - 10}{5 - 0} = 13$$

4th order Divided Difference is given by,

$$f(x_0, x_1, x_2, x_3, x_4) = \frac{13 + 14}{5 - (-4)} = 3$$

Applying Newton's divided difference formula,

$$\begin{aligned} y(x) = f(x) = & f(x_0) + (x - x_0) f(x_0, x_1) \\ & + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ & + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \\ & + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

$$\begin{aligned}
f(x) &= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + \\
&\quad + (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3) \\
&= 1245 - 404x - 1616 + (94) [x^2 + 5x + 4] \\
&\quad - 14x [x^2 + 5x + 4] + 3x [(x^2 + 5x + 4)(x - 2)] \\
f(x) &= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 \\
&\quad - 70x^2 - 56x + 3x [x^3 - 2x^2 + 5x^2 - 10x + 4x - 8] \\
f(x) &= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 \\
&\quad - 70x^2 - 56x + 3x^4 - 6x^3 + 15x^3 - 30x^2 + 12x^2 - 24x \\
&= 3x^4 - 5x^3 + 6x^2 - 14x + 5
\end{aligned}$$

Problem 4.13.4. By means of Newton's divided difference formula, find $f(8)$ from the following data-

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: $x_0 = 4, x_1 = 5, x_2 = 7, x_4 = 10, x_5 = 11, x_6 = 13$

$f(x_0) = 48, f(x_1) = 100, f(x_2) = 294, f(x_3) = 900, f(x_4) = 1210,$
 $f(x_5) = 2028$

First order Divided Differences are give by,

$$\begin{aligned}
f(x_0, x_1) &= \frac{100 - 48}{5 - 4} = 52 \\
f(x_1, x_2) &= \frac{294 - 100}{7 - 5} = 97 \\
f(x_2, x_3) &= \frac{900 - 294}{10 - 7} = 202 \\
f(x_3, x_4) &= \frac{1210 - 900}{11 - 10} = 310 \\
f(x_4, x_5) &= \frac{2028 - 1210}{13 - 11} = 409
\end{aligned}$$

Second order Divided Differences are give by,

$$f(x_0, x_1, x_2) = \frac{97 - 52}{7 - 4} = 15$$

$$f(x_1, x_2, x_3) = \frac{202 - 97}{10 - 5} = 21$$

$$f(x_2, x_3, x_4) = \frac{310 - 202}{11 - 7} = 27$$

$$f(x_3, x_4, x_5) = \frac{409 - 310}{13 - 10} = 33$$

Third order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3) = \frac{21 - 15}{10 - 4} = 1$$

$$f(x_1, x_2, x_3, x_4) = \frac{27 - 21}{11 - 5} = 1$$

$$f(x_2, x_3, x_4, x_5) = \frac{33 - 27}{13 - 7} = 1$$

4th order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3, x_4) = \frac{1 - 1}{11 - 4} = 0$$

$$f(x_1, x_2, x_3, x_4, x_5) = \frac{1 - 1}{13 - 5} = 0$$

5th order Divided Differences are give by,

$$f(x_0, x_1, x_2, x_3, x_4, x_5) = 0$$

Applying Newton's divided difference formula,

$$\begin{aligned} y(x) = f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) \\ &+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \\ &+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

Substituting the $x = 8$ and divided differences in the above equation,

$$\begin{aligned} f(8) &= 48 + (8 - 4)(52) + (8 - 4)(8 - 5)(15) \\ &+ (8 - 4)(8 - 5)(8 - 7)(1) \\ &+ (8 - 4)(8 - 5)(8 - 7)(8 - 10)(0) \\ &= 48 + (4)(52) + (4)(3)(15) + (4)(3)(1)(1) + 0 \\ &= 448 \end{aligned}$$

Problem : Find the equation of the polynomial which passes through the points $(4, -43)$, $(7, 83)$, $(9, 327)$ and $(12, 1053)$ using Newton's divided difference formula

$$\text{Ans: } f(x) = x^3 - 4x^2 - 7x - 15$$

Practice Problem :

Construct the interpolation polynomial for the data given below using Newton's divided difference formula.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

Hence find $f(9)$

$$\text{Ans: } f(x) = 2x^3 - 3x^2 + 5x - 4 \text{ and } f(9) = 1256$$

Practice Problem :

Given,

x	-4	-1	0	2	5
y	1245	33	5	9	1335

Determine $f(x)$ as a polynomial in x for the following data using Newton's difference formula.

$$\text{Ans: } y = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Practice Problem :

Using Newton's divided difference formula find $f(8)$, $f(15)$ from the following data

X	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

$$\text{Ans: } f(8) = 448, f(15) = 3150$$

4.14 Numerical Integration :

Let $I = \int_a^b y dx$, where y takes the values $y_0, y_1, y_2, \dots, y_n$ for $x = x_0, x_1, x_2, \dots, x_n$

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \frac{b-a}{n}$ so that $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n =$

$$x_0 + nh = b$$

$$\therefore I = \int_{x_0}^{x_0+nh} f(x) dx$$

Numerical Integration is the process of obtaining approximately the value of the definite integral $I = \int_a^b y dx$ without actually integrating the function but only using the values of y at some points of x equally spaced over $[a, b] = [x_0, x_0 + nh]$.

4.15 Three rules to obtain the value of the definite integral:

- Trapezoidal rule :

$$I = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})]$$

- Simpson's (1/3)rd rule:

$$I = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

While using this formula, the given interval of integration must be divided into an even number of sub-intervals (i.e. n =even)

- Simpson's (3/8)th rule:

$$I = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})]$$

While using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

4.16 Working procedure for problems:

Step 1: Given the definite integral $I = \int_a^b y dx$ for evaluation, first divide the interval $[a, b]$ into n equal parts (strips) of width $h = \frac{(b-a)}{n}$

Step 2: Prepare a table consisting the values of x and the corresponding computed values of y

Step 3: Substitute values from this table into the appropriate rule to obtain the approximate value of the given definite integral.

Note: Number of ordinates = $n + 1$ where n is the number of sub intervals.

Problem 4.16.1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

- (i) Simpson's 1/3 rule,
(ii) Simpson's 3/8 rule,

Solution:: Here, $a = 0$ and $b = 6$.

Let $n = 6$.

Hence $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$.

Thus Divide the interval $(0, 6)$ into six parts each of width $h = 1$.

The values of $y = f(x) = \frac{1}{1+x^2}$ are given below :

Computational table:

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.05884	0.0385	0.027
$= y$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Simpson's 1/3 rule,

$$\begin{aligned} \int_0^6 \frac{1}{1+x^2} &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{1}{3} [(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] \\ &= 1.3662 \end{aligned}$$

(ii) By Simpson's 3/8 rule,

$$\begin{aligned} \int_0^6 \frac{1}{1+x^2} &= \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3] \\ &= \frac{3}{8} [(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] \\ &= 1.3571 \end{aligned}$$

Actual value : Let us evaluate the integral explicitly.

$$\begin{aligned}\int_0^6 \frac{1}{1+x^2} dx &= [\tan^{-1}x]_0^6 \\ &= \tan^{-1}(6) - \tan^{-1}(0) = 1.4056\end{aligned}$$

(Note : Keep the Calculator in Radian Mode)

Problem 4.16.2. Calculate the value of the integral

$$\int_4^{5.2} \log x dx \text{ by}$$

(a) Simpson's 1/3 rule (b) Simpson's 3/8 rule Compare the answers with exact solution.

Solution: Here $a = x_0 = 4$, $b = x_n = 5.2$

Let $n = 6$. Then $h = \frac{x_n - x_0}{n} = 0.2$.

Taking $h = 0.2$, divide the range of integration (4, 5.2) into six equal parts.

The values of $\log x$ for each point of sub-division are given below:

Computational table:

X	$y = f(x) = \log x = \ln(x)$
$x_0 = 4$	$y_0 = 1.3863$
$x_1 = x_0 + h = 4.2$	$y_1 = 1.4351$
$x_2 = x_0 + 2h = 4.4$	$y_2 = 1.4816$
$x_3 = x_0 + 3h = 4.6$	$y_3 = 1.5261$
$x_4 = x_0 + 4h = 4.8$	$y_4 = 1.5686$
$x_5 = x_0 + 5h = 5.0$	$y_5 = 1.6094$
$x_6 = x_0 + 6h = 5.2$	$y_6 = 1.6487$

(a) By Simpson's 1/3 rule, we have

$$\begin{aligned}\int_4^{5.2} \log x dx &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{0.2}{3} [(1.3863 + 1.6487) + 4(1.4351 + 1.5261 + 1.6094) + 2(1.4816 + 1.5686)] \\ &= 1.8278472.\end{aligned}$$

(b) By Simpson's $3/8$ rule, we have

$$\begin{aligned}\int_4^{5.2} \log x dx &= \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3] \\ &= \frac{3(0.2)}{8} [(1.3863 + 1.6487) + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2(1.520)] \\ &= 1.8278\end{aligned}$$

Exact value of the integral is

$$\begin{aligned}\int_4^{5.2} \log x dx &= [x(\log x - 1)]_4^{5.2} \\ &= [5.2(\log 5.2 - 1) - 4(\log 4 - 1)] = 3.3730249 - 1.5451774 = 1.8278475\end{aligned}$$

Problem 4.16.3. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $1/3^{\text{rd}}$ rule taking $h = \frac{1}{4}$

Solution : Given: $f(x) = \frac{1}{1+x^2}$ and $a = 0, b = 1, h = \frac{1}{4}$
 $\therefore n = \frac{b-a}{h} = \frac{1-0}{1/4} = 4$

Computational table:

x	$y_x = f(x) = \frac{1}{1+x^2}$
$x_0 = a = 0$	$y_0 = \frac{1}{1+0^2} = 1$
$x_1 = x_0 + h = 0 + 1/4 = 1/4$	$y_1 = \frac{1}{1+(1/4)^2} = 0.9411$
$x_2 = x_0 + 2h = 0 + 2(1/4) = 1/2$	$y_2 = \frac{1}{1+(1/2)^2} = 0.8$
$x_3 = x_0 + 3h = 0 + 3(1/4) = 3/4$	$y_3 = \frac{1}{1+(3/4)^2} = 0.64$
$x_4 = x_0 + 4h = 0 + 4(1/4) = 1$	$y_4 = \frac{1}{1+1^2} = 0.5$

By Simpson's $1/3$ rule:

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &= \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)] \\ &= \frac{1}{12} [(1 + 0.5) + 4(0.9411 + 0.64) + 2(0.8)] \\ &= 0.7853\end{aligned}$$

Problem 4.16.4. Evaluate $\int_0^6 \frac{1}{1+x} dx$ by using (i) Simpson's $\frac{1}{3}$ rule (ii) Simpson's $\frac{3}{8}$ rule and compare the result with its actual value.

Solution : All the formulae are applicable if n , the number of intervals is a multiple of six. So we divide the interval $(0, 6)$ into equal parts each of width, $h = \frac{6-0}{6} = 1$. The values of $y = f(x)$ are given below.

Computational table:

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$	1	0.5	0.3333	0.25	0.2	0.1666	0.1428
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Simpson's $\frac{1}{3}$ rule,

$$\begin{aligned} \int_0^6 \frac{1}{1+x} dx &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{1}{3} [(1 + 0.1428) + 4(0.5 + 0.25 + 0.1666) + 2(0.3333 + 0.2)] \\ &= \frac{1}{3} (1.1428 + 3.6664 + 1.0666) \\ &= \frac{1}{3} (5.8758) \\ &= 1.9586 \end{aligned}$$

(ii) By Simpson's $\frac{3}{8}$ rule,

$$\begin{aligned} \int_0^6 \frac{1}{1+x} dx &= \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3] \\ &= \frac{3}{8} [(1 + 0.1428) + 3(0.5 + 0.3333 + 0.2 + 0.1666) + 2(0.25)] \\ &= \frac{3}{8} [1.1428 + 3.5997 + 0.5] = \frac{3}{8} (5.2425) \\ &= 1.9659 \end{aligned}$$

By actual integration,

$$\begin{aligned} \int_0^6 \frac{1}{1+x} dx &= [\log(1+x)]_0^6 = \log 7 - \log 1 \\ &= \log 7 \\ &= 1.94591 \end{aligned}$$

From the above problem, we conclude that Simpson's $1/3$ rule is more accurate than the Simpson's $3/8$ rule.

Problem 4.16.5. Evaluate $\int_0^{\pi/2} \sqrt{\cos x} dx$ by using the trapezoidal rule by taking 7 ordinates.

$$\therefore f(x) = \sqrt{\cos x}$$

$$a = 0, \quad b = \pi/2, \quad \text{no. of ordinates, } n + 1 = 7$$

$$\Rightarrow n = 6.$$

$$h = \frac{\frac{\pi}{2} - 0}{6} \Rightarrow h = \frac{\pi}{12}$$

Computational table:

x	0	$\frac{\pi}{12} = 15^\circ$	30°	45°	60°	75°	90°
$y = \sqrt{\cos x}$	1	0.9828	0.9306	0.8409	0.7071	0.5087	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Here only in table, we used x values in degrees, obtained y values using the calculator in degree mode. However, in the trapezoidal rule the value of h to be substituted is $h = \frac{\pi}{12} = 0.26179$ (by directly using π key)

By Trapezoidal rule,

$$\begin{aligned} \int_0^{\pi/2} \sqrt{\cos x} dx &= \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)] \\ &= \frac{0.26179}{2} [(1 + 0) + 2(0.9828 + 0.9306 + 0.8409 + 0.7071 + 0)] \\ &= \frac{0.26179}{2} [8.9402] \\ &= 1.1702 \end{aligned}$$

Problem 4.16.6. Evaluate $\int_0^1 \frac{1}{1+x} dx$ by taking 7 ordinates and by using simpson's 3/8 th rule.

Here,

$$y = f(x) = \frac{1}{1+x}, \quad a = 0, b = 1, \quad \text{no. of ordinates, } n + 1 = 7 \Rightarrow n = 6$$

$$h = \frac{b - a}{n} = \frac{1}{6}$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$y = \frac{1}{1+x}$	1	0.8571	0.75	0.6667	0.6	0.5455	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ rule,

$$\begin{aligned}\int_0^6 \frac{1}{1+x} dx &= \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3] \\ &= \frac{3 \times \frac{1}{6}}{8} [(1 + 0.5) + 3(0.8571 + 0.75 + 0.6 + 0.5455) + 2(0.6667)] \\ &= \frac{3}{48} (11.0912) \\ &= 0.6932\end{aligned}$$

Problem 4.16.7. Use Simpson's 1/3 rd rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking seven ordinates.

Solution: Divide the interval $(0, 0.6)$ into $n = 6$ parts each of width $h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$.

The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
$y = e^{-x^2}$	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rd rule, we have

$$\begin{aligned}\int_0^{0.6} e^{-x^2} dx &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{0.1}{3} [(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)] \\ &= \frac{0.1}{3} [1.6977 + 10.7308 + 3.6258] \\ &= \frac{0.1}{3} (16.0543) \\ &= 0.5351.\end{aligned}$$

4.17 Question Bank : Module 4 - Numerical Methods I

Regula Falsi and Newton Raphson Methods

1. Using Newton-Raphson method, find the real root of the equation $3x = \cos x + 1$ correct to four decimal places. (VTU Model 2022)
2. Find a real root of $x^3 - 9x + 1 = 0$ in $(2, 3)$ by the Regula-Falsi method in four iterations (VTU Model 2022)
3. Use Regula Falsi method to find a real root of the equation $x^3 - 2x - 5 = 0$, correct to 3 decimal places. [VTU - June 2018]
4. Find a real root of $xe^x - \cos x = 0$ correct to three decimal places lying in the interval $(0.5, 0.6)$, using Regula-Falsi method. [VTU-Model 2018]
5. Use the Regula-falsi method to find a real root of the equation $\cos x = xe^x$, which lies in $(0, 1)$. Write the answer correct to 4 decimal places. (VTU Model 2022)
6. Use the Regula-falsi method to find a real root of $x \log_{10} x - 1.2 = 0$ [VTU:Jan 2020, June 2019, Model 2018, Dec 2010/July16]
7. Use the Regula-falsi method to find a real root of $\cos x = 3x - 1$ [VTU: Dec 2012/July 13/July 15] Ans: **0.607**
8. Use the Regula-falsi method to find a real root of $xe^x - 2 = 0$ [VTU- Jan17]
9. Use the Regula-falsi method to find a real root of $xe^x - 3 = 0$, correct to three decimal places. [VTU Jan 2018]
10. Use the regula falsi method to find the forth root of 12 correct to 3 decimal places [VTU: Dec2015/Jan17]
11. Use the Regula Falsi method to find the root of the equation $2x - \log_{10} x = 7$ which lies between 3.5 and 4 [VTU-June 12/June17] Ans: **3.7893**

12. Find a real root of the equation $x^3 + x^2 + 3x + 4 = 0$ near $x = -1$, correct to four decimal places using Newton- Raphson method. [VTU – Model 2018]
13. Use Newton Raphson method to find a real root of $x \sin x + \cos x = 0$, near $x = \pi$. Carryout the iterations upto four decimal places of accuracy. [VTU – Model 2022, Jan 2020, June 2019, Model 2018, Dec13/Dec14/Jan15/June17]
14. Find the real root of the equation $xe^x - 2 = 0$ correct to three decimal places using Newton- Raphson method. [VTU- Jan17]
15. Using Newton-Raphson method, find the root that lies near $x = 4.5$ of the equation $\tan x = x$ correct to four decimal places. [VTU- Jan 2018, Jan17]
16. Using Newton-Raphson method find the value of cube root of 18 correct to 2 decimals, assuming 2.5 as the initial approximation. [VTU – June17]
17. Find the real root of the equation $x \log_{10} x - 1.2 = 0$ using Newton-Raphson method [VTU – June 2018]
18. Compute one positive root of $2x - \log_{10} x = 7$ by Newton-Raphson method correct to four decimal places.
Ans : 3.7892
19. Use Newton-Raphson method to find a root of the equation $x^3 - 3x - 5 = 0$
Ans : 2.279
20. Find the negative root of the equation $x^3 - 4x + 9 = 0$ from Regula Falsi method. Ans: -2.7065
21. Find the root of the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by the method of false position. Ans : 2.7473
22. Find the root of the equation $4 \sin x = e^x$, that lies between 0 and 0.5. Correct to 4 places of decimals, using Regula-Falsi method. Ans :0.3706
23. Find the root of the equation $xe^x - 3 = 0$, that lies between 1 and 2, correct to 3 places of decimals, using the method of false position. Ans : 1.049

24. Use the Newton-Raphson method to find a root of the equation $xe^x - 2 = 0$ correct to 3 decimal places. (VTU 2005) Ans :0.853

25. Use the Newton-Raphson method to find a root of the equation $\cos x = xe^x$ correct to 3 decimal places. Ans : 0.518

Newton's Forward and Backward Interpolation formulas

Using Newton's forward interpolation find y at $x = 5$ from the data

1.

x	4	6	8	10
y	1	3	8	16

 (VTU Model 2022)

2. Using Newton's backward interpolation formula find the value of y when $x = 6$ from the given table

x	1	2	3	4	5
y	1	-1	1	-1	1

(VTU Model 2022)

3. Given $f(40) = 184$, $f(50) = 204$, $f(60) = 226$, $f(70) = 250$, $f(80) = 276$, $f(90) = 304$, find $f(42)$ using Newton's forward interpolation formula. [VTU Jan 2020]

4. Given $f(40) = 184$, $f(50) = 204$, $f(60) = 226$, $f(70) = 250$, $f(80) = 276$, $f(90) = 304$, find $f(42)$ and $f(85)$ using suitable interpolation formulae. [VTU: - Model 2018, June 2018, June 12/Jan 16]

5. Given $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$ Find $\sin 57^\circ$ using an appropriate interpolation formula. [VTU: - June 2018]

6. From the following table of half-yearly Premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age	45	50	55	60	65
Premium (in Rupees)	114.84	96.16	83.32	74.48	68.48

[VTU Jan

2018]

7. From the following table find the number of students who have obtained (a) less than 45 marks (b) between 40 and 45 marks.

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

[VTU- June 2019,

July15/Jan17]

8. From the following table find the number of students who have obtained less than 70 marks.

Marks	0-19	20-39	40-59	60-79	80-99
No. of students	41	62	65	50	17

[VTU – Jan 17]

9. Estimate the probable number of persons with daily income 20 to 25 Rs from the following table.

Income per day(Rs)	Under 10	10- 20	20- 30	30- 40	40- 50
No. of person	20	45	115	210	115

[VTU:

Jan14]

10. Estimate the probable number of getting wages below Rs 35 from the following data :

Wages (in Rs)	0-10	10- 20	20- 30	30- 40
Frequency	9	30	35	42

[VTU Jan 2018]

11. The population of the town is given by the table. Using Newton's forward and backward interpolation formula, calculate the increase in the population from the year 1955 to 1985

Year	1951	1961	1971	1981	1991
Population in 1000	19.96	39.65	58.81	77.21	94.61

[VTU-

Jan16/June17]

12. Using Newton's backward interpolation formula find the interpolating polynomial for the function given by the following data. Hence find $f(12.5)$

x	10	11	12	13
f(x)	22	24	28	34

[VTU - Jan17]

13. Using Newton's forward interpolation formula find y at $x=160$ for the following data.

x	100	150	200	250	300	350	400
$f(x)$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

[VTU: June13]

14. Use an appropriate interpolation formula to compute using the following data:

x	1.7	1.8	1.9	2.0	2.1	2.2
$f(x)$	5.474	6.050	6.686	7.389	8.166	9.025

[VTU Model 2018]

15. Estimate the value of $f(22)$ from the following available data:

x	20	25	30	35	40	45
y	354	332	291	260	231	204

Ans : 352.22304

16. The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x :	100	150	200	250	300	350	400
y :	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Use Newton's forward formula to find y when $x = 218$ ft. Ans : 15.6993 nautical miles.

17. Find the number of men getting wages between Rs. 10 and Rs. 15 from following table

Wages in R s :	0 – 10	10 – 20	20 – 30	30 – 40
Frequency :	9	30	35	42

Ans : 15

18. Estimate the value of $f(42)$ from the following available data:

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

Ans :219

19. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105.

Ans : 8666

20. Find a cubic polynomial in x which takes on the values $-3, 3, 11, 27, 57$ and 107 , when $x = 0, 1, 2, 3, 4$ and 5 respectively.

Ans:

$$y = x^3 - 2x^2 + 7x - 3$$

21. Find the cubic polynomial which takes the following values.

x	0	1	2	3
y	1	2	1	10

$$\text{Ans: } 2x^3 - 7x^2 + 6x + 1$$

Newton's Divided Difference interpolation Formula

1. Using Newton's divided difference formula find $f(9)$ from the following data

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

[VTU Model 2022, Jan 2020]

2. Construct the interpolation polynomial for the data given below using Newton's divided difference formula.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

Hence find $f(9)$. [VTU- Model 2018,

June 2018, June17] Ans: $f(x) = 2x^3 - 3x^2 + 5x - 4$, $f(3) = 38$ and $f(9) = 1256$

3. Find the equation of the polynomial which passes through the points $(4, -43)$, $(7, 83)$, $(9, 327)$ and $(12, 1053)$ using Newton's divided difference formula [VTU- Jan17]

$$\text{Ans: } f(x) = x^3 - 4x^2 - 7x - 15$$

4. Using Newton's divided difference formula find $f(82)$, $f(98)$ from the following data

x	80	85	90	95	100
f(x)	5026	5674	6362	7088	7854

[VTU: Dec12]

5. Given,

x	-4	-1	0	2	5
y	1245	33	5	9	1335

Determine $f(x)$ as a polynomial in x for the following data using Newton's difference formula. [VTU- Jan17]

Ans:

$$y = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

6. Using Newton's divided difference formula, find a polynomial for the data :

x	3	7	9	10
F(x)	168	120	72	63

Hence find y at $x = 8$

[VTU- Jan 2018]

7. Using Newton's divided difference formula, fit an interpolating polynomial for the data given below and hence find y at $x = 2$

x	0	1	4	8	10
f(x)	-5	-14	-125	-21	355

[VTU: June13]

8. Fit an interpolating polynomial for the data $u_{10} = 355, u_0 = -5, u_8 = -21, u_1 = -14, u_4 = -125$ by using Newton's general interpolation formula and hence find u_2

Ans:

$$f(x) = 2x^3 - 17x^2 + 6x - 5, u_2 = -45$$

9. Using Newton's divided difference formula find $f(8), f(15)$ from the following data

X	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Ans: $f(8) = 448, f(15) = 3150$

Lagrange's interpolation formula

1. Using Lagrange's interpolation formula, fit a polynomial which passes through the points $(-1, 0), (1, 2), (2, 9)$ and $(3, 8)$ and hence estimate the value of y when $x = 2.2$
- (VTU Model 2022)
2. Use Lagrange's interpolation formula to find y at $x = 10$ given

x	5	6	9	11
y	12	13	14	16

[VTU- Jan17]

3. Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
f(x)	-12	0	6	12

Hence estimate y at $x = 2$

[VTU: - June 2018, Jan16]

4. Using Lagrange's formula find the interpolating polynomial that approximates to the function described by the following table.

x	0	1	2	5
f(x)	2	3	12	147

[VTU: Jan 2018, July16]

5. Using Lagrange's formula find y at $x = 4$ by using the following data.

x	0	1	2	5
f(x)	2	3	12	147

[VTU: June 2019],

6. The following table gives the premium payable at ages in years completed. Interpolate the premium payable at age 35 completed, using Lagrange's formula.

Age completed	25	30	40	60
Premium in Rs	50	55	70	95

[VTU- Jan17]

7. Use Lagrange's interpolation formula to find $f(5)$. Given

x	0	1	2	3	4
f(x)	3	6	11	18	27

[VTU- July15]

8. Using Lagrange's interpolation formula to fit a polynomial for the following data:

x	2	10	17
y	1	3	4

[VTU-Model 2018]

9. Use Lagrange's interpolation formula to find $f(4)$ given

x	0	2	3	6
$f(x)$	-4	2	14	158

Ans: $f(4) = 40$]

10. Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
$f(x)$	-12	0	6	12

Hence estimate y at $x = 2$

[VTU: Jan16] [Ans:

$$f(x) = x^3 - 7x^2 + 18x - 12; f(2) = 4]$$

11. Using Lagrange's interpolation method, find the value of $f(x)$ at $x = 5$ given the values

x	1	3	4	6
$f(x)$	3	9	30	132

[Ans: $f(5) = 69.4$]

12. Using Lagrange's interpolation formula to fit a polynomial for the following data:

x	2	10	17
y	1	3	4

[VTU-Model 2018]

Numerical Integration

1. Evaluate $\int_0^5 \frac{1}{4x+5} dx$ by dividing the interval into 10 equal parts. [Ans: 1.61] (Model 2022)
2. Evaluate $\int_0^6 \frac{dx}{(1+x^2)}$ by using Simpson's (3/8)th rule, taking 7 ordinates. [VTU June 2019]
3. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos \theta} d\theta$ by using Simpson's (1/3)th rule by dividing $[0, \frac{\pi}{2}]$ into 6 equal parts. (VTU Model 2022)
4. Use Simpson's (1/3)th rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking 6 sub intervals. [VTU-Jan 2018, Jan17]
5. Evaluate $\int_0^1 \frac{dx}{(1+x)}$ by using Simpson's (3/8)th rule taking seven ordinates and hence deduce the value of $\log_e 2$ [VTU: Jan17]
6. Evaluate $\int_0^1 \frac{x dx}{(1+x^2)}$ by using Simpson's one third rule taking seven ordinates and hence find $\log_e 2$ [VTU: Model 2018, July 2018, Dec13/Jan16/Jan17]
7. Evaluate $\int_4^{5.2} \log_e x dx$ taking 6 equal strips by applying Simpson's (3/8)th rule. [VTU : Model 2022, Jan 2020, June 2019, June13, Jan17]
8. Use Simpson's (3/8)th rule to obtain the approximate value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x) dx$ by considering 6 parts. [VTU: Jan 2018, Jan17]
9. Using Simpson's (3/8)th rule, Evaluate $\int_0^3 \frac{dx}{(1+x)^2}$ taking 4 equidistant ordinates. [VTU: Model 2018]
10. Evaluate $\int_0^1 \frac{dx}{(1+x^2)}$ by using Simpson's (1/3)th rule, Simpson's (3/8)th rule [VTU: Model 2022, Jan 2020, Dec12/Jan14]

11. Evaluate $\int_0^1 \frac{x dx}{(1+x^2)}$ by using Simpson's (3/8)th rule, dividing the interval into 3 equal parts, and hence find $\log_e \sqrt{2}$ [VTU: July15]
 12. Using Simpson's 1/3 rule with 7 ordinates, evaluate $\int_2^8 \frac{1}{(\log_{10} x)} dx$ [VTU: – June 2018]
 13. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx$ using Simpson's $\frac{1}{3}$ - rule by taking 10 equal parts. [VTU Model 2019]
 14. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$ using Simpson's (3/8)th rule by dividing the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ into 6 equal parts. [VTU Model 2019]
 15. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's (1/3)th rule taking four equal strips and hence deduct an approximate value of Π . [Ans: 0.7854]
 16. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos \theta} d\theta$ by using Simpson's (1/3)th rule by dividing $[0, \frac{\pi}{2}]$ into 6 equal parts. Ans: = 1.1873
 17. Use Simpson's (1/3)th rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking 6 sub intervals. [VTU-Jan17] [Ans: 0.5351]
1. Use Simpson's (1/3)th rule to find $\int_2^8 \frac{1}{\log_{10} x} dx$ by taking 7 ordinates. [Ans: 9.7203]
 2. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using (i) Simpson's 1/3 rule (ii) Simpson's (3/8)th rule taking seven ordinates and hence deduce the value of $\log_e 2$ [VTU: Jan17] [Ans: 0.6932, 0.693]
 3. Evaluate $\int_0^1 \frac{x dx}{1+x^2}$ by using Weddle's rule taking seven ordinates and hence find $\log_e 2$ [Ans: 0.3466, $\log_e 2 = 0.6932$] [VTU: Dec13/Jan16/Jan17]
 4. Evaluate $\int_4^{5.2} \log_e x dx$ taking 6 equal strips by applying using (i) Simpson's (1/3)th rule, (ii) Simpsons 3/8th rule and (iii) weddles rule [Ans: 1.8278, 1.8278, 1.8279] [VTU: June13/Jan17]
 5. Use Simpson's (3/8)th rule to obtain the approximate value of $\int_0^{0.3} (1 - 8x^3)^{\frac{1}{2}} dx$ by considering 3 equal intervals. [Ans: 0.2916]

6. Use Simpson's $(3/8)^{\text{th}}$ rule to obtain the approximate value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x) dx$ by considering 6 parts. [VTU: Jan17] [Ans: 4.053]
10. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $(1/3)^{\text{th}}$ rule. Simpson's $(3/8)^{\text{th}}$ rule, Weddle's rule. [VTU: Dec12/Jan14] [Ans: 1.3662, 1.3571, 1.3735]
7. Evaluate $\int_0^1 \frac{x dx}{1+x^2}$ by using Simpson's $(3/8)^{\text{th}}$ rule, dividing the interval into 3 equal parts, and hence find $\log_e \sqrt{2}$ [VTU: July15]
8. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Simpson's $(1/3)^{\text{th}}$ rule, (ii) Simpson's $3/8^{\text{th}}$ rule and (iii) Weddle's rule. Compare the results with actual value. [VTU 2013]
Ans: (i) 1.3662, (ii) 1.3571 (iii) 1.3735
13. Evaluate $\int_0^5 \frac{1}{4x+5} dx$ by dividing the interval into 10 equal parts. [Ans: 1.61]
14. Evaluate $\int_0^\pi \sin x dx$ using 11 ordinates. [Ans: 2.0009]
15. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx$ using Simpson's $1/3$ rule taking 10 equal parts. [VTU Model 2019] [Ans: 1.3028]
1. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$ by dividing the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ into 6 equal parts. [VTU Model 2019]



Lecture Notes

BMATS201

Mathematics-II for CSE Stream

Module 5: Numerical Methods -II

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Module 5

Numerical Methods II

Syllabus

Introduction to various numerical techniques for handling Mechanical Engineering applications

Numerical Solution of Ordinary Differential Equations (ODEs): Numerical solution of ordinary differential equations of first order and first degree - Taylor's series method, Modified Euler's method, Runge-Kutta method of fourth order and Milne's predictor-corrector formula (No derivations of formulae). Problems.

Self-Study: Adam-Bashforth method.

Applications: Finding approximate solutions to solve mechanical engineering problems.

5.1 Numerical solution of first order differential equations :

Consider the first order differential equation,

$$\frac{dy}{dx} = f(x, y), \text{ with initial condition, } y(x_0) = y(0),$$

Such problems in which all the initial conditions are given at the initial point only, are called **initial value problems**.

Let $y(x_0), y(x_1), \dots, y(x_n)$ be the solution values at the equidistant points $x_0, x_1,$

..., x_n . Computation of the approximate values to these solution values is known as Numerical solution of the Differential equation.

Analytical methods, when available, generally enable to find the value of y for all values of x . Numerical methods, on the other hand, lead to the values of y corresponding only to some finite set of values of x . Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution involves some error and is an approximate solution.

5.2 Taylors series method :

Taylor's series is a numerical method used to approximate the value of a function $f(x)$ at a specific point x , by using a series of terms that are derived from the function's derivatives evaluated at that point.

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ then the Taylor's series expansion of $y(x)$ at $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \frac{(x - x_0)^3}{3!}y'''_0 + \frac{(x - x_0)^4}{4!}y^{iv}_0 + \dots$$

Problem 5.2.1. Find $y(0.1)$ for $y' = x^2y - 1$, $y(0) = 1$, using Taylor Series method.

Solution::

Given $y' = x^2y - 1$, $y(0) = 1$, $y(0.2) = ?$

Here, $x_0 = 0$, $y_0 = 1$

Differentiating successively, we get

$$y' = x^2y - 1$$

$$y'' = 2xy + x^2y'$$

$$y''' = 2y + 4xy' + x^2y''$$

$$y^{iv} = 6y' + 6xy'' + x^2y'''$$

Now substituting, we get

$$y'_0 = x_0^2y_0 - 1 = -1$$

$$y''_0 = 2x_0y_0 + x_0^2y'_0 = 0$$

$$y'''_0 = 2y_0 + 4x_0y'_0 + x_0^2y''_0 = 2$$

$$y_0^{iv} = 6y'_0 + 6x_0y''_0 + x_0^2y_0''' = -6$$

Putting these values in Taylor's Series, we have

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \frac{(x - x_0)^3}{3!}y_0''' + \frac{(x - x_0)^4}{4!}y_0^{IV} + \dots$$

$$Y(0.1) = 1 + 0.1 \cdot (-1) + \frac{(0.1)^2}{2!} \cdot (0) + \frac{(0.1)^3}{3!} \cdot (2) + \frac{(0.1)^4}{4!} \cdot (-6) + \dots$$

$$= 1 - 0.1 + 0 + 0.00033 + 0 + \dots$$

$$= 0.90031$$

$$\therefore y(0.1) = 0.90031$$

Problem 5.2.2. Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.

Solution: We have $y' = 2y + 3e^x$; $x_0 = 0, y_0 = 0$

Hence $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0, y = 0$ we get

$$y'' = 2y' + 3e^x,$$

$$y''(0) = 2y'(0) + 3 = 9$$

$$y''' = 2y'' + 3e^x,$$

$$y'''(0) = 2y''(0) + 3 = 21$$

$$y^{(iv)} = 2y''' + 3e^x,$$

$$y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$\begin{aligned} y(x) &= y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \dots \\ &= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \dots \\ &= 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{15}{8}x^4 + \dots \end{aligned}$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.2)^4 + \dots = 0.8110$

Exact solution : Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x

Its I.F. being e^{-2x} , the solution is

$$ye^{-2x} = \int 3e^x e^{-2x} dx + c = -3e^{-x} + c \text{ or } y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$,

$$\therefore c = 3.$$

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2, y = 3(e^{0.4} - e^{0.2}) = 0.8112$

Comparing (i) and (ii), it is clear that (i) approximates to the exact value up to three decimal places.

Problem 5.2.3. Solve $y' = x + y, y(0) = 1$ by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.

Solution: Differentiating successively, we get

$$y' = x + y, \quad y'(0) = 1 \quad [\because y(0) = 1]$$

$$y'' = 1 + y', \quad y''(0) = 2$$

$$y''' = y'', \quad y'''(0) = 2$$

$$y^{iv} = y''', \quad y^{iv}(0) = 2, \text{ etc.}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \dots$$

Here $x_0 = 0, y_0 = 1$

$$\therefore y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(4) \dots$$

Thus

$$\begin{aligned} y(0.1) &= 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!} \dots \\ &= 1.1103 \end{aligned}$$

$$\begin{aligned} \text{and } y(0.2) &= 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \dots \\ &= 1.2427 \end{aligned}$$

Problem 5.2.4. Solve $\frac{dy}{dx} = y^2 + x^2$ with $y(0) = 1$. Use Taylor's method at $x = 0.2$ and 0.4 .

Solution: Given $\frac{dy}{dx} = y^2 + x^2$ & $y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$, Since $[y(x_0) = y_0]$

Taylor series formula is

$$y(x) = y_0 + \frac{(x - x_0)}{1!}y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \frac{(x - x_0)^3}{3!}y'''_0 + \frac{(x - x_0)^4}{4!}y^{iv}_0 + \dots$$

$$y' = y^2 + x^2$$

$$y'_0 = y_0^2 + x_0^2 = 1 + 0 = 1$$

$$y'' = 2yy' + 2x$$

$$\begin{aligned} y''_0 &= 2y_0y'_0 + 2x_0 \\ &= 2(1)(1) + 2(0) = 2 \end{aligned}$$

$$y''' = 2yy'' + 2y'y' + 2$$

$$\begin{aligned} y'''_0 &= 2y_0y''_0 + 2(y'_0)^2 + 2 \\ &= 2(1)2 + 2(1) + 2 = 8 \end{aligned}$$

$$y^{iv} = 2yy''' + 2y'y'' + 4y'y'' = 2yy''' + 6y'y''$$

$$\begin{aligned} y^{iv}_0 &= 2y_0y'''_0 + 6y'_0y''_0 \\ &= 2(1)8 + 6(1)2 = 28 \end{aligned}$$

Therefore equation (1) becomes,

$$\begin{aligned} y(x) &= y_0 + \frac{(x-x_0)}{1!}y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \frac{(x-x_0)^4}{4!}y^{iv}_0 + \dots \\ &= 1 + \frac{(x-0)}{1}(1) + \frac{(x-0)^2}{2}(2) + \frac{(x-0)^3}{6}(8) + \frac{(x-0)^4}{24}(28) + \dots \end{aligned}$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(8) + \frac{x^4}{24}(28)$$

To find y at $x = 0.2$

$$\begin{aligned} \therefore y(0.2) &= 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6}(8) + \frac{(0.2)^4}{24}(28) \\ &= 1 + 0.2 + 0.02 + 0.010667 + 0.00186667 \end{aligned}$$

$$\therefore y(0.2) = 1.23253$$

To find y at $x = 0.4$

$$\begin{aligned} \therefore y(0.4) &= 1 + 0.4 + \frac{(0.4)^2}{2} + \frac{(0.4)^3}{6}(8) + \frac{(0.4)^4}{24} \\ &= 1 + 0.4 + 0.08 + 0.085333 + 0.0298667 \end{aligned}$$

$$\therefore y(0.4) = 1.5952$$

Problem 5.2.5. Use Taylor's series method to find $y(0.1)$. Given $y' = e^x - y^2$, $y(0) = 1$

Solution : Given : $y' = e^x - y^2, x_0 = 0, y_0 = 1, h = x - x_0 = 0.1$

$$y' = e^x - y^2$$

$$y'_0 = e^{x_0} - y_0^2 = 1 - 1 = 0$$

$$y'' = e^x - 2yy'$$

$$\begin{aligned} y_0'' &= e^{x_0} - 2y_0y_0' \\ &= 1 - 2(1)(0) = 1 \end{aligned}$$

$$\begin{aligned} y''' &= e^x - 2[yy'' + y'y'] \\ &= e^x - 2yy'' - 2(y')^2 \end{aligned}$$

$$\begin{aligned} y_0''' &= e^{x_0} - 2y_0y_0'' - 2(y_0')^2 \\ &= 1 - 2(1)(1) - 0 = -1 \end{aligned}$$

$$\begin{aligned} y^{iv} &= e^x - 2[yy''' + y''y'] - 4y'y'' \\ &= e^x - 2yy''' - 2y''y' - 4y'y'' \\ &= e^x - 2yy''' - 6y'y'' \end{aligned}$$

$$\begin{aligned} y_0^{iv} &= e^{x_0} - 2y_0y_0''' - 6y_0'y_0'' \\ &= 1 - 2(1)(-1) - 0 = 3 \end{aligned}$$

$$y(x) = y_0 + \frac{(x - x_0)}{1!}y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \frac{(x - x_0)^3}{3!}y'''_0 + \frac{(x - x_0)^4}{4!}y_0^{iv} + \dots$$

$$y(0.1) = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y_0^{iv} + \dots$$

$$= 1 + (0.1)(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(-1) + \frac{(0.1)^4}{24}(3) + \dots$$

$$= 1 + 0 + 0.005 - 0.00017 + 0.00001 + \dots$$

$$= 1.0049891 = 1.005 \text{ [correct to 3 decimal places]}$$

5.3 Modified Euler's Method :

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

To find y_1 i.e. $y(x_1)$ where $x_1 = x_0 + h$, we proceed as follows.

Step 1: We first obtain an initial approximation for $y_1 = y(x_1)$ by applying Euler's formula $y_1^{(0)} = y_0 + hf(x_0, y_0)$

Step 2: Since the accuracy is very poor in Euler's formula, the formula is modified and is given by,

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

In general

$$y_1^{(n+1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(n)})]$$

and called as modified Euler's formula.

Problem 5.3.1. Given that $dy/dx = \log_{10}(x + y)$, $y(0) = 1$, find $y(0.1)$ and $y(0.2)$ using modified Euler's method.

Solution:

Given differential equation is,

$$\frac{dy}{dx} = \log_{10}(x + y) = f(x, y)$$

with initial condition, $x_0 = 0$, $y_0 = 1$ and $x_1 = 0.1$

Taking, $h = 0.1$, such that

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$\text{and } x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$$

First Stage:

By Using Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= y_0 + h [\log_{10}(x_0 + y_0)]$$

$$= 1 + 0.1 [\log_{10}(0 + 1)] = 1 \quad \text{at } x_1 = x_0 + h = 0.1$$

Applying Euler's modified formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(n)})]$$

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})] = 1.0020$$

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1.0021 \text{ and}$$

$$y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})] = 1.0021$$

clearly, $y_1^{(3)} = y_1^{(2)} = 1.0021 = y_1$ (Improved value) at $x_1 = 0.1$

Second Stage: Taking $y_1 = 1.0021$ and $x_1 = 0.1$

By Using Euler's formula,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$\Rightarrow y_2 = y_1 + h [\log_{10}(x_1 + y_1)]$$

$$= 1.0021 + 0.1 [\log_{10}(0.1 + 1.0021)]$$

$$= 1.0063, \text{ at } x_2 = 0.2$$

Applying Euler's modified formula,

$$y_2^{(n+1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(n)})]$$

$$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.0124$$

$$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.0125$$

And

$$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})] = 1.0125$$

Clearly,

$$y_2^{(3)} = y_2^{(2)} = 1.0125 = y_2$$

at $x_2 = 0.2$

Hence the required value of y at $x = 0.2$ is 1.0125 .

Problem 5.3.2. Using modified Euler's method find y at $x = 0.2$ given $\frac{dy}{dx} = 3x + \frac{y}{2}$ with $y(0) = 1$ taking $h = 0.1$. Perform two iteration at each step.

Solution: : Given differential equation is,

$$\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y)$$

with initial condition, $x_0 = 0, y_0 = 1$. Let $x_1 = 0.1$

Taking, $h = 0.1$, such that

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$\text{and } x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$$

Stage 1: To find $y(0.1)$:

By Using Euler's formula

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ &= y_0 + h \left[3x_0 + \frac{y_0}{2} \right] \\ &= 1.05 \end{aligned}$$

Applying Euler's modified formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(n)}) \right]$$

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] \\ &= y_0 + \frac{h}{2} \left\{ \left[3x_0 + \frac{y_0}{2} \right] + \left[3x_1 + \frac{y_1^{(0)}}{2} \right] \right\} \\ &= 1 + \frac{0.1}{2} \left\{ \left[3(0) + \frac{1}{2} \right] + \left[3(0.1) + \frac{1.05}{2} \right] \right\} \\ &= 1.06625 \end{aligned}$$

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] \\ &= y_0 + \frac{h}{2} \left\{ \left[3x_0 + \frac{y_0}{2} \right] + \left[3x_1 + \frac{y_1^{(1)}}{2} \right] \right\} \\ &= 1 + \frac{0.1}{2} \left\{ \left[3(0) + \frac{1}{2} \right] + \left[3(0.1) + \frac{1.06625}{2} \right] \right\} \\ &= 1.0667 \end{aligned}$$

Thus $y_1 = y(0.1) = 1.0667$

Stage 2: To find $y(0.2)$:

By Using Euler's formula

$$\begin{aligned} y_2^{(0)} &= y_1 + hf(x_1, y_1) \\ &= y_1 + h \left[3x_1 + \frac{y_1}{2} \right] \\ &= 1.0667 + (0.1) \left[3(0.1) + \frac{1.0667}{2} \right] \\ &= 1.15 \end{aligned}$$

Applying Euler's modified formula,

$$y_2^{(n+1)} = y_1 + \frac{h}{2} \left[f(x_1, y_1) + f(x_2, y_2^{(n)}) \right]$$

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2} \left[f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \\ &= y_1 + \frac{h}{2} \left\{ \left[3x_1 + \frac{y_1}{2} \right] + \left[3x_2 + \frac{y_2^{(0)}}{2} \right] \right\} \\ &= 1.0667 + \frac{0.1}{2} \left\{ \left[3(0.1) + \frac{1.0667}{2} \right] + \left[3(0.2) + \frac{1.15}{2} \right] \right\} \\ &= 1.1671 \end{aligned}$$

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2} \left[f(x_1, y_1) + f(x_2, y_2^{(1)}) \right] \\ &= y_1 + \frac{h}{2} \left\{ \left[3x_1 + \frac{y_1}{2} \right] + \left[3x_2 + \frac{y_2^{(1)}}{2} \right] \right\} \\ &= 1.0667 + \frac{0.1}{2} \left\{ \left[3(0.1) + \frac{1.0667}{2} \right] + \left[3(0.2) + \frac{1.1671}{2} \right] \right\} \\ &= 1.1675 \end{aligned}$$

Thus $y_2 = y(0.2) = 1.1675$

Problem 5.3.3. Using Euler's modified method, obtain a solution of the equation

$$dy/dx = x + |\sqrt{y}|$$

with initial conditions $y = 1$ at $x = 0$, for the range $0 \leq x \leq 0.4$ in steps of 0.2 .

Solution:: Given differential equation is,

$$\frac{dy}{dx} = x + |\sqrt{y}| = f(x, y) = x + \sqrt{y}$$

with initial condition, $x_0 = 0, y_0 = 1$. Let $x_1 = 0.2$

Taking, $h = 0.2$, such that

$$x_1 = x_0 + h = 0 + 0.2 = 0.2$$

and $x_2 = x_0 + 2h = 0 + 2(0.2) = 0.4$ The various calculations in this method are arranged as follows:

First Stage :

Step 1: We first obtain an initial approximation for $y_1 = y(x_1)$ by applying Euler's formula

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ y_1^{(0)} &= 1 + (0.2)(x_0 + \sqrt{y_0}) \\ &= 1 + (0.2)(0 + \sqrt{1}) \\ &= 1.2 \end{aligned}$$

Step 2: Since the accuracy is very poor in Euler's formula, the formula is modified and is given by,

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})] \\ &= 1 + \frac{0.2}{2} [f(0, 1) + f(0.2, 1.2)] \\ &= 1 + \frac{0.2}{2} [(0 + \sqrt{1}) + (0.2 + \sqrt{1.2})] \\ &= 1.2295 \\ y_1^{(2)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \\ &= 1 + \frac{0.2}{2} [(0 + \sqrt{1}) + (0.2 + \sqrt{1.2295})] \\ &= 1.2309 \\ y_1^{(3)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})] \\ &= 1 + \frac{0.2}{2} [(0 + \sqrt{1}) + (0.2 + \sqrt{1.2309})] \\ &= 1.2309 \end{aligned}$$

Thus $y_1 = 1.2309$ **Second Stage :**

Step 1: We first obtain an initial approximation for $y_2 = y(x_2)$ by applying Euler's

formula

$$\begin{aligned}
 y_2^{(0)} &= y_1 + hf(x_1, y_1) \\
 y_2^{(0)} &= y_1 + (0.2)(x_1 + \sqrt{y_1}) \\
 &= 1.2309 + (0.2)(0.2 + \sqrt{1.2309}) \\
 &= 1.4928
 \end{aligned}$$

Step 2:

$$\begin{aligned}
 y_2^{(1)} &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(0)})] \\
 &= 1.2309 + \frac{0.2}{2} [f(0.2, 1.2309) + f(0.4, 1.4928)] \\
 &= 1.2309 + \frac{0.2}{2} [(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.4928})] \\
 &= 1.524 \\
 y_2^{(2)} &= y_2 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})] \\
 &= 1.2309 + \frac{0.2}{2} [f(0.2, 1.2309) + f(0.4, 1.524)] \\
 &= 1.2309 + \frac{0.2}{2} [(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.524})] \\
 &= 1.5253 \\
 y_2^{(2)} &= y_2 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})] \\
 &= 1.2309 + \frac{0.2}{2} [f(0.2, 1.2309) + f(0.4, 1.5253)] \\
 &= 1.2309 + \frac{0.2}{2} [(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.5253})] \\
 &= 1.5254
 \end{aligned}$$

Thus $y_2 = 1.5254$

Problem 5.3.4. Using modified Euler's method find $y(0.1)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ taking $h = 0.1$.

Solution: : Given differential equation is,

$$\frac{dy}{dx} = x - y^2 = f(x, y)$$

with initial condition, $x_0 = 0$, $y_0 = 1$. Let $x_1 = 0.1$

Taking, $h = 0.1$, such that

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

By Using Euler's formula

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ &= y_0 + h[x_0 - y_0^2] \\ &= 1 + 0.1[0 - 1^2] \\ &= 0.9 \end{aligned}$$

Applying Euler's modified formula,

$$\begin{aligned} y_1^{(n+1)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(n)})] \\ y_1^{(1)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})] \\ &= y_0 + \frac{h}{2} [(x_0 - y_0^2) + (x_1 - y_1^2)] \\ &= 1 + \frac{0.1}{2} [(0 - 1^2) + (0.1 - 0.9^2)] \\ &= 0.9145 \end{aligned}$$

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \\ &= 1 + \frac{0.1}{2} [(0 - 1^2) + (0.1 - 0.9145^2)] \\ &= 0.9132 \end{aligned}$$

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})] \\ &= 1 + \frac{0.1}{2} [(0 - 1^2) + (0.1 - 0.9132^2)] \\ &= 0.9133 \end{aligned}$$

Thus $y(0.1) = 0.9133$

5.4 R.K. Method of order 4:

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

We need to find $y(x_1) = y_1$ With the help of formula

$$y_1 = y_0 + k$$

where

$$k = \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

and

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Problem 5.4.1. Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ $y(0) = 1$ compute $y(0.2)$ by taking $h = 0.2$ using Runge-kutta method of fourth order.

Solution:: By data $f(x, y) = 3x + \frac{y}{2}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

We shall first compute k_1, k_2, k_3, k_4

$$k_1 = hf(x_0, y_0)$$

$$= (0.2)f(0, 1)$$

$$= (0.2) \left[(3)(0) + \frac{1}{2} \right]$$

$$= 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2)f\left[0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right]$$

$$= (0.2)f(0.1, 1.05)$$

$$= (0.2) \left[3(0.1) + \frac{1.05}{2} \right]$$

$$= 0.165$$

$$\begin{aligned}
 k_3 &= hf \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} \right) \\
 &= (0.2) f \left[0 + \frac{0.2}{2}, 1 + \frac{0.165}{2} \right] \\
 &= (0.2) f(0.1, 1.0825) \\
 &= (0.2) \left[3(0.1) + \frac{1.0825}{2} \right] \\
 &= 0.16825
 \end{aligned}$$

$$\begin{aligned}
 k_4 &= hf(x_0 + h, y_0 + k_3) = (0.2) f \left[0 + \frac{0.2}{2}, 1 + \frac{0.16825}{2} \right] \\
 &= (0.2) f(0.2, 1.16825) = (0.2) \left[3(0.2) + \frac{1.16825}{2} \right] \\
 &= 0.236825
 \end{aligned}$$

We have,

$$y(x_0 + h) = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

Therefore,

$$y(0.2) = 1 + \frac{1}{6} [0.1 + 2(0.165 + 2(0.16825) + 0.236825)] = 1.1672208$$

Problem 5.4.2. Using the Runge-Kutta method of fourth order, find $y(0.1)$. Given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$.

Solution:: We have $f(x, y) = \frac{y-x}{y+x}$

To find $y(0.1)$ Hence

$$x_0 = 0, y_0 = 1, h = 0.1$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1f(0, 1)$$

$$= 0.1 \frac{(1 - 0)}{(1 + 0)} = 0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= 0.1 \times f(0.05, 1.05)$$

$$= (0.1) \times \frac{(1.05 - 0.05)}{(1.05 + 0.05)}$$

$$= 0.091$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= (0.1) \times f(0.05, 1.0455)$$

$$= (0.1) \times \frac{(1.04555 - 0.05)}{(1.04555 + 0.05)}$$

$$= 0.0909$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.1)f(0.1, 1.0909)$$

$$= (0.1) \times \frac{(1.0909 - 0.1)}{(1.0909 + 0.1)}$$

$$= 0.0832$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1 + 2(0.091) + 2(0.0909) + 0.0832]$$

$$= 0.0912$$

Hence $y(0.1) = y_1 = y_0 + k = 1.0912$.

Problem 5.4.3. Using the Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.

Solution: We have $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$ Hence

$$x_0 = 0, y_0 = 1, h = 0.2$$

$$k_1 = hf(x_0, y_0) = 0.2f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196$.

To find $y(0.4)$:

$$\text{Here } x_1 = 0.2, y_1 = 1.196, h = 0.2.$$

$$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$.

Problem 5.4.4. Apply the Runge-Kutta method to find the approximate value of y for $x = 0.2$, in steps of 0.1 , if $dy/dx = x + y^2$, $y = 1$ where $x = 0$.

Solution:: Given $f(x, y) = x + y^2$. Here we take $h = 0.1$ and carry out the calculations in two steps. Step I. $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1f(0.05, 1.1) = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1f(0.05, 1.1152) = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1168) = 0.1347$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\therefore = \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165$

Step II.

$$x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$$

$$\therefore k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1165) = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1f(0.15, 1.194) = 0.1576$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(0.2, 1.1576) = 0.1823$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$

Problem 5.4.5. Using the Runge-Kutta method of order 4, find y for $x = 0.1, 0.2$ given that $dy/dx = xy + y^2, y(0) = 1$.

Solution:: We have $f(x, y) = xy + y^2$.

To find $y(0.1)$:

Here $x_0 = 0, y_0 = 1, h = 0.1$.

$$\therefore k_1 = hf(x_0, y_0) = (0.1) \times f(0, 1) = 0.1000$$

$$k_2 = hf \left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 \right) = (0.1) \times f(0.05, 1.05) = 0.1155$$

$$k_3 = hf \left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 \right) = (0.1) \times f(0.05, 1.0577) = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) \times f(0.1, 1.1172) = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$

To find $y(0.2)$: Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1$

$$k_1 = hf(x_1, y_1) = (0.1) \times f(0.1, 1.1169) = 0.1359$$

$$k_2 = hf \left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 \right) = (0.1) \times f(0.15, 1.1848) = 0.1581$$

$$k_3 = hf \left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 \right) = (0.1) \times f(0.15, 1.1959) = 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) \times f(0.2, 1.2778) = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773$.

5.5 Predictor–Corrector methods

To solve a differential equation over an interval (x_n, x_{n+1}) , using previous single-step methods, only the values of y at the beginning of interval is required. However, in the following methods, four prior values are needed for finding the value of y_{n+1} at a given value of x . Also the solution at y_{n+1} is obtained in two stages. This method of refining an initially crude estimate of y_{n+1} by means of a more accurate formula is known as **Predictor–Corrector method**. A Predictor formula is used to predict the value of y_{n+1} and then a Corrector Formula is applied to calculate a still better approximation of y_{n+1} . Now we study one such method namely **Milne's method**.

5.6 Milne's method

Milne's predictor formula :

$$y_4^{(p)} = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3)$$

Milne's corrector formula :

$$y_4^{(c)} = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4)$$

Problem 5.6.1. Given that $dy/dx = x - y^2$ and the data $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. Compute y at $x = 0.8$ by applying Milne's method

Solution:: Given, $y' = x - y^2$ in the range $0 \leq x \leq 1$ for the boundary conditions $y = 0$ at $x = 0$.

	$x_0 = 0.0,$	$y_0 = 0.0000,$	$f_0 = 0.0000$
and	$x_1 = 0.2,$	$y_1 = 0.020,$	$f_1 = 0.1996$
	$x_2 = 0.4,$	$y_2 = 0.0795,$	$f_2 = 0.3937$
	$x_3 = 0.6,$	$y_3 = 0.1762,$	$f_3 = 0.5689$

Using the predictor, $y_4^{(p)} = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3)$

$$x = 0.8, \quad y_4^{(p)} = 0.3049, \quad f_4 = 0.7070$$

and the corrector,

$$y_4^{(c)} = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4), \text{ yields}$$

$$y_4^{(c)} = 0.3046,$$

Problem 5.6.2. Given that $dy/dx = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$. Find y at $x = 0.4$ using Milne's method.

Solution:: We have $f(x, y) = xy + y^2$.

Now the starting values for the Milne's method are:

x	y	$f = y'$
$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the predictor

$$y_4^{(p)} = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3)$$

$$x_4 = 0.4 \quad y_4^{(p)} = 1.8344$$

$$f_4 = x_4 y_4 + y_4^2 = 4.0988$$

and the corrector,

$$y_4^{(c)} = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4)$$

$$\begin{aligned} y_4^{(c)} &= 1.2773 + \frac{0.1}{3} [1.8869 + 4(2.7132) + 4.098] \\ &= 1.8397 \end{aligned}$$

$$f_4 = x_4 y_4 + y_4^2 = 4.1159.$$

Again using the corrector,

$$\begin{aligned} y_4^{(c)} &= 1.2773 + \frac{0.1}{3} [1.8869 + 4(2.7132) + 4.1159] \\ &= 1.8391 \end{aligned}$$

$$f_4 = x_4 y_4 + y_4^2 = 4.1182$$

Again using the corrector,

$$\begin{aligned} y_4^{(c)} &= 1.2773 + \frac{0.1}{3} [1.8869 + 4(2.7132) + 4.1182] \\ &= 1.8392 \text{ which is same as (i)} \end{aligned}$$

Hence $y(0.4) = 1.8392$.

Problem 5.6.3. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$, $y(0.3) = 2.090$ find $y(0.4)$ correct to 4 decimal places by Milnes predictor corrector method. Apply the corrector formula twice.

Solution : Given, $y' = 2e^x - y$ with the initial condition, $y = 2$ at $x = 0$.

	x	y	$f(x, y) = 2e^x - y$
\therefore	$x_0 = 0.0,$	$y_0 = 2,$	$f_0 = 0.0000$
and	$x_1 = 0.1,$	$y_1 = 2.010,$	$f_1 = 0.2003$
	$x_2 = 0.2,$	$y_2 = 2.040,$	$f_2 = 0.4028$
	$x_3 = 0.3,$	$y_3 = 2.090,$	$f_3 = 0.6097$
	$x_4 = 0.4,$	$y_4 = ?$	$f_4 = ?$

Here, $h=0.1$

Using the predictor

$$\begin{aligned} y_4^{(p)} &= y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \\ &= 2 + \frac{4(0.1)}{3} [2(0.2003) - 0.4028 + 2(0.6097)] \\ &= 2.1623 \end{aligned}$$

$$x_4 = 0.4 \quad y_4^{(p)} = 2.1623 \quad f_4 = 2e^{0.4} - 2.1623 = 0.8213$$

and the corrector,

$$\begin{aligned} y_4^{(c)} &= y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4) \\ y_4^{(c)} &= 2.040 + \frac{0.1}{3} [0.4028 + 4(0.6097) + 0.8213] \\ &= 2.1621 \end{aligned}$$

Problem 5.6.4. Given $dy/dx = x^3 + y$, $y(0) = 2$. The values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$.

$x_0 = 0$	$y_0 = 2$
$x_1 = 0.2$	$y_1 = 2.073$
$x_2 = 0.4$	$y_2 = 2.452$
$x_3 = 0.6$	$y_3 = 3.023$

Solution:Here,

$$y' = f(x, y) = x^3 + y$$

$$f_1 = y'_1 = x_1^3 + y_1 = (0.2)^3 + 2.073 = 2.081$$

$$f_2 = y'_2 = x_2^3 + y_2 = (0.4)^3 + 2.452 = 2.516$$

$$f_3 = y'_3 = x_3^3 + y_3 = (0.6)^3 + 3.023 = 3.239$$

By Milne's predictor formula, we have

$$\begin{aligned} y_4^{(p)} &= y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \\ &= 2 + \frac{4(0.2)}{3} [2(2.081) - 2.516 + 2(3.239)] \\ &= 2 + \frac{0.8}{3} [8.124] = 2 + 2.1664 \\ &= 4.1664 \end{aligned}$$

Hence $f_4 = x_4^3 + y_4 = (0.8)^3 + 4.1664 = 4.6784$

Using Milne's corrector formula,

$$\begin{aligned} y_4^{(c)} &= y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4) \\ &= 2.452 + \frac{0.2}{3} [2.516 + 4(3.239) + 4.6784] \\ &= 2.452 + \frac{0.2}{3} [20.1504] \\ &= 3.79536 \end{aligned}$$

Hence Corrected value of y at $x = 0.8$ is 3.79536.

Problem 5.6.5. Given $\frac{dy}{dx} = \frac{1}{x+y}$, Given that $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$ and $y(0.6) = 2.2493$. Find the value of y at $x = 0.8$ using Milnes Predictor and Corrector Formulae.

Solution : Here

$x_0 = 0$	$y_0 = 2$	$h = 0.2$
$x_1 = 0.2$	$y_1 = 2.0933$	
$x_2 = 0.4$	$y_2 = 2.1755$	
$x_3 = 0.6$	$y_3 = 2.2493$	
$x_4 = 0.8$	$y_4 = \dots$	

To find $y(0.8)$:

$$y' = \frac{1}{x + y}$$

$$f_1 = y'_1 = \frac{1}{x_1 + y_1} = \frac{1}{0.2 + 2.0933} = 0.4361$$

$$f_2 = y'_2 = \frac{1}{x_2 + y_2} = \frac{1}{0.4 + 2.1755} = 0.3883$$

$$f_3 = y'_3 = \frac{1}{x_3 + y_3} = \frac{1}{0.6 + 2.2493} = 0.3510$$

By Milne's predictor formula, we have

$$\begin{aligned} y_4^{(p)} &= y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \\ &= 2 + \frac{4(0.2)}{3} [2(0.4361) - 0.3883 + 2(0.3510)] \\ &= 2.3162 \end{aligned}$$

$$\text{Hence } f_4 = \frac{1}{x_4 + y_4} = \frac{1}{0.8 + 2.3162} = 0.321$$

Using Milne's corrector formula,

$$\begin{aligned} y_4^{(c)} &= y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4) \\ &= 2.1755 + \frac{0.2}{3} [0.3883 + 4(0.351) + 0.321] \\ &= 2.3164 \\ \therefore y(0.8) &= 2.3164 \end{aligned}$$

Problem 5.6.6. Given $dy/dx = x^2(1+y)$ and $y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979$, evaluate $y(1.4)$ by using Milne's method.

Solution : Given: $y' = f(x, y) = x^2(1 + y)$

Here, $h = 0.1$

$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$	$x_4 = 1.4$
$y_0 = 1$	$y_1 = 1.233$	$y_2 = 1.548$	$y_3 = 1.979$	$y_4 = ?$

Here $h = 0.1$

$$f = y' = x^2(1 + y)$$

$$f_1 = y'_1 = x_1^2(1 + y_1) = (1.1)^2(1 + 1.233) = 2.702$$

$$f_2 = y'_2 = x_2^2(1 + y_2) = (1.2)^2(1 + 1.548) = 3.669$$

$$f_3 = y'_3 = x_3^2(1 + y_3) = (1.3)^2(1 + 1.979) = 5.035$$

By Milne's predictor formula, we have

$$\begin{aligned} y_4^{(p)} &= y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \\ &= 1 + \frac{4(0.1)}{3} [2(2.702) - 3.669 + 2(5.035)] \\ &= 2.574 \end{aligned}$$

$$\text{Hence } f_4 = x_4^2(1 + y_4) = (1.4)^2(1 + 2.574) = 7.005$$

Using Milne's corrector formula,

$$\begin{aligned} y_4^{(c)} &= y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4) \\ &= 1.548 + \frac{0.1}{3} [3.669 + 4(5.035) + 7.005] \\ &= 2.575 \end{aligned}$$

Problem 5.6.7. Given $2\frac{dy}{dx} = (1+x^2)y^2$ and $y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12, y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's method.

Solution :

$$\text{Given : } x_0 = 0, x_1 = 0.1, x_2 = 0.2, \quad x_3 = 0.3, x_4 = 0.4$$

$$y_0 = 1, y_1 = 1.06, y_2 = 1.12, y_3 = 1.21$$

$$h = 0.1$$

$$\text{Given : } 2\frac{dy}{dx} = (1 + x^2)y^2.$$

$$\therefore f(x, y) = y' = \frac{1}{2} (1 + x^2) y^2$$

$$f_0 = y_0' = \frac{1}{2} (1 + x_0^2) y_0^2 = \frac{1}{2} (1 + 0)(1) = \frac{1}{2} = 0.5$$

$$f_1 = y_1' = \frac{1}{2} (1 + x_1^2) y_1^2 = \frac{1}{2} [1 + (0.1)^2] [1.06]^2 = 0.5674$$

$$f_2 = y_2' = \frac{1}{2} [1 + x_2^2] y_2^2 = \frac{1}{2} [1 + (0.2)^2] [1.12]^2 = 0.6522$$

$$f_3 = y_3' = \frac{1}{2} [1 + x_3^2] y_3^2 = \frac{1}{2} [1 + (0.3)^2] [1.21]^2 = 0.7979$$

By Milne's predictor formula, we have

$$\begin{aligned} y_4^{(p)} &= y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \\ &= 1 + \frac{4(0.1)}{3} [(2)(0.5764) - (0.6522) + (2)(0.7979)] \\ &= 1.2771 \end{aligned}$$

Hence $f_4 = \frac{1}{2} (1 + x_4^2) y_4^2 = \frac{1}{2} (1 + (0.4)^2) (1.2771)^2 = 0.9460$

Using Milne's corrector formula,

$$\begin{aligned} y_4^{(c)} &= y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4) \\ &= 1.12 + \frac{0.1}{3} [0.6522 + 4(0.7979) + 0.9460] \\ &= 1.2797 \end{aligned}$$

$$\therefore y(0.4) = 1.2797$$

Problem 5.6.8. Using Milne's method find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0143; y(4.4) = 1.0187$.

Solution: We have $y' = (2 - y^2) / 5x = f(x, y)$

Then the starting values of the Milne's method are

$x_0 = 4,$	$y_0 = 1,$	$f_0 = \frac{2-1^2}{5 \times 4} = 0.05$
$x_1 = 4.1,$	$y_1 = 1.0049,$	$f_1 = 0.0485$
$x_2 = 4.2,$	$y_2 = 1.0097,$	$f_2 = 0.0467$
$x_3 = 4.3,$	$y_3 = 1.0143,$	$f_3 = 0.0452$
$x_4 = 4.4,$	$y_4 = 1.0187,$	$f_4 = 0.0437$

Here, $h=0.1$

Since y_5 is required, we use the predictor

$$\begin{aligned} y_5^{(p)} &= y_1 + \frac{4h}{3} (2f_2 - f_3 + 2f_4) \\ x = 4.5, y_5^{(p)} &= 1.0049 + \frac{4(0.1)}{3} (2 \times 0.0467 - 0.0452 + 2 \times 0.0437) = 1.023 \\ f_5 &= \frac{2 - y_5^2}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424 \end{aligned}$$

Now using the corrector $y_5^{(c)} = y_3 + \frac{h}{3} (f_3 + 4f_4 + f_5)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3} (0.0452 + 4 \times 0.0437 + 0.0424) = 1.023$$

Hence $y(4.5) = 1.023$

Question Bank : Module 5 - Numerical Methods II

Taylor's series method

1. Use Taylor's series method to find y at $x = 0.1, 0.2, 0.3$ considering terms up to the third degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$ [VTU-Dec 2018, Dec 2012]
2. From Taylor's series method, find $y(0.1)$ considering up to fourth degree term if $y(x)$ satisfies the equation $\frac{dy}{dx} = x - y^2, y(0) = 1$ [VTU-Jan 2015]
3. Employ Taylor's series method to find y at $x = 0.1$ and 0.2 correct to four places of decimal in step size of 0.1 given the linear differential equation $\frac{dy}{dx} - 2y = 3e^x$ whose solution passes through origin. Also find $y(0.1)$ and $y(0.2)$ by analytical method. [VTU-Jan 2018, Jan 2014, July 2013]
4. Using Taylor's series method for $y' = \sqrt{x^2 + y}, y(0) = 0.8$, find $y(0.1)$. consider up to third order derivative terms. [VTU-July 2017]
5. Applying Taylor's series method, find y at $x = 0.1$. Given $\frac{dy}{dx} = x + y^2, y(0) = 1$. [VTU-Jan 2014]
6. Using Taylor's series method solve $\frac{dy}{dx} = x^2y + 1, y(0) = 0$. Find the third order solution at $x = 0.4$ [VTU-June 2012]
7. Using Taylor's series method find $y(0.1)$, Given $\frac{dy}{dx} = x^2y - 1, y(0) = 1$ [VTU- Jan 2021, July 2017, June 2012, July 2011]
8. Use Taylor's series method to solve $\frac{dy}{dx} = x^2y - 1, y(0) = 1$ at $x = 0.2$. Consider upto 4th degree terms. [VTU Model 2022, July 2017, July 2016, Jan 2016, Dec 2012]

9. Find $y(0.1)$ correct to 6 decimal places by using Taylor's series method. Given $dy = (xy + 1)dx, y(0) = 1.1$ [VTU-Jun 2010]
10. Use Taylor's series method to find $y(4.1)$. Given that $(x^2 + y)y' = 1$ and $y(4) = 4$ [VTU Jan 2018]
11. Use Taylor's series method to find $y(0.1)$. Given that $y' + y + 2x = 0$ and $y(0) = -1$. Consider up to third order derivative term. [VTU July 2019]
12. Use Taylor's series method to find $y(0.1)$ from $y' = 3x + y^2$ and $y(0) = 1$. Consider up to fourth derivative term. [VTU Model 2022, July 2019]
13. Use Taylor's series method to find $y(0.1)$ and $y(0.2)$. Given $y' = x + y, y(0) = 1$ (VTU Sept 2020)
14. Use Taylor's series method to find $y(1.1)$ from $y' = e^x - y, y(0) = 2$ (VTU Jan 2021, Sept 2020)
15. Use Taylor's series method to find $y(0.2)$ taking $h=0.1$. Given $y' = e^x - y^2, y(0) = 1$ (VTU Model 2022)

Modified Euler's method

1. Using modified Euler's method find $y(20.2)$ and $y(20.4)$ given that $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ With $y(20) = 5$ taking $h = 0.2$. [VTU-July 2017]
2. Apply modified Euler's method to solve the following initial value problems by considering the accuracy up to two approximations in every step $\frac{dy}{dx} = x + |\sqrt{y}|$ in the range $0 \leq x \leq 0.4$ by taking $h = 0.2$ given that $y = 1$ at $x = 0$ initially. [VTU-Jan 2021, Dec 2018, July 2017]
3. Given $\frac{dy}{dx} = \frac{1}{1+x^2} - 2y^2, y(0) = 0$. Find $y(0.5)$, by taking $h = 0.25$ using Euler's modified method. [VTU-July 2017, Dec 2011]

4. Solve by using modified Euler's method to obtain $y(1.2)$. Given $y' = \frac{y+x}{y-x}$, $y(1) = 2$. [VTU-July 2015]
5. Solve by using modified Euler's method to obtain y at $x=0.2$ taking $h=0.1$. Given $y' = \frac{y-x}{y+x}$, $y(0) = 1$. Carryout 3 iterations. [VTU- Model 2022, Dec 2018, Dec 2012]
6. Solve by using modified Euler's method to find $y(0.4)$ by taking $h = 0.2$. Given $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ [VTU-Jan 2014, 2015]
7. Determine the value of y when $x = 0.1$ given that $y(0) = 1$ and $y'' = x^2 + y^2$, using modified Euler's method. Take $h = 0.05$ [VTU-Jan 2014]
8. Given $\frac{dy}{dx} = x + y$, $y(0) = 1$. compute $y(0.2)$ by using modified Euler's method. [VTU-Jan 2013]
9. Use modified Euler's method to find y at $x=0.1$. Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, $h = 0.1$ [VTU Sept 2020, Jan 2018]
10. $\frac{dy}{dx} = 1 + \frac{y}{x}$ with $y(1) = 2$. Find the value of y at $x=1.2$ using Eulers Modified method. (VTU - Dec 2018)
11. Find $y(0.2)$ by using modified Euler's method, given that given that $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Take $h = 0.1$ and carry out two modifications at each step. [VTU-July 2019]
12. Use modified Euler's method to find y at $x = 0.4$. Given $\frac{dy}{dx} = x + \sin y$, $y(0) = 1$, $h = 0.2$ (VTU Jan 2020)
13. Use modified Euler's method to find y at $x = 1.2$. Given $\frac{dy}{dx} = 1 + \frac{y}{2}$, $y(1) = 2$, $h = 0.2$ (VTU Sept. 2020)
14. Use modified Euler's method to find y at $x = 0.1$. Given $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$, $h = 0.05$ (VTU Model 2022, Jan 2021)

Runge-kutta method

1. Apply Runge-kutta method of order 4 to find an approximate value of y for $x = 0.2$ in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$ [VTU-June, July 2014]
2. Use fourth order Runge-kutta method to solve $(x + y)\frac{dy}{dx} = 1, y(0.4) = 1$ at $x = 0.5$ correct to four decimal places. [VTU-July 2013]
3. Use fourth order Runge-kutta method to find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$ taking $h = 0.2$. [VTU-Dec 2018, Jan 2018, Jan 2014, Dec 2012, Dec 2011]
4. Solve $(y^2 - x^2)dx = (y^2 + x^2)dy$ for $x = 0.1$. Given that $y = 1$ at $x = 0$ initially, by applying Runge-kutta method of order 4. [VTU- Model 2022, July 2017, Dec 2012, June 2012, July 2011]
5. Use fourth order Runge-kutta method to solve $\frac{dy}{dx} = x + y$ at $x = 0.2$ given that $y(0) = 1$ [VTU-Jan 2013]
6. Use fourth order Runge-kutta method to solve $\frac{dy}{dx} = -xy^2, y(0) = 2$. Compute $y(0.2)$ by taking $h = 0.1$ [VTU-June 2012]
7. Use fourth order Runge-kutta method to solve the following initial value problem
 $\frac{dy}{dx} = 3x + \frac{y}{2}, y(0) = 1$.compute $y(0.2)$ with $h = 0.1$ [VTU-Model 2022, Jan 2021, Dec 2015, Jan 2016, July 2016]
8. Given $\frac{dy}{dx} = x + y, y(0) = 1$. compute $y(0.2)$ by using fourth order Runge-kutta method. (take $h=0.2$) [VTU-Jan 2013]
9. Use fourth order Runge-kutta method to solve the following initial value problem

10. $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$.compute $y(0.2)$ with $h = 0.1$ [VTU Dec 2010]
10. Use Runge-kutta method to find $y(0.1)$ from $\frac{dy}{dx} = x^2 + y$ given that $y(0) = -1$ [VTU- July 2019]
11. Use fourth order Runge-kutta method to solve the following initial value problem
 $\frac{dy}{dx} + y + xy^2 = 0, y(0) = 1$ Find y at $x = 0.1$ (Jan 2020)
12. Use fourth order Runge-kutta method to solve the following initial value problem
 $\frac{dy}{dx} = 3e^x + 2y, y(0) = 1$ Find y at $x = 0.1, h = 0.1$ (VTU : Model 2022, Sept 2020)

Milne's Method

1. Applying Milne's Predictor - Corrector method, to find $y(1.4)$, from $\frac{dy}{dx} = x^2 + \frac{y}{2}$, given that $y(1) = 2, y(1.1) = 2.2156, y(1.2) = 2.4549, y(1.3) = 2.7514$ (VTU - Model 2022)
2. Applying Milne's Predictor-Corrector method, find $y(0.8)$, from $\frac{dy}{dx} = x^3 + y$, given that $y(0) = 2, y(0.2) = 2.073, y(0.4) = 2.452, y(0.6) = 3.023$ (VTU - Model 2022)
3. Using Milne's Predictor-Corrector method, find $y(4.5)$, given $\frac{dy}{dx} = \frac{2-y^2}{5x}$ and $y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0143, y(4.4) = 1.0187$ (VTU - Model 2022)
4. Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762$. Compute y at $x = 0.8$ by applying Milne's method. [VTU-Jan 2021, Sept 2020, July 2015, Jan 2016, June 2010]

5. The following table gives the solution of $5xy' + y^2 - 2 = 0$. find the value of y at $x = 4.5$ using Milne's predictor and corrector formulae. Use the corrector formula twice.

X	4	4.1	4.2	4.3	4.4
Y	1	1.0049	1.0097	1.0143	1.0187

[VTU-July 2017]

6. Given $\frac{dy}{dx} = xy + y^2$; $y(0)=1$, $y(0.1)=1.1169$, $y(0.2)=1.2773$, $y(0.3)=1.5049$ find $y(0.4)$ correct to three decimal places using Milne's predictor corrector method. Apply the corrector formula twice. [VTU Jan 2014, Dec 2012, June 2012]
7. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$, $y(0.3) = 2.090$ find $y(0.4)$ correct to 4 decimal places by Milnes predictor corrector method. Apply the corrector formula twice. [VTU Jan 2018, July 2013]
8. Given $2\frac{dy}{dx} = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's method. [VTU- Model 2022, Dec 2011]
9. Given $\frac{dy}{dx} = \frac{x+y}{2}$, Given that $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$ Find the value of y at $x = 2$ using Milnes Predictor and Corrector Formulae. ANS: 6.8732
10. Given $\frac{dy}{dx} = \frac{1}{x+y}$, Given that $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$ and $y(0.6) = 2.2493$. Find the value of y at $x = 0.8$ using Milnes Predictor and Corrector Formulae. [VTU-July 2019]
11. Apply Milne's predictor-corrector formulae to compute $y(0.3)$ given, $\frac{dy}{dx} = x + y^2$ with
- | | | | | |
|---|--------|--------|--------|--------|
| x | 0.0 | 0.1 | 0.2 | 0.3 |
| y | 1.0000 | 1.1000 | 1.2310 | 1.4020 |

(VTU Model 2019)