MODULE -1

LASERS & OPTICAL FIBERS

Syllabus

LASER: Basic properties of a LASER beam, Interaction of Radiation with Matter, Einstein's A and B Coefficients, Laser Action, Population Inversion, Metastable State, Requisites of a laser system, Semiconductor Diode Laser, Applications: Bar code scanner, Laser Printer, Laser Cooling. Numerical problems.

Optical Fiber: Principle and structure, Acceptance angle and Numerical Aperture (NA) and derivation of Expression for NA, Classification of Optical Fibers, Attenuation and Fiber Losses, Applications: Fiber Optic networking, Fiber Optic Communication. Numerical Problems

LASERS

Laser is the acronym of Light Amplification by Stimulated Emission of Radiation.

Properties of Lasers

1. The laser light is very nearly monochromatic.
2. The laser light is coherent with the waves all exactly in phase with one another.
3. Directionality**:** Laser beam is highly directional. It hardly diverges. This property isuseful to measure long distance with higher accuracy.
4. Intensity: Laser is extremely intense; hence by Laser we can achieve very high energy density.

Interaction of Radiation with Matter

When radiation interacts with matter, it leads to the transition of a quantum system such as atom or molecule from one energy state to another.

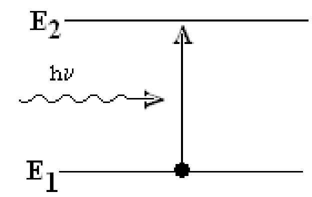
Consider two energy states E_1 and E_2, $(E_2 > E_1)$ of a system. An electron at the energy state E_1 is excited to E_2, when it absorbs a light photon of energy $\Delta E = (E_2 - E_1)$. If an electron makes a transition from the higher energy state E_2 to E_1, a light of photon of energy $\Delta E = E_2 - E_1$ is emitted. In both the case the frequency of the photon involved is $\nu = \frac{\Delta E}{h} = \frac{E_2 - E_1}{h}$

There are three possible ways in which **interaction of radiation and matter** can take place.

1. Induced (stimulated) absorption:

"The process in which an atom in a lower energy state is raised to a higher energy state by absorbing a suitable photon is called stimulated absorption."

Consider two energy states with energies E_1 and E_2. Let a photon of energy, $\Delta E = E_2 - E_1$ be incident on the atom. The atom absorbs the energy of the photon and its energy becomes equal to $E_1 + \Delta E = E_2$. Hence it makes a transition to the excited state E_2. This is called induced absorption.



Induced Absorption

Atom + photon → atom*

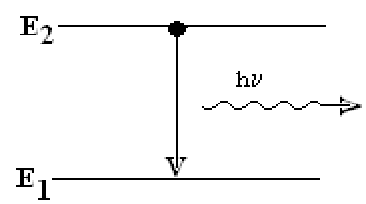
2. Spontaneous emission:

The process in which an atom in the higher energy state falls to the lower state by emitting a photon on its own is called spontaneous emission.

Consider an atom in the excited state, the atom voluntarily emits a photon of energy ΔE equal to $(E_2 - E_1)$ and falls to the energy state E_1. The emission where an atom emits a photon without any aid by external agency is called spontaneous emission.

The photons emitted may have any direction and phase.

Hence they are incoherent.

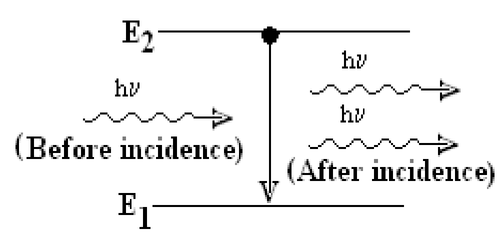


Spontaneous Emission

Atom*→ atom + photon

3. Stimulated emission:

"The process of the emission of a photon by a system under the influence of a incident photon of suitable energy, due to which the system transits from a higher energy state to a lower energy state is called stimulated emission."



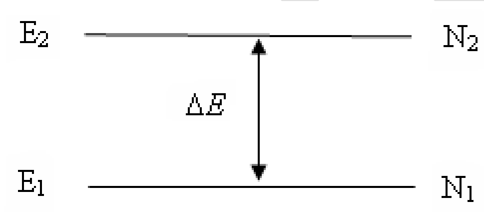
Stimulated Emission

Consider an atom in the exited state with energy E_2. Let a photon of energy $\Delta E = E_2 - E_1$ interacts with this atom. As a result, the atom emits a photon and transits to the lower energy state. The emitted photon will have same phase, energy and direction of movement as that of the incident photon.” The electromagnetic waves associated with the two photons will have same phase and thus they are coherent.

This kind of emission is responsible for laser action.

Atom* + photon →Atom+ 2 photon

Einstein’s A and B coefficients (Expression for Energy Density):



Consider two energy states E_1 and E_2 of a system of atoms ($E_1 < E_2$). Let N_1 be the number of atoms in the energy state E_1 and N_2 be in E_2 per unit volume of the system. Then N_1 and N_2 are called the number density of atoms in the energy states E_1 and E_2 respectively. Let $U_\upsilon d\nu$ be the energy of the incident radiation per unit volume of the system where radiations lie in the frequency range ν to $\nu + d\nu$, then 'U_υ' is called the energy density of frequency "ν".

<u>Case (1): Induced absorption</u>:

The number of such absorptions per unit time per unit volume is called the rate of absorption.

$\therefore$ Rate of absorption $\propto N_1 U_\upsilon$

Or Rate of absorption = $B_{12} N_1 U_\upsilon \rightarrow$ (1)

Where B_{12}→ Einstein’s Coefficient of induced absorption.

<u>Case (2): Spontaneous emission:</u>

Rate of spontaneous emission = $A_{21} N_2$ ------------ (2)

where A_{21}→ Einstein’s coefficient of spontaneous emission.

Case 3: Stimulated emission:

The rate of stimulated emission $= B_{21}N_2U_\upsilon$ --------------- (3)

where B_{21} is the Einstein's coefficient of stimulated emission.

At thermal equilibrium,

The rate of absorption= Rate of spontaneous emission+ Rate of stimulated emission.

ie $B_{12}N_1U_\upsilon = A_{21}N_2 + B_{21}N_2U_\upsilon$

ie $U_\upsilon(B_{12}N_1 - B_{21}N_2) = A_{21}N_2$

$$\therefore U_\upsilon = \frac{A_{21}N_2}{B_{12}N_1 - B_{21}N_2}$$

Or $$U_\upsilon = \frac{A_{21}}{B_{21}}\left[\frac{1}{\frac{B_{12}}{B_{21}}\frac{N_1}{N_2} - 1}\right]$$ ---------------------- (4)

According to Boltzmann's law we have

$$N_2 = N_1 e^{-\left(\frac{E_2 - E_1}{KT}\right)} = N_1 e^{-\frac{h\upsilon}{KT}}$$

$$\therefore \frac{N_1}{N_2} = e^{h\nu/kT}$$ -------------(5)

Therefore equation (4) becomes

$$U_\upsilon = \frac{A_{21}}{B_{21}}\left[\frac{1}{\frac{B_{12}}{B_{21}}e^{\frac{h\upsilon}{KT}} - 1}\right]$$ ----------------- (6)

According to Planck's law of radiation

$$U_\nu = \frac{8\pi h\nu^3}{C^3}\left[\frac{1}{e^{\frac{h\nu}{KT}} - 1}\right] \quad \text{-------------------- (7)}$$

Comparing equation (6) and (7) we have

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{C^3}, and\ \frac{B_{12}}{B_{21}} = 1$$

➔ $B_{12}=B_{21}=B$ ---------- (8) and $A_{12}=A$

The identity (8) implies that the probability of induced absorption is equal to the probability of stimulated emission.

Therefore we can write the expression for energy density in terms of Einstein's A & B coefficients as

$$U_\upsilon = \frac{A}{B\left[e^{\frac{h\nu}{hT}} - 1\right]}$$

Condition for Laser action:

1. **Population inversion:**

"It is the state of a system in which the number of atoms in the higher energy level is greater than the number of atoms in the lower energy state."

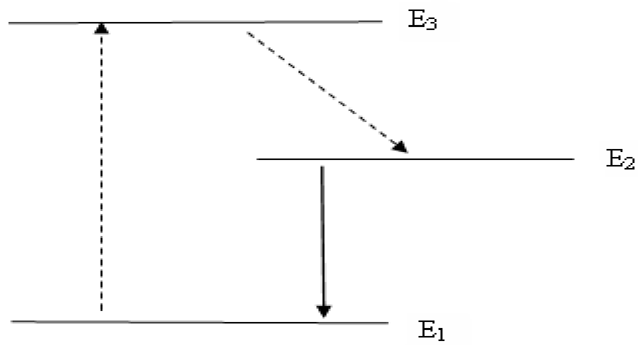
Under normal condition, the population is more in lower state. But for stimulated emission and hence for lasing action more atoms must be present in the exited state. This can be achieved by some artificial means i.e. by providing energy in to the active medium of the laser system.

2. Metastable state:

Under normal condition, population inversion doesn't exist. However, it is possible to achieve the population inversion in certain systems which possess a special exited state called **metastable state.**

It is an exited state different from the ordinary exited state. The atoms which are excited to the higher energy states remain for a short duration of 10^{-8} sec and return to lower energy state. In case the state at which the atom is exited is a metastable state, then it stays there for a longer time of about 10^{-3} to 10^{-2} sec. This property helps in achieving population inversion.

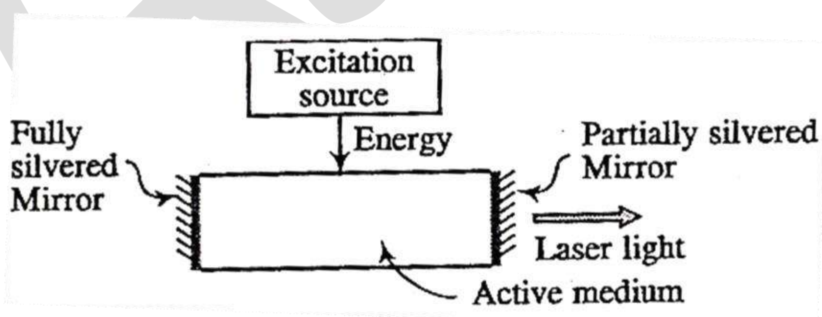
Laser Action



Consider 3 energy levels E_1, E_2, and E_3 of an atomic system. Let E_2 be a metastable state. By supplying suitable external energy, the atoms are exited from E_1 to E_3. The atoms in E_3 undergo spontaneous transition to E_1 and E_2 rapidly. Since E_2 is a metastable state, the atoms in this state stay for a longer time duration because of which the population in E_2 increases and population inversion is created.

Once the population of E_2 exceeds that of E_1, the stimulated emission takes place. The photons emitted are all identical in respect of phase, wavelength and direction; grow to a large number which is the laser light.

Requisites of Laser System



1) **Excitation mechanism:** An excitation mechanism/source provides energy in an appropriate from for pumping the atoms to higher energy levels (xenon flash lamp). If the pumping is achieved by light energy

input, then it is called optical pumping (Ruby laser). If the pumping is achieved by electrical energy input, then it is called electrical pumping (He-Ne laser).

2) **Active medium:** A medium in which light gets amplified is known an active medium. The medium may be solid, liquid or a gas. A part of the input energy is absorbed by the active medium in which population inversion occurs at a certain state.

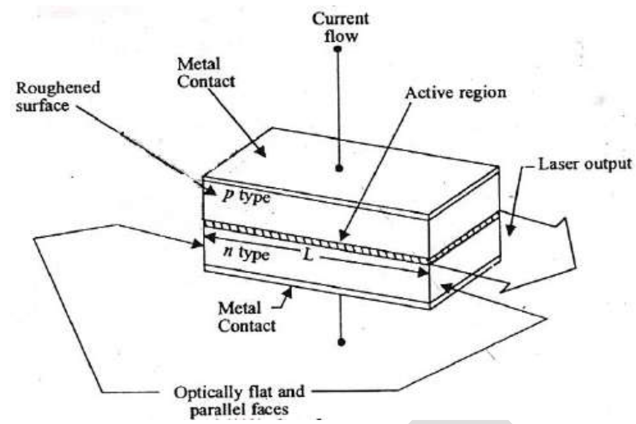
3) Laser cavity: It provides the feedback necessary to tap certain permissible part of laser energy from the active medium. It consists two mirrors along with the active medium called cavity. The mirrors reflect the photons to and fro through the active medium. A laser device consists of an active medium bound between two parallel mirrors of high reflectivity. Inside the cavity two types of waves exist, one moving towards the right and other to the left. These waves interfere constructively or destructively depending on the phase difference.

In order to arrange for constructive interference, the distance 'L' between the two mirrors should be such that the cavity should support an integral number of half wavelength, i.e. $L = m\frac{\lambda}{2}$, where L is the distance between the two mirrors. m is an integer >0. This results in the amplification of stimulated emission of radiation which is the laser light.

Semiconductor Diode laser:

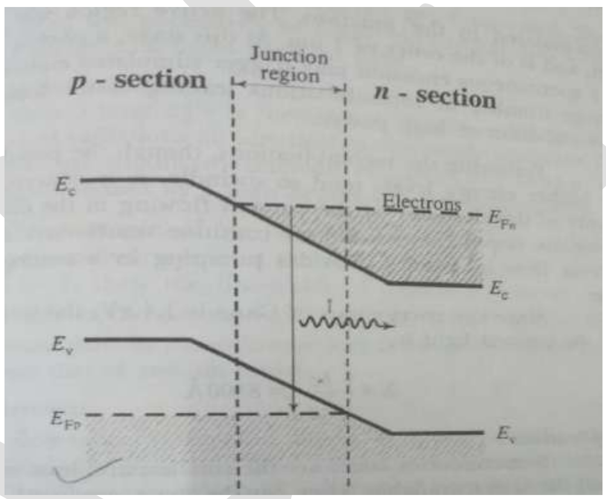
Construction:

A semiconductor diode laser is a specially fabricated p-n junction device that emits coherent light when it is forward biased. The diode is extremely small in size with sides of the order of 1mm. The junction lies in the horizontal plane through the centre. The top and bottom faces are metalized and ohmic contacts are provided to pass current through the diode. The front and rear faces are polished parallel to each other and perpendicular to the plane of the junction. The other two faces are roughened to prevent lasing action in that direction. The active region consists of a layer of about 1 μ m thickness.



Working:

The population inversion can be achieved in a semiconductor by using it in the form of a heavily doped p-n junction and forward biasing it. With very high doping on n-side, the donar levels as well as a portion of the conduction band (CB) are occupied by electrons and the Fermi level lies within the CB.

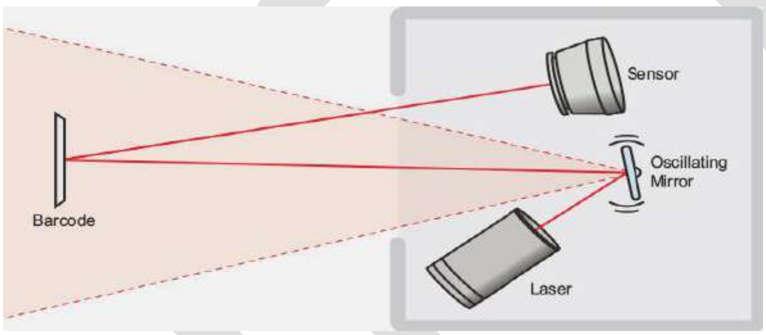


When a forward bias is applied, the fermi energy levels shifts and the new distribution is as shown in figure. Electrons and holes are injected in to the depletion region, where they appear in large number. At low forward current, the electron-hole recombination causes spontaneous emission of photons. As the current increased and reaches a threshold value, the carrier-concentration in the depletion region will reach very high values. The upper levels in the depletion region are having high population of electrons while the lower levels are having large number of holes. This is the state of population inversion. The narrow region where the state of population inversion is achieved is called inversion or active region. Thus the forward bias current plays the role of pumping agent in semiconductor laser. The photons that propagate in the junction plane induce the conduction electrons to jump in to the vacant state of VB. The stimulated electron hole recombination causes emission of coherent radiation.

Applications of Laser:

Bar code scanner:

A barcode, consisting of bars and spaces of varying width that can be read with an optical (Laser) barcode scanner. A barcode scanner/reader is a device with lights, lenses and a sensor that decodes and captures the information contained in barcodes Laser light is incident on the Bar Code label surface and its reflection is captured by a sensor (laser photo detector) to read a bar code. A laser beam is reflected off a mirror and swept left and right to read a bar code. Reflections of laser light are strong in white areas and weak in black areas. A sensor receives reflections to obtain analog waveforms. The analog signal is converted into a digital signal via an A/D converter. (Binarization). Data retrieval is achieved when a code system is determined from the digital signal obtained. (Decoding process).



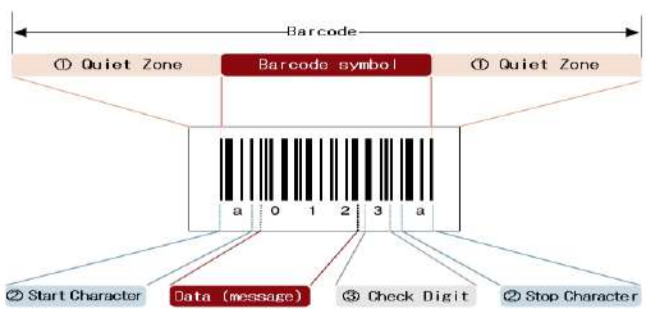


Figure: A typical Barcode

LASER Printer

Laser Printers were invented in 1969 by Gary Starkweather. The laser printers are digital printing device that can be used to create high quality text and images on the plain paper.

Working:

1. Laser beam projects an image of the page to be printed onto an electrically charged rotating photo sensitive drum coated with selenium
2. Photo conductivity allows charge to leak away from the areas which are exposed to laser light and the area gets positively charged.
3. Toner particles are then electro statically picked up by the drum's charged areas which have been exposed to light.
4. The drum then prints the image onto paper by direct contact.
5. The paper finally passes through two hot rollers, known as the fuser unit which fuses the ink to the paper.

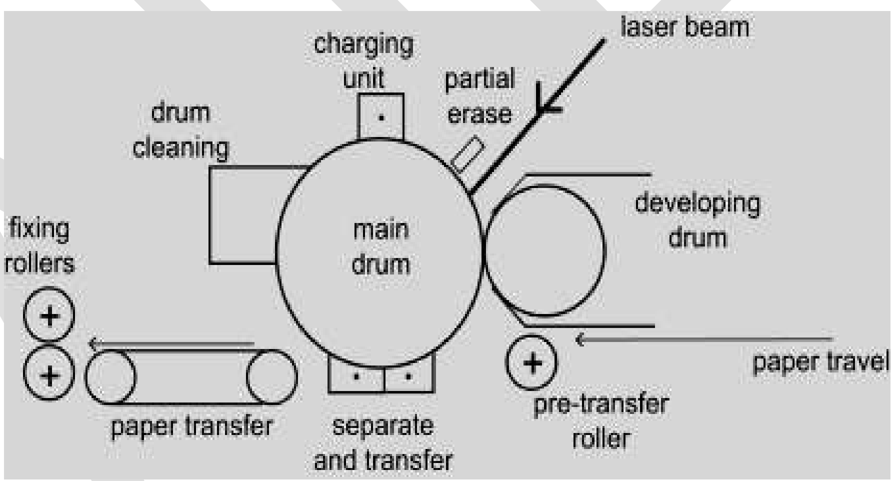


Figure: Schematic Diagram of Laser Printer

Laser Cooling

Laser cooling is the use of dissipative light forces for reducing the random motion and thus the temperature of small particles, typically atoms or ions. Here the incident laser light is used by the atoms for absorption and subsequent spontaneous emission of photons. The amount of light used in these processes depends on the velocity or momentum of an atom or ion i.e, $E = \frac{h}{P} = \frac{h}{mv}$. The optical frequency of chosen laser light should be higher than the atomic resonance, so that only the fastest atoms can absorb photons. Subsequently, the laser frequency is reduced so that slower and slower atoms participate in the interaction, and finally all atoms have a greatly reduced speed (at least in one dimension). This corresponds to a lower temperature, assuming that thermal equilibrium can be re-established. Using, this phenomenon their different types of laser cooling, such as Doppler Cooling, Sisyphous Cooling etc.

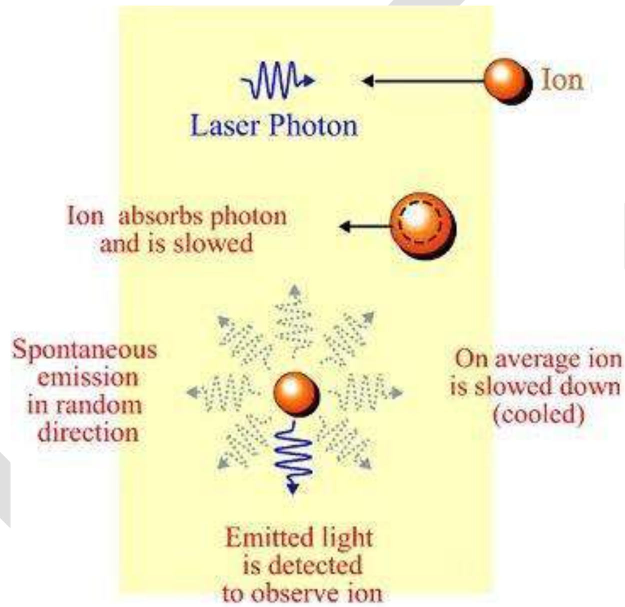


Figure: Laser Cooling of atoms

OPTICAL FIBER

Introduction

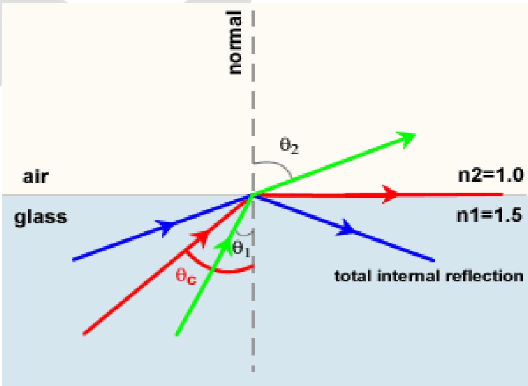
A conventional method of long distance communication uses radio waves (10^6 Hz) and micro waves (10^{10} Hz) as carrier waves. A light beam acts as carrier waves which is capable of carrying far more information since optical frequencies are extremely large (10^{15} Hz).

Soon after the discovery of laser, some preliminary experiments in propagation of information carrying light waves through the open atmosphere wave carried out, but it was realized that the unwanted elements such as rain, fog etc, leads to adverse effects. Thus in order to have an efficient and dependable communication system one would require a guiding medium in which the information carrying light waves could be transmitted. This resulted in the development of optical fiber which is an efficient guiding medium for laser light.

Optical fibers are essentially light guides used in optical communication as waveguides. Use of light waves in place of radio and microwave has improved the speed of communication.
The principle behind the transmission of light waves in an optical fiber **is total internal reflection**.

Total internal reflection

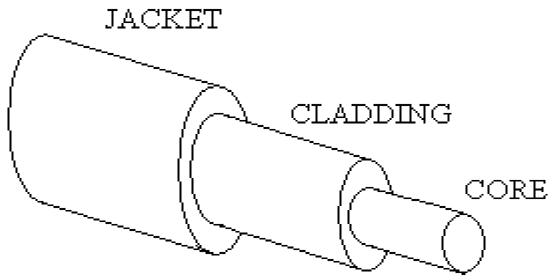
The basic principle of optical fiber is multiple total internal reflection. When a ray of light travel from denser to a rarer medium, at an angle of incidence greater that critical angle Θ_c, the ray is not refracted but it is reflected into the same denser medium. This property is called **total internal reflection**. Light signals are transmitted through optic fibers by multiple total internal reflections.



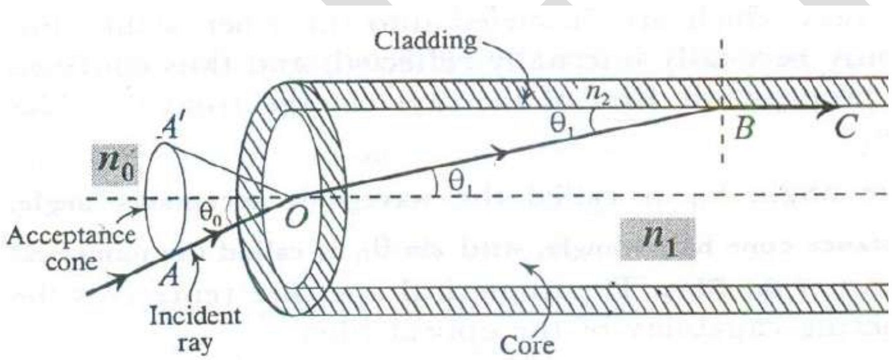
[Light ray's incident on a high to low refractive index interface (e.g. glass–air)]

Structure of Optical Fiber:

An optical fiber consists of a very thin transparent cylindrical core having refractive index n_1 surrounded by a cylindrical shell called cladding of slightly lower refractive index n_2. The core cladding system is surrounded by plastic jackets



Expression for Angle of Acceptance and Numerical Aperture:



Consider an optical fiber into which light is launched at one end from a medium of RI $\mathbf{n_0}$. Let $\mathbf{n_1}$be the RI of core and $\mathbf{n_2}$be that of the cladding. Assume that a ray of light enters the fiber at an angle θ_0 with respect to the axis of the fiber. The light ray refracts at an angle θ_1 and strikes the core – cladding interface at an angle of $(90-\theta_1)$. If $(90-\theta_1)$ Is greater than the critical angle foe the core – cladding interface, the ray undergoes total internal reflection at the interface.

It is clear from the fig that any light ray which enters the core at an angle less than θ_0 will undergo total internal reflection at core – cladding interface and propagates along the fiber. The angle θ_0 is called the **acceptance angle.** “It is the maximum angle that a light ray can have relative to the axis of the fiber and propagates down the fiber”. Sine of the acceptance angle θ_0 , $\sin\theta_0$ is called the **numerical aperture (NA)** of the fiber. It represents the light gathering capacity of the optical fiber.

Applying Snell's law at 'O'

$$\frac{\sin\,\theta_0}{\sin\,\theta_1} = \frac{n_1}{n_0} \rightarrow (1)$$

Applying Snell's law at 'B'

$$n_1 \sin(90 - \theta_1) = n_2 \sin 90$$

$$\cos\theta_1 = \frac{n_2}{n_1} \rightarrow (2)$$

Rewriting equation (1),

$$\sin\,\theta_0 = \frac{n_1}{n_0}\sin\,\theta_1$$

$$= \frac{n_1}{n_0}\sqrt{1-\cos^2\theta_1}$$

$$= \frac{n_1}{n_0}\sqrt{1-\frac{n_2^2}{n_1^2}}$$

$$\sin\,\theta_0 = \frac{\sqrt{n_1^2 - n_2^2}}{n_0} \rightarrow (3)$$

If the medium surrounding the fiber is air, then $n_0 = 1$

$$\therefore\ \sin\,\theta_0 = \sqrt{n_1^2 - n_2^2}$$

If θ_i is the angle of incidence of an incident ray w.r.t. the axis of the fiber, then ray will be able to propagate,

if $\quad \theta_i \langle \theta_0$

if $\quad \sin\,\theta_i \langle \sin\,\theta_0$

or $\quad \boxed{\sin\,\theta_i \langle N.A.}$

This is the **condition for propagation.**

Fractional Index Change (Δ):

It is the ratio of the refractive index (RI) difference between the core and cladding to the RI of core of an optical fiber, i.e. $\Delta = \frac{n_1 - n_2}{n_1}$; Where; n_1 and n_2 is the RI of core and cladding.

Relation between Numerical aperture (NA) and Fractional index change(Δ)

We have numerical aperture,

$$N.A. = \sqrt{n_1^2 - n_2^2}$$

$$= \sqrt{(n_1 + n_2)(n_1 - n_2)}$$

But Fractional Index Change, $\Delta = \frac{n_1 - n_2}{n_1}$

i.e. $n_1\Delta = n_1 - n_2$

$\therefore$ $N.A. = \sqrt{(n_1 + n_2)n_1\Delta}$

Since $n_1 \approx n_2$; $n_1 + n_2 \approx 2n_1$

Hence $N.A. = \sqrt{2n_1 n_1 \Delta} = n_1\sqrt{2\Delta}$

i.e. $\mathbf{N.A = n_1\sqrt{2\Delta}}$

Modes of propagation:

Light propagates as an electromagnetic wave through an optical fiber. All waves having ray directions above the critical angle will be trapped within the fiber due to total internal reflection. But all such waves do not propagate along the fiber. There are certain ray directions allowed for the propagation. These allowed ray directions or possible number of path of light in an optical fibers are known as modes of propagation. The paths are zigzag paths excepting the axial direction. The number of modes that a fiber will support depends on the diameter of the core and wavelength of the wave being transmitted.

Types of optical fibers:

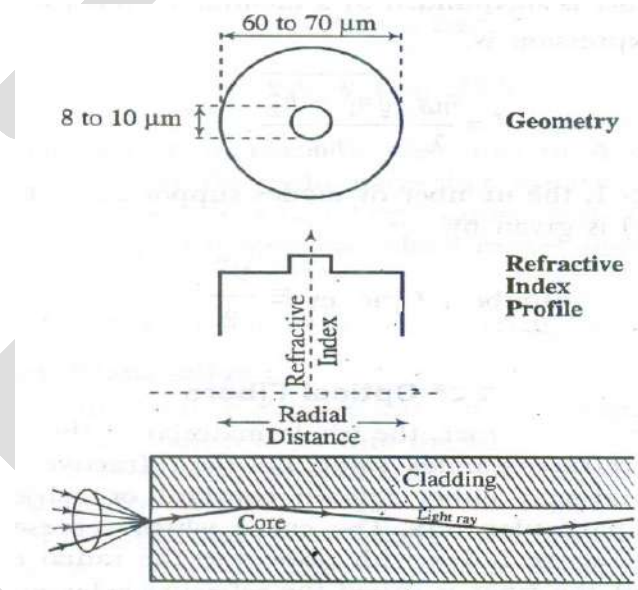
Refractive index profile is a curve which represents the variation of refractive index with respect to the radial distance from the axis of the fiber.

Depending on the RI profile and number of modes that a fiber can support, we have three types of optical fibers. They are

1) Step index single mode fiber, 2) Step index multi mode fiber 3) Graded index multi mode fiber

1. Step index single mode fiber:

A step index single mode fiber has a core diameter of about 8 to 10 μm and external diameter of cladding is 60 to 70 μm. The RI of the core has a uniform value. The cladding also has a uniform RI but slightly lesser than that of the core. The RI of the fiber changes abruptly at the core – cladding interface. Hence it is called a step index fiber. This fiber can support only one mode of propagation along its axis. Hence it is called a single mode fiber.

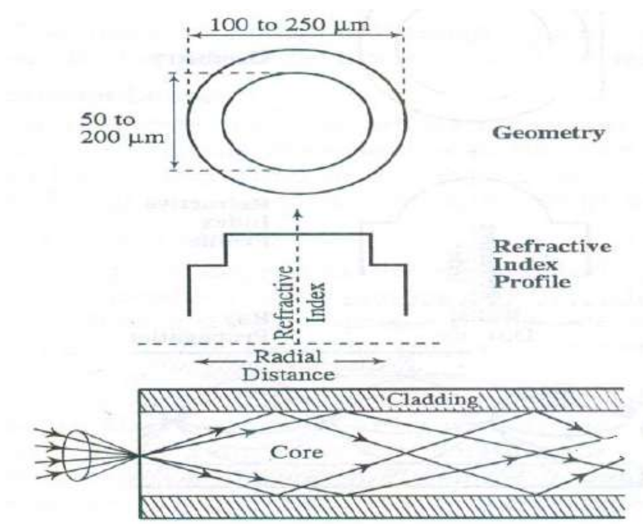


Due to narrow diameter of the fiber only laser can be used as the source of light with these fibers. There is no intermodal dispersion in the fiber.

These are widely used in submarine cable systems.

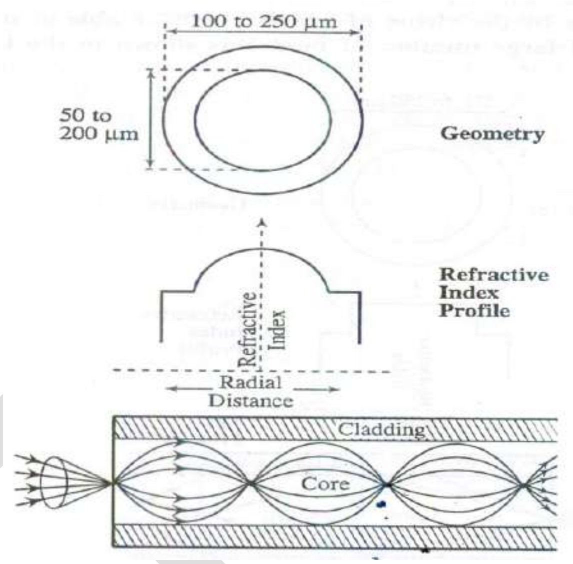
2. Step index Multi mode fiber:

This fiber has a core diameter of 100 μm. The RI remains uniform in the core and cladding region. But the RI changes abruptly at the core – cladding interface. Because of larger diameter, this fiber allows many modes to propagate through it.



The step index multimode fiber can accept either a laser or LED as source of light. It is used in data links which has lower band width requirements.

3. Graded Index Multimode fiber:



It is a multimode fiber with a core consisting of concentric layers of different refractive indices. Therefore, RI of the core decreases with distance from the fiber axis. The RI of the cladding remains uniform. The RI profile and the modes of propagation are shown in fig. such a RI profile causes a periodic focusing of light propagating through the fiber. Either a laser or a LED can be the source for these fibers.

Normalized frequency (V – number):

It is the relation between fiber size, the refractive indices and the wavelength of light propagating through the fiber. It is given by,

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}$$

Where, $d \rightarrow$ diameter of the core; $n_1 \rightarrow$ RI of the core; $n_2 \rightarrow$ RI of the cladding;

$\lambda \rightarrow$ Wavelength of light.

Since $\sqrt{n_1^2 - n_2^2}$ = NA, we can write,

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}$$

The number of modes supported by a fiber is given by, $M_N = \frac{V^2}{2}$

Attenuation:

The loss of power suffered by the optical signal as it propagates through the fiber is called attenuation.

The attenuation or fiber loss is due to the following factors:

1. Absorption losses
2. Scattering losses
3. Radiation or bending losses

1. **<u>Absorption Losses:</u>**

The loss of signal strength occurs due to absorption of photons during its propagation. Photons are absorbed by

a) Impurities in the silica glass of which the fiber is made of.
b) Intrinsic absorption by the glass material itself.

a. Absorption by impurities:

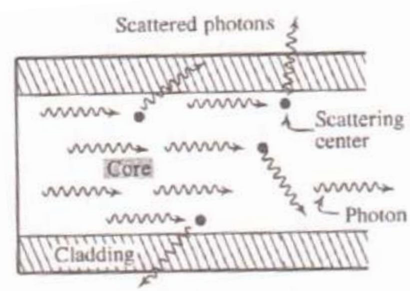
The impurities that are generally present in fiber glass are iron, chromium, cobalt, copper etc. During signal propagation when photons interact with these impurities, the electrons absorb the photons and get excited to higher energy levels. Later these electrons give up their absorbed energy in the form of light photons. But this is of no use, since these photons differ in wavelength and phase.

b. Intrinsic Absorption

The fiber material itself has a tendency to absorb light energy however small it may be. Hence there will be a loss and is known as intrinsic absorption.

2. **<u>Scattering Losses:</u>**

The optical power is lost due to the scattering of photons. This scattering is due to the non- uniformity in the density of the fiber material, which leads to the variation in the RI of the fiber. Structural inhomogenities and defects created in the fiber can also cause scattering. The loss of light energy by scattering is found to be wavelength dependent. It decreases with increase in the wavelength of light to be transmitted through the fiber.

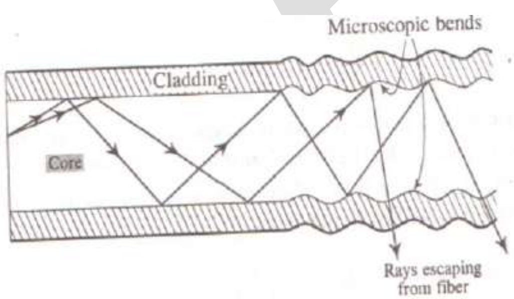


RAYLEIGH SCATTERING
Fig. 8

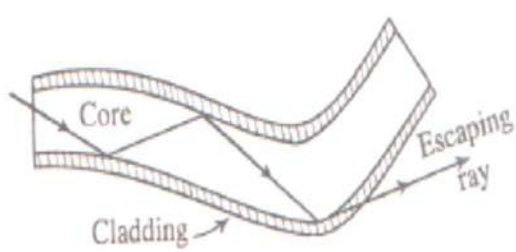
3. **<u>Bending Losses (Radiation Losses):</u>**

Radiation losses occur due to bending of fiber. There are two types of bends

a) Microscopic bends
b) Macroscopic bends



MICROSCOPIC BENDING LOSS



RADIATION LEAKS

Microscopic bends are caused during manufacturing as well as due to the applied stress on the fiber. Macroscopic bending arises during the installation of the fiber. At the point of a bend, light will escape to the surrounding medium due to the fact that the angle of incidence at that point becomes lesser than the critical angle.

To minimize these losses, the optical fiber has to be laid without sharp bends and they should be freed from the external stresses by providing mechanical strength through external coverage.

Attenuation Co-efficient:

Attenuation is the loss of power suffered by the optical signal as it propagates through the fiber. It is also referred as fiber loss.

Attenuation co-efficient or attenuation, $\alpha = -\frac{10}{L}\log(\frac{P_o}{P_i})$ dB/km

Where P_i is the optical power launched at the input and P_o the output power after traveling a distance L km.

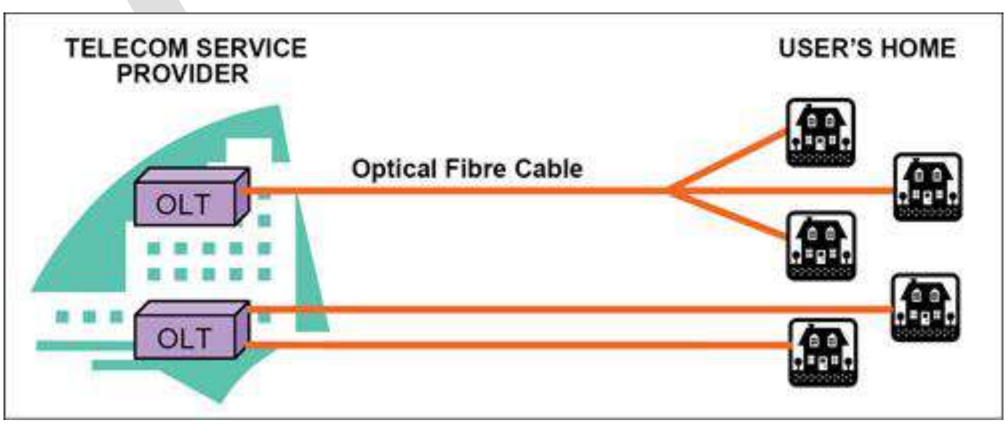
Application:

Fiber optic networking

Fiber optic networking consists of super-fast pulses of light sent through the optical fiber. These pulses are detected by the sensor kept on the other end of the fiber, which converts the pulse into electrical signal output. This is performed by a special piece of equipment called the optical network terminal, which then sends the signal through an Ethernet connection to the user. The stretch between the main fiber network line and the end user is referred to as the "last mile". The fiber connections that run all the way to the end user's home, business or desktop computer. This is the fastest and most expensive "last mile" option, as it brings the full speed and reliability of signal straight to the consumer.

Nowadays, even copper cables are used to carry the internet connection from a terminal called a "street cabinet" to a whole housing block, campus or residential building. This option is less expensive, but a small amount of the signal speed is lost in the "last mile."

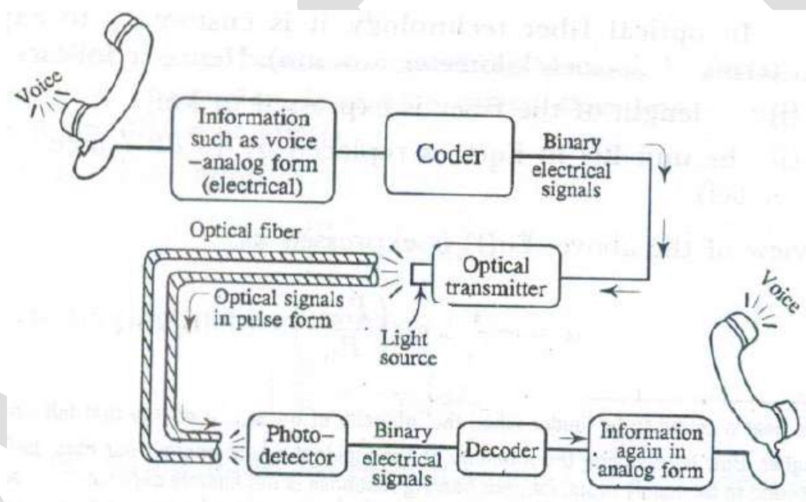
The main difference is that fiber optic networking doesn't utilize electric current like other types of internet connections instead it uses light (Laser/ LED) which are transmitted through the fiber glass core.



Fiber Optic Communication (Point – point communication system using optical fibers):

In a point - point communication system, we have analog information such as voice of a telephone user. The voice gives rise to electrical signals in analog form coming out of the transmitter section of the telephone. With the help of a coder, the analog signal is converted into binary data. The binary data in the form of a stream of electrical pulses are converted into pulses of optical power by modulating the light emitted by an optical source such as a laser diode or LED. This unit is called optical transmitter, from which the optical power is launched into the fiber.

During the propagation of the signal, attenuation or losses occurs. This may reach a limiting stage beyond which it may not be possible to retrieve the information from the light signal. Hence a repeater is needed in the transmission path. A repeater consists of a receiver and a transmitter. The receiver converts the optical signal into corresponding electrical signal and then it is amplified. These electrical signals are again converted into optical signals and fed into the optical fiber.



At the receiving end the optical signal from the fiber is fed into a photo detector. Hence signal is converted to pulses of electric current. This is then fed to a decoder which converts the binary data into an analog signal, which will be the same information such as voice; which was there at the transmitting end.

Merits and demerits of optical communication:

The advantages of optical fiber communication include the following

- The large amount of information that can be transmitted per unit time in a fiber.
- Low attenuation and allows data transmission for longer distances
- The optical cable is resistance for electromagnetic interference
- The size of the fiber cable is 4.5 times better than copper wires

- These cables are lighter, thinner, and occupy less area compare with metal wires.
- Installation is very easy due to less weight.
- The optical fiber cable is very hard to tap because they don't produce electromagnetic energy. These cables are very secure while carrying or transmitting data.
- A fiber optic cable is very flexible, easily bends, and opposes most acidic elements that hit the copper wire.

The disadvantages of optical fiber communication include

- The optical fiber cables are very difficult to merge & there will be a loss of the beam within the cable while scattering.
- The installation of these cables is cost-effective. They are not as robust as the wires. Special test equipment is often required to the optical fiber.
- Fiber optic cables are compact and highly vulnerable while fitting
- These cables are more delicate than copper wires.
- Special devices are needed to check the transmission of fiber cable.

QUESTIONS:

1. Discuss the possible ways through which radiation and matter interaction can takes place. (6)
2. Explain the terms (i) Resonant Cavity (ii) Stimulated Emission (iii) Metastable state (6)
3. Explain the requisites of a laser system. (4)
4. Discuss the need of population inversion. (4)
5. Obtain an expression for energy density of radiation under equilibrium condition in terms of Einstein co-efficient. (6)
6. Discuss the different vibrational modes of a carbon dioxide laser along suitable diagrams. (6)
7. Explain the construction and working of a semiconductor laser. (6)
8. Explain the working of BAR code reader. (4)
9. Explain working the Laser Printer. (4)
10. Mention any four applications of laser. (4)
11. Define Acceptance angle and Numerical Aperture and hence derive an expression for NA in terms of RIs core, cladding and surrounding.
12. Discuss the types of optical fibers based on Modes of Propagation and RI profile.

PROBLEMS

1. The average output power of laser source emitting a laser beam of wavelength 632.8nm is 5 mW. Find the number of photons emitted per second by laser source.

Hint:

$\Delta E = h\nu = hc/\lambda$

$N \times \Delta E = 5 \times 10^{-3} W$

(Ans: $\Delta E = 3.143 \times 10^{-19}$ J, $N = 1.59 \times 10^{16}$)

2. A pulsed laser emits photons of wavelength 780nm with 20 mW average power/pulse. Calculate number of photons contained in each pulse if pulse duration is 10ns.

Hint: Energy of each photon $\Delta E = h\nu = hc/\lambda$

Energy, $E = P \times t$

$N X \Delta E = E$

(ANS: $\Delta E = 2.55 \times 10^{-19}$J, $E = 2 \times 10^{-10}$J, $N = 7.86 \times 10^{8}$)

3. A medium in thermal equilibrium at temperature 300K has two energy levels with wavelength separation of 1μm. Find ratio of population densities of upper and lower levels.

Hint: Boltzmann factor $\frac{N_2}{N_1} = e^{\frac{-\Delta E}{kT}} = e^{\frac{-hc}{\lambda kT}}$

(ANS: $\frac{N_2}{N_1} = 1.365 \times 10^{-21}$)

4. A pulse from laser with power 1mW lasts for 10ns .If the number of photons emitted per second is 3.491×10^{7}, Calculate the wavelength of laser.
Hint: Energy of each photon $\Delta E = h\nu = hc/\lambda$

Energy, $E = P \times t$

$N X \Delta E = E$

(ANS: $\lambda = 6943.6 \times 10^{-10}$m)

5. Find the ratio of population of two energy levels in a medium in thermal equilibrium if transition between them produces light of wavelength 694.3nm.Assume the ambient temperature as27^{o}C.

Data: wavelength of light $\lambda = 694.3$nm

Temperature ,T=27^{o}C=273+27= 300K

(ANS: $\frac{N_2}{N_1}$= 8.87x10^{-31})

6. The refractive indices of core and cladding are 1.50 and 1.48 respectively in an optical fiber. Find the numerical aperture and angle of acceptance.

Hint: N.A.=$\sqrt{(n_1^2 - n_2^2)}$

N.A= sinθ_o

(ANS: N.A. =0.244 and θ_o=14.4^o)

7 An optical fiber has a core material with RI 1.55 and its cladding material has RI 1.50. The light is launched into it in air. Calculate its N.A, acceptance angle and fractional index change.

Hint: N.A.=$\frac{\sqrt{(n_1^2 - n_2^2)}}{n_o}$

Δ=$\frac{n_1 - n_2}{n_o}$

(ANS: N.A= 0.39, θ_o=23^o, Δ =0.032)

8. The angle of acceptance of an optical fiber is 30^o when kept in air. Find angle of acceptance when it is in a medium of refractive index 1.33.

Hint: sin θ_o=$\frac{\sqrt{(n_1^2 - n_2^2)}}{n_o}$

sin θ_o^I=$\frac{\sqrt{(n_1^2 - n_2^2)}}{n_0^I}$

(ANS:θ_o=0.5, θ_o^I=22^o)

9. Find the attenuation in an optical fiber of length 500m, when a light signal of power 100mW emerges out of the fiber with a power of 90mW.

Hint: Fiber attenuation, α= -$\frac{10}{L} log_{10}(\frac{P_{out}}{P_{in}})$dB/km

(ANS: α=0.915dB/km)

10.The attenuation of light in an optical fiber is 3.6dB/km.What fraction of its initial intensity remains after i)1km, ii) after 3km ?

Hint: Fiber attenuation, $\alpha = -\frac{10}{L} log_{10}(\frac{P_{out}}{P_{in}})$dB/km

11. An optical glass fiber of refractive index 1.50 is to be clad with another glass to ensure internal reflection that will contain light traveling within 5^o of the fiber axis. What maximum index of refraction is allowed for the cladding?

Hint : $n_1 sini = n_2 sinr$ Ans: $n_2 < 1.49$

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Module 2

Quantum Mechanics

Syllabus: *de Broglie Hypothesis and Matter Waves, de Broglie wavelength and derivation of expression by analogy, Phase Velocity and Group Velocity, Heisenberg's Uncertainty Principle and its application (Nonexistence of electron inside the nucleus - Non Relativistic), Principle of Complementarity, Wave Function, Time independent Schrödinger wave equation (Derivation), Physical Significance of a wave function and Born Interpretation, Expectation value, Eigen functions and Eigen Values, Particle inside one dimensional infinite potential well, Quantization of Energy States, Waveforms and Probabilities. Numerical Problems.*

Wave Particle Dualism:

According to wave theory, light waves leave a source with their energy spread out continuously through the wave pattern. According to the quantum theory, light consists of a stream of photons each small enough to be absorbed by a single electron. Both views have strong experimental support. Hence we can think of light as having a dual character. "The property of light of behaving as both a particle and a wave is called wave – particle duality."

De – Broglie's Hypothesis:

De-Broglie extended the wave particle dualism of light to the material particles. This is known as de-Broglie hypothesis. According to this hypothesis, material **particles in motion possess a wave character**. The waves associated with material particles are called **matter waves or de-Broglie waves**.

Expression for de-Broglie wavelength:

Consider a photon of frequency 'ν'. Then its energy is given by

$$E = h\nu \rightarrow (1) \; ; \qquad h \rightarrow \text{Planck's constant.}$$

If this photon is treated as a particle of mass 'm' moving with a velocity 'c', its energy is given by

$$E = mc^2 \rightarrow (2)$$

From (1) and (2),

$$h\nu = mc^2$$

$$\frac{hc}{\lambda} = mc^2$$

Or $mc = \frac{h}{\lambda}$

i.e. $\lambda = \frac{h}{p} \rightarrow (3)$; where p=mc is the momentum of the photon.

According to de – Broglie, the equation (3) must be a universal relation applicable to photons as well as to any material particle. Hence a particle of mass 'm' moving with a velocity 'v' must be associated with a wave of wavelength. $\boxed{\lambda = \frac{h}{mv}} \rightarrow$; Where mv = p, momentum of the particle.

This equation is known as de-Broglie equation.

De –Broglie wavelength of electrons:

If the electrons are accelerated by a potential difference of 'V' so that electrons acquire a velocity 'v' then the work done on the electrons is 'eV'. As a result the kinetic energy gained by the electrons is $\frac{1}{2}mv^2$.

$\because$ We can write $eV = \frac{1}{2}mv^2 \rightarrow (1)$

Or $2eVm = m^2v^2$

$\Rightarrow mv = \sqrt{2meV}$

From de-Broglie's equation, $\lambda = \frac{h}{mv}$

$$\boxed{\lambda = \frac{h}{\sqrt{2meV}} \rightarrow (2)}$$

Or $\boxed{\lambda = \frac{h}{\sqrt{2mE}} \rightarrow (3)}$; (from (1)

Where $E = \frac{1}{2}mv^2$ is the kinetic energy of the electrons.

Substituting $m = 9.1\times10^{-31}kg$; $e = 1.6\times10^{-19}C$ and $h = 6.625\times10^{-34}J-s$ in equation (2) we get

$$\lambda = \frac{6.625\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times1.6\times10^{-19}\times V}} = \frac{12.26\times10^{-10}}{\sqrt{V}} = \frac{12.26\text{A}^0}{\sqrt{V}}$$

Mater waves:

The wave associated with the moving particles is called matter waves or de-Broglie waves.

Characteristics of Matter waves:

1. Matter waves are the waves associated with moving material particles.
2. Matter waves produce interference and diffraction effects similar to electromagnetic waves.
3. The amplitude of the matter waves at a particular region and time gives the probability of finding the particle at the same region and time.
4. Since the wavelength of matter waves is inversely proportional to the velocity of the body, a body at rest has an infinite wavelength and one traveling with high velocity has a lower wavelength.
5. Wavelength of matter waves decreases with increase in mass of the body. Due to this reason, the wavelike behavior of heavier bodies is not very evident whereas wave nature of subatomic particles could be observed experimentally.

Phase velocity (V_p):

"The phase velocity 'V_p' of a wave is the velocity with which a definite phase of the wave is propagated in a medium".

For a progressive wave with no damping, the equation for displacement is given by

$y = A\sin(\omega t - kx)$ -----------------(1)

where ω is angular frequency, k is propagation constant, and A is amplitude

In equation (1), (ωt-kx) gives the phase of the vibrating particles. For all uniphase points in a periodic wave (ωt-kx) will be same.

i.e, $\frac{d}{dt}(\omega t - kx) = 0$

i.e. $\omega - k\frac{dx}{dt} = 0$

or $\frac{dx}{dt} = \frac{\omega}{k}$ -----------------(2)

A point marked on a wave can be considered as representing a particular phase for the wave at that point. If 'dx' is the distance moved by such a point in time 'dt', then $\frac{dx}{dt}$ gives the phase velocity.

$\therefore$ Phase velocity, $V_p = \frac{\omega}{k}$

Group Velocity (V_g):

Group velocity is the velocity with which a wave packet moves which is formed due to superposition of two or more traveling waves of slightly different wavelengths and same amplitude.

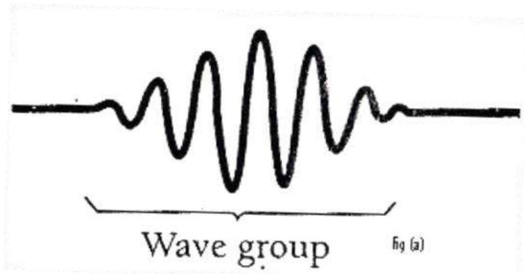
$$V_g = \frac{d\omega}{dk}$$

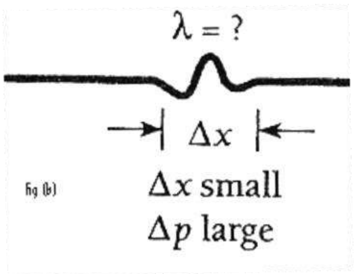
Note: The relation between group velocity and phase velocity is, V_g x $V_{phase} = C^2$

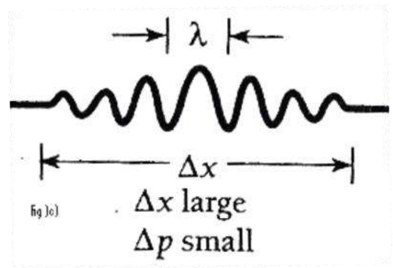
Heisenberg's uncertainty principle:

At the atomic scale of quantum mechanics, measurement of physical parameters becomes very difficult. At any instant, the position and momentum of a classical body can be measured with very high accuracy. However, in the case of a quantum particle, there are uncertainties associated with the position and momentum of the wave packet, which represent the partic

Consider the wave group shown below fig (a)







The particle corresponding to this wave group may be located anywhere within the group at a given time. The narrower its wave group, the more precisely a particles position can be specified (fig (b)). But the wavelength λ of the waves in a narrow wave packet is not well defined. This means that, since $\lambda = \frac{h}{p}$, the particle's momentum is not a precise quantity.

On the other hand, a wide wave packet (fig (c)) has a clearly defined wavelength. The momentum that corresponds to this wavelength is therefore a precise quantity. But the width of the group is too large for us to be able to say exactly where the particle is at a given time.

Thus we have the uncertainty principle given by Heisenberg.

"It is impossible to know both the exact position and exact momentum of an object at the same time."

Mathematically $\Delta x \, . \Delta p \geq \frac{h}{4\pi}$

Where Δx → uncertainty in the position, Δp → uncertainty in the momentum

ie "**In any simultaneous determination of the position and momentum of the particle, the product of the corresponding uncertainties inherently present in the measurement is equal to or greater than ($\frac{h}{4\pi}$).**"

If we arrange matter, so that Δx is small, corresponding to narrow wave group then Δp will be large. If we reduce Δp in some way, a broad wave group is inevitable and Δx will be large.

Energy-Time uncertainty principle:

It states that "In an simultaneous measurement of energy and time in a physical process, the product of the corresponding uncertainties inherently present in the measurement is equal to or greater than ($\frac{h}{4\pi}$)."

ie $$\Delta E\ .\Delta t \geq \frac{h}{4\pi}$$

Where ΔE → uncertainty in the energy, Δt → uncertainty in the time

Angular displacement and Angular momentum uncertainty principle:

It states that "In an simultaneous determination of angular momentum and angular displacement in a physical process the product of the corresponding uncertainties inherently present in the measurement is equal to or greater than ($\frac{h}{4\pi}$)."

ie $$\Delta L\ .\Delta\theta \geq \frac{h}{4\pi}$$

Where ΔL → uncertainty in the momentum, $\Delta\theta$ → uncertainty in the displacement

Physical Significance:

The physical significance of the Heisenberg's uncertainty principle is that one should not think of the exact position or an accurate value for momentum of a particle. Instead one should think of the probability of finding the particle at a certain position or of the most probable value for the momentum of the particle. Similar interpretation is made for the conjugate pair ΔE and Δt and ΔL and $\Delta\theta$.

Applications of uncertainty principle:

Non-Existence of electrons in the nucleus:

Energy of the body can be E = $\frac{P^2}{2m}$

Heisenberg's uncertainty principle states that $\Delta x\ .\Delta p \geq \frac{h}{4\pi}$ -------------- (1)

The diameter of the nucleus is of the order of 10^{-14}m. If the electron is present inside the nucleus, then the uncertainty in its position is almost equal to the diameter of the nucleus i.e $\Delta x = 10^{-14}$m.

Then, from (1) $\Delta p \geq \frac{h}{4\pi.\Delta x}$

$$\geq \frac{6.625\times10^{-34}}{4\times3.14\times10^{-14}}$$

$\Delta p \geq 0.527\times10^{-20}\,kg.m/s$ is the uncertainty in the momentum of the electron.

Then the momentum of the electron must at least be equal to the uncertainty in the momentum

i.e, $p = 0.527\times10^{-20}\,kg.m/s$

Mass of the electron, m = 9.1×10^{-31} Kg,

In order that the electron may exist within the nucleus, its energy must be such that

$$E \geq \frac{P^2}{2m} \qquad \text{i.e, } E \geq \frac{(0.527\times10^{-20})^2}{2\times9.1\times10-31} \qquad \geq 1.372 \times10^{-11}\text{ J}$$

$$E \geq (1.372 \times10^{-11} /(1.6x10^{-19})$$

i.e. $E \geq 85$MeV

This means that in order that an electron may exist inside the nucleus, its kinetic energy must be greater than or equal to 85 MeV. But experiments show that the electrons emitted by certain unstable nuclei never have more than a small fraction of this energy (about 3MeV to 4MeV). From this we conclude that electron cannot exist within the nucleus.

Principle of complementarity

When the particle nature of the matter is measured, the wave nature of the matter is necessarily suppressed and vice versa. The inability to observe the wave nature and the particle nature of the matter simultaneously is known as the complementarity principle. It was first explained by Niels Bohr in the year 1928 and hence it is familiarly known as the Bohr's Complementarity principle.

Complementarity principle states that **a single quantum can exhibit either exhibit particle-like or wave-like behaviour, but never both at the same time**. These are mutually exclusive and complementary aspects of the quantum system.

Wave function (ψ):

The quantity whose variations make up matter waves is called the wave function ψ (psi). This wave function accounts for the wave-like properties of a particle and contain all possible information about the state of the system. The value of the wave function associated with a moving body at a particular point x, y, z in space at the time is related to the likelihood of finding the body there at that time.

Physical significance of wave function ψ :

1. In matter waves, the quantity that varies periodically is called the **wave function ψ** (psi)**.**
2. The wave function ψ is a complex quantity that means it includes both real and imaginary parts. Hence ψ by itself cannot be an observable quantity.
3. Even though the wave function ψ by itself has no physical meaning, the **probability density**, $|\psi|^2 = \psi.\psi^*$ is a real and positive quantity and is measurable, where ψ* is a complex conjugate of ψ
4. Probability density $|\psi|^2$ is the probability of finding the particle in a unit volume and it can have values anywhere between 0 and 1. (Intermediate probabilities say 0.3 means that there is a 30% chance of finding the particle. But the amplitude of a wave can be positive as well as negative. A negative probability says -0.2 is meaningless.)
5. Higher values of probability density mean the probability of finding the particle within a given volume is more
6. Probability 1 corresponds to the certainty of finding the particle, and probability 0 corresponds to the certainty of not finding the particle. That is,

$$\iiint \psi.\psi^* \, dv = 1, if\ particle\ is\ present$$

$$\iiint \psi.\psi^* \, dv = 0, if\ particle\ is\ absent$$

where $dv = dxdydz\ is\ the\ volume\ of\ element$

Normalization:

If ψ is the wave function associated with a particle, then the probability of finding the particle in a volume dv is $|\psi|^2\, dv$. Since the particle is present somewhere in a particular region, the integral of $|\psi|^2\, dv$ over all the space must be finite. If the particle is certainly to be found, in certain region of space then

$$\int_{-\infty}^{+\infty} \left|\psi^2\right| dv = 1$$ ------------------- (1)

Any wave function satisfying the equation (1) is said to be normalized wave function.

Very often ψ is not a normalized wave function, i.e. the result of $\int |\psi|^2 dv$ will not be unity, but involves a constant that existed in the equation for ψ. However, the actual result obtained is equated to unity and the value of the constant is determined. It is then substituted in the equation for ψ. This process is called **normalization.**

Properties of wave function:

A system is defined by its energy, position, momentum etc. It is postulated in quantum mechanics that a wave function (ψ) corresponding to a system contains all possible information about the system. In order to describe behavior of a system, the Schrödinger's equation has to be solved. Since it is a second order differential equation it has many solutions. All of them may not be the correct wave functions which correspond meaningfully to a physical system. Only those wave functions which satisfy the following criteria are acceptable wave functions.

1. ψ is single valued, everywhere. There should be only one probability for the particle to be in a specific location at a specific time.

2. ψ must be finite everywhere.

3. ψ and its first derivatives with respect to its variables must be continuous and single valued everywhere.

4. ψ must be normalized, which means that ψ must go to zero as x→$\pm\infty$, y→ $\pm\infty$, z→$\pm\infty$ in order that $\int |\psi|^2 dV$ over all space be a finite constant.

Eigen functions:

Eigen functions are those wave functions of quantum mechanics which possess the properties that they are single valued and finite everywhere and also their first derivatives with respect to their variables are continuous everywhere.

Eigen values:

The values of a physical observable such as energy, momentum etc for which Schrödinger's wave equation can be solved are called **Eigen values**.

A wave function ψ contains all the information regarding the state of a system. The wave functions can be obtained by solving Schrödinger's wave equation. Once the correct wave functions called Eigen functions are known, quantum mechanical operators could be used to evaluate the physical observables like energy. But as postulated in quantum mechanics only those values 'λ' for a physical quantity are possible which satisfy the operator equation $\hat{A}\psi = \lambda\psi$ where $\hat{A} \rightarrow$ operator for the physical quantity , ψ → Eigen function

Thus the Eigen functions should be such that the operator operating on it produces back the wave function multiplied by a constant 'λ' such values (λ) for a physical quantity for which Schrödinger's equation can be solved are called Eigen values.

Time Independent Schrödinger wave equation:

Consider a particle, moving freely in the positive x-direction in a stationary potential field. The wavelength of the associated deBroglie wave is given by

$$\lambda = \frac{h}{p} \qquad \rightarrow *$$

The wave equation for de Broglie wave associated with such a particle can be written in complex notation as

$$\psi = Ae^{-i(wt-kx)} \qquad A \rightarrow \text{Amplitude of the wave}$$

$$\psi = Ae^{+i(kx-\omega t)} \qquad \rightarrow (1)$$

Since the particle is moving in a stationary or steady field, the potential energy of the particle does not depend on time but it varies only with the position of the particle.

Differentiating (1) w.r.t x twice we get

$$\frac{d\psi}{dx} = Ae^{i(kx-\omega t)}(ik)$$

$$\frac{d^2\psi}{dx^2} = Ae^{i(kx-\omega t)}(ik)^2$$

$$= -k^2\psi$$

But $k = \frac{2\pi}{\lambda}$, substituting in the above equation we get

$$\frac{d^2\psi}{dx^2} = -\frac{4\pi^2\psi}{\lambda^2}$$

Or $\frac{1}{\lambda^2} = -\frac{1}{4\pi^2\psi}\cdot\frac{d^2\psi}{dx^2}$ ------------ (2)

If 'm' is the mass of the particle moving with a velocity 'v' then K.E is

K.E= $\frac{1}{2}mv^2 = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$ ----------- (3)

Where $p = mv$, momentum of the particle. But $p = \frac{h}{\lambda}$; from (*)

$\therefore$ K.E = $\frac{h^2}{\lambda^2 \cdot 2m} = \frac{h^2}{2m}\cdot\frac{1}{\lambda^2}$

$$= \frac{h^2}{2m} \times -\frac{1}{4\pi^2\psi}\cdot\frac{d^2\psi}{dx^2}$$

K.E $= -\frac{h^2}{8\pi^2 m\psi}\cdot\frac{d^2\psi}{dx^2}$ ------------------------ (4)

Let 'V' be the potential energy of the particle which depends on the position of the particle in the field. Then total energy 'E' of the particle is

$$E = K.E + P.E$$

$$E = -\frac{h^2}{8\pi^2 m\psi}\times\frac{d^2\psi}{dx^2} + V$$

ie $\dfrac{d^2\psi}{dx^2} = -\dfrac{8\pi^2 m}{h^2}(E - V)\psi$

or
$$\boxed{\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0}$$

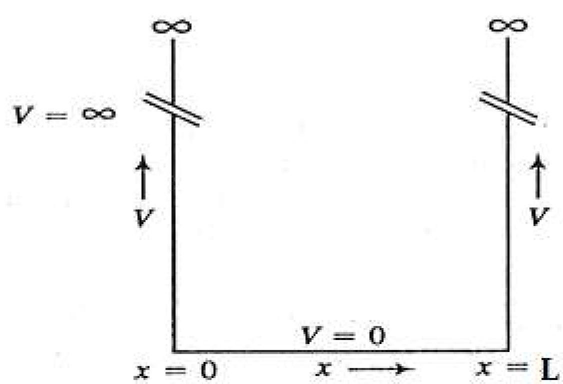
This is the time independent Schrödinger's wave equation in one dimension.

In three dimensions, it becomes

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0$$

<u>Applications of Schrodinger's wave equation</u>:

Energy Eigen values of a particle in one dimensional, infinite potential well (*particle in a box):*



Consider a particle, which is free to move in the x-direction only in the region x = 0 and x = L. Outside this region the potential energy is taken to be infinite and within this region it is zero ie V = 0 for $0<x<L$ and $V = \infty$ for $x \geq \infty$ and $x \leq 0$

We have the Schrödinger's time independent wave equation in one dimension

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \quad \text{------------ (1)}$$

Outside the well, the equation (1) become

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E-\infty)\psi = 0 \qquad \text{Since } V = \infty$$

This equation holds good only if ψ = 0 for all points outside the well, ie $|\psi|^2 = 0$ which means that the particle cannot be found at all outside the well.

Inside the well, equation (1) becomes

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi = 0 \quad \text{(Since V =0)}$$

Let $\frac{8\pi^2 m}{h^2}E = k^2$ ---------------- (2)

Then; $\frac{d^2\psi}{dx^2} + k^2\psi = 0$ ------------- (3)

The solution for the above equation is

$$\psi = A\sin kx + B\cos kx \quad \text{------------- (4)}$$

At x = 0, ψ = 0, substituting in (4) we get

$$0 = A\sin 0 + B\cos 0$$

$$\Rightarrow B = 0$$

At x=L, ψ = 0 and equation (4) becomes

$$0 = A\sin kL + B\cos kL$$

$$\Rightarrow A\sin kL = 0 \qquad \text{(Since B=0)}$$

Here 'A' need not be zero

$$\therefore \qquad \sin kL = 0$$

i.e. $kL = n\pi$, where n=0, 1, 2, 3………..is an integer called quantum number.

$$\therefore \qquad k = \frac{n\pi}{L}$$

Substituting the values of B and k in (4)

$$\boxed{\psi_n = A\sin\frac{n\pi}{L}x} \quad \text{------------------ (5)}$$

which represents the permitted solutions.

Since there is only one particle and at any time it is present somewhere inside the well only, the integral of the wave function over the entire space in the well must be equal to unity.

i.e. $\int_0^L |\psi_n|^2 dx = 1$

$$\int_0^L \left|A\sin\frac{n\pi}{L}x\right|^2 dx = 1$$

$$\int_0^L A^2 \sin^2(\frac{n\pi x}{L})dx = 1$$

$$A^2\left[\frac{1}{2}\int_0^L dx - \frac{1}{2}\int_0^L \cos\frac{2n\pi}{L}x\,dx\right] = 1 \qquad [\because \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)]$$

Or $\frac{A^2}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}x\right)\right]_0^L = 1$

Or $\frac{A^2}{2}\left[L - \frac{L}{2n\pi}\sin(2n\pi) - 0\right] = 1$

$$\Longrightarrow \frac{A^2 L}{2} = 1$$

$$\Longrightarrow \boxed{A = \sqrt{\frac{2}{L}}}$$

Substituting in (5) we get

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right)x \quad \text{--------------------} \quad (6)$$

which are the normalized wave functions of a particle in a one-dimensional infinite potential well.

Since 'n' can take all possible integral values, there is more than one wave function.

Quantization of energy

Energy could be gained or lost only in integral multiples of some smallest unit of energy, a quantum (the smallest possible unit of energy). Energy can be gained or lost only in integral multiples of a quantum. T. This is known as quantization of energy.

Equation (6) represents the Eigen functions of the particle inside the potential well. For each Eigen function we associate an Eigen value i.e. the allowed energies associated with that particle.

Substituting the value of k in equation (2),

$$k^2 = \left(\frac{n\pi}{L}\right)^2 = \frac{8\,\pi^2 mE}{h^2}$$

$$\text{i.e., } \mathbf{E_n = \frac{n^2h^2}{8mL^2}}$$

Since 'n' is restricted, particle energy is restricted to certain values. 'n' is called quantum number. The energy values E_n are called Eigen values (or quantization of energy). Lowest energy of the particle is called **zero point energy** or **ground state energy** and is given by,

$$\mathbf{E_1 = \frac{h^2}{8mL^2}}$$

Wave functions, probability densities and Energy levels for particle in an infinite potential well:

The normalized wave functions of a particle in a one dimensional potential well of width 'L' are given by

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right)x \quad \text{----------------------} \ (1) \qquad n = 0,1,2\ldots\ldots$$

Case1: For n=1:

This is the ground state and the particle is normally found in this state. The Eigen function corresponding to this state

$$\psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}\right)x \; ; \quad \text{from (1)}$$

Here $\psi_1 = 0$ for x = 0 and x = L and is maximum for $x = \frac{L}{2}$ (fig a)

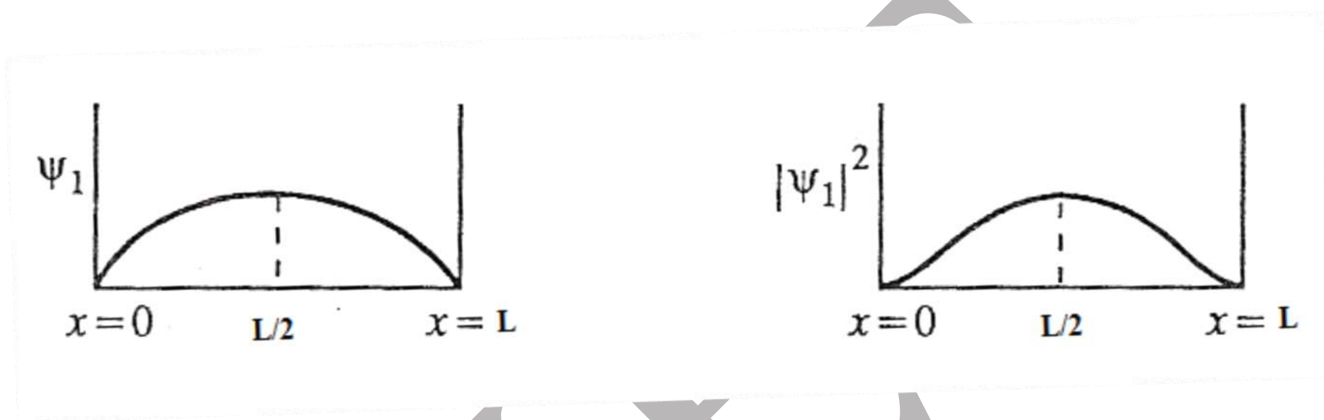


Fig (a) Fig (b)

A plot of $|\psi_1|^2$, the probability density verses x is shown in fig (b). It indicates the probability of finding the particle at different locations inside the well. $|\psi_1|^2 = 0$ at x = 0 and x = L and is maximum at $x = \frac{L}{2}$. This means that in the ground state the particle cannot be found at the walls of the box, and the probability of finding it is maximum at the central region.

The energy of the particle in the ground energy state is

$$E_1 = \frac{h^2}{8mL^2} = E_0 \text{, Zero-point energy}$$

<u>Case 2</u>: For n=2

This is the first exited state. The Eigen function for this state is

$$\psi_2 = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi}{L}\right)x$$

Here $\psi_2 = 0$ for x = 0, $\frac{L}{2}$, L and maximum for $x = \frac{L}{4}$ and $\frac{3L}{4}$ (fig c)

The plot of $|\psi_1|^2$ verses x is shown in fig (d). As can be seen from the plot, the particle cannot be observed either at the walls or at the center.

The energy of the particle in this state is

$$E_2 = \frac{4h^2}{8mL^2} = 4E_0 \quad ; \qquad \because E_n = \frac{n^2h^2}{8mL^2}$$

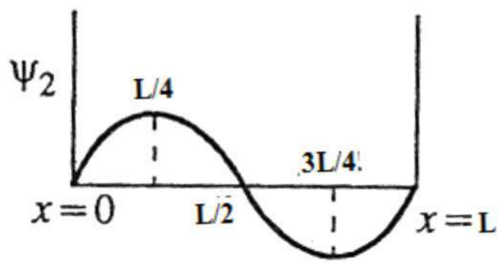


Fig (c)

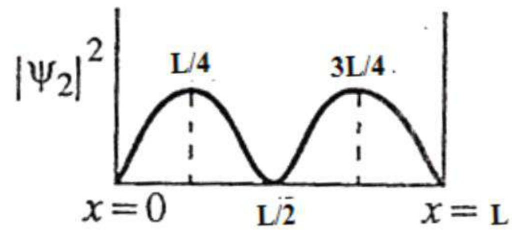


Fig (d)

<u>Csae3:</u> For n=3 (second exited state)

The Eigen function for the second exited state is

$$\psi_3 = \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi}{L}\right) x$$

Here $\psi_3 = 0$ for x=0, $\frac{L}{3}, \frac{2L}{3}$ and L and maximum for $x = \frac{L}{6}, \frac{L}{2}, \frac{5L}{6}$. The plot of ψ_3 and $|\psi_3|^2$ verses x are shown in fig (e) and fig(f). The energy of the particle in this energy state is

$$E_3 = \frac{9h^2}{8mL^2} = 9E_0$$

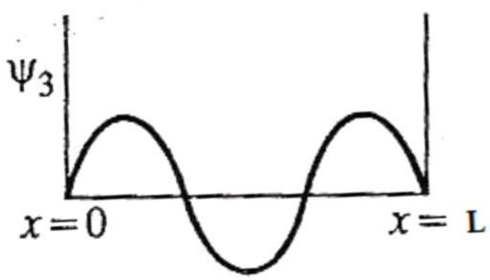


Fig (e)

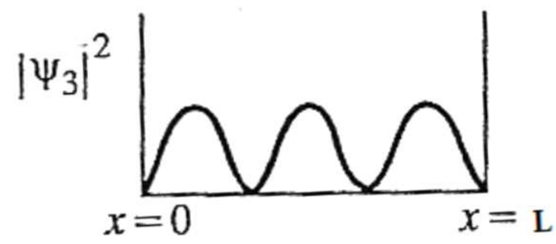


Fig (f)

Important Questions

1. State de Broglie hypothesis and explain the phase velocity and group velocity of de Broglie waves (5)
2. Derive the expression for de Broglie wavelength for the moving particle. (5)
3. Show that electrons cannot exist inside the nucleus of an atom. (4)
4. State and explain Heisenberg's uncertainty principle (5)
5. Set up one dimensional time independent Schrödinger wave equation.(7)
6. What are wave functions and give the properties of a wave function? (6)
7. Find the Eigen functions and energy Eigen values for a particle in one dimensional potential well of infinite height and discuss the solutions. (8)
8. State and explain Heisenberg's uncertainty principle. Give its physical significance (7)
9. Assuming the Schrödinger wave equation, find the energy Eigen values of a particle in an infinite potential well. Comment on zero point energy.(8)
10. Explain the term normalization in quantum mechanics. (3)
11. What are Eigen functions and Eigen values?(4)
12. Starting from Schrödinger wave equation, derive the expression for wave function for a particle inside an infinite height potential well? (8)

Problems

1. Determine the debroglie avelength of an electron accelerated by a potential difference of (i) 150V (ii) 5000V (iii) 400V.

   Hint: Equation to be used

$$\lambda = \frac{h}{\sqrt{2meV}}$$

   Ans: (i) 1.0018×10^{-10}m; (ii) 0.17353×10^{-10}m; (iii) 61.36×10^{-10}m

2. A neutron of mass 1.675×10^{-27} kg is moving with the kinetic energy 10KeV. Calculate the de Broglie wavelength associated with it.

   Hint: Equation to be used Ans: 2.86×10^{-13}m

$$\lambda = \frac{h}{\sqrt{2mE}}$$

3. The position and momentum of an electron with energy of 1keV are simultaneously determined. If its position is located within 1Ǻ what is the minimum percentage uncertainty in its momentum?

   Hint: Equation to be used; $\Delta x\ .\Delta p \geq \frac{h}{4\pi}$ and $p = \sqrt{2mE}$;

   Ans: 3.1%

4. A spectral line of wavelength 5461Å has a width of 10^{-4}Å. . Evaluate the minimum time spent by the electrons in the upper energy state .

   Hint: Equation to be used; $\Delta E\ .\Delta t \geq \frac{h}{4\pi}$

   $\lambda = 5461\text{x}10^{-10}$m

9 An electron is bond in a one-dimensional potential well of width 1Ǻ, but of infinite height. Find its energy values in the ground state and also in the first two excited states.

   Hint: Equation to be used;

   Ans: In ground state, n = 1, E_1 = 37.64eV.

   In first excited state, n = 2, E_2 = 150. 54eV.

   In second excited state, n = 3, E_3 = 338. 7eV

10 A spectral line of wavelength 5461 Ǻ. Evaluate the minimum time spent by the electrons in the energy state between the excitation and de-excitation.

Hint: Equation to be used, E = hc/λ and $\Delta x . \Delta p \geq \frac{h}{4\pi}$

Ans: Δt = 8nS.

11 A quantum particle confined to a one-dimensional box of width 'a' is in its first excited state. What is the probability of finding the particle over an interval of (a/2) marked symmetrically at the centre of the box.

Hint: Equation to be used' $P = \int_{x_1}^{x_2} |\psi|^2 dx$ and $\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right)x$

Ans: 0.5 or 50%

12

The position and momentum of an electron with energy 0.5 keV are determined. What is the minimum percentage uncertainty in its momentum if the uncertainty in the measurement of its position is 0.5. (**Ans.: 8.69%)**

13 Calculate the wavelength associated with an electron having K.E. 100eV. (**Ans.: 1.23**)

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Quantum Computing

Syllabus

Introduction to Quantum Computing, Moore's law & its end, Differences between Classical & Quantum computing. Concept of qubit and its properties. Representation of qubit by Bloch sphere. Single and Two qubits. Extension to N qubits. Dirac representation and matrix operations: Matrix representation of 0 and 1 States, Identity Operator I, Applying I to $|0\rangle$ *and* $|1\rangle$ *states, Pauli Matrices and its operations on* $|0\rangle$ *and* $|1\rangle$ *states, Explanation of i) Conjugate of a matrix and ii) Transpose of a matrix. Unitary matrix U, Examples: Row and Column Matrices and their multiplication (Inner Product), Probability, and Quantum Superposition, normalization rule. Orthogonality, Orthonormality. Numerical Problems.*

Quantum Gates: Single Qubit Gates: Quantum Not Gate, Pauli – X, Y and Z Gates, Hadamard Gate, Phase Gate (or S Gate), T Gate Multiple Qubit Gates: Controlled gate, CNOT Gate, (Discussion for 4 different input states). Representation of Swap gate, Controlled -Z gate, Toffoli gate

Introduction to Quantum computing

Quantum Computing is a new kind of computing based on Quantum mechanics that deals with the physical world that is probabilistic and unpredictable in nature. Quantum mechanics being a more general model of physics than classical mechanics give rise to a more general model of computing- quantum computing that has more potential to solve problems that cannot be solved by classical ones. To store and manipulate the information, they use their own quantum bits also called '**Qubits**' unlike other classical computers which are based on classical computing that uses binary bits 0 and 1 individually. The computers using such type of computing are known as 'Quantum Computers'. It uses the subatomic particles like atoms, electrons, photons, and ions as their bits along with their information of spins and states. They can be superposed and can give more combinations. Therefore, they can run in parallel using memory efficiently and hence is more powerful.

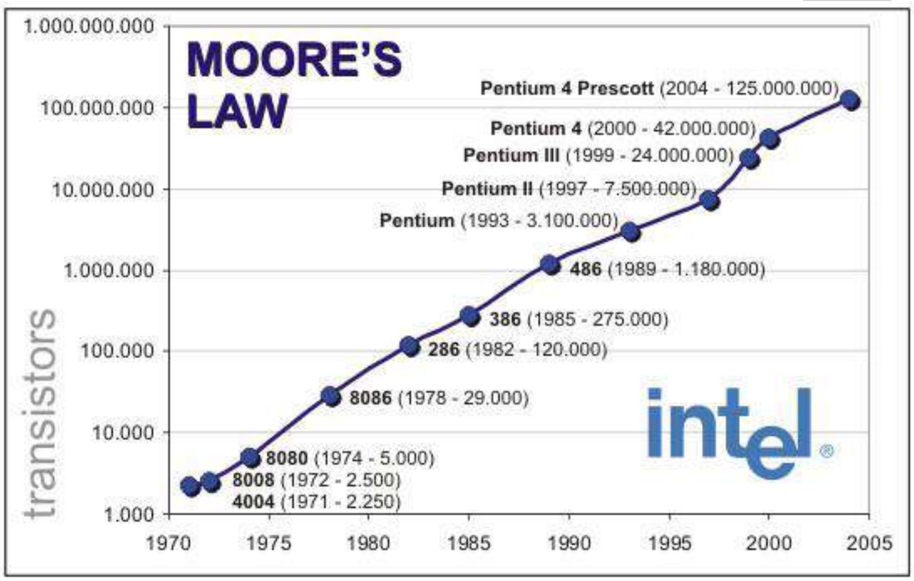
The algorithms are also written based on quantum principles in which, Shor's algorithm for factorization and Grover's search algorithm are basic. (Grover is an Indian born Physicist working in Bell Labs). The process of computation is incredibly fast but it has to be done by the help of quantum computers which are yet to be realized in practice.

Moore's law & its end

Moore's law is a term used to refer to the observation made by Gordon Moore in 1965. He stated that "the number of transistors in a dense integrated circuit (IC) doubles about every two years".

Ex: In 1978, the Intel 8086 came with a transistor count of 29,000. Then, the Intel 8051 came in 1980 with 50,000 transistors, followed by the Intel 80186 with 55,000 transistors in 1982. Finally, in 1985, the Intel 80386 had a 275,000 transistor count and so on.

Moore’s Law predicted to coming to its end due to the development of more robust computer systems and we are unable to develop chips with smaller containing more transistors. Computer chips need new developmental architectures implemented into them in order to be as efficient if more transistors are to be utilized. The creation of more powerful computers is regarded as the most important aspect of a computer system, energy efficiency and device lifetime is just as important, requiring more effective utilization of large numbers of transistors, especially when it comes to large cloud data centers which power large portions of online web applications.



Differences between Classical & Quantum computing

Comparison key	Classical computing	Quantum computing
Basis of computing	Large scale integrated multipurpose computer based on classical physics	High speed parallel computer based on quantum mechanics
Information storage	Bit based information storage using voltage/ charge	Quantum bit (qubit) based information storage using electron spin
Bit values	Bits having a value of either 0 or 1 and can have a single value at any instant	Qubits having a value of 0,1 or sometimes negative and can have both values at the same time
Number of possible states	The number of possible states is 2 which is either 0 or 1	The number of possible states is infinite since it can hold combinations of 0 or 1 along with some complex information
Output	Deterministic- (repetition of computation on the same input gives the same output)	Probabilistic- (repetition of computation on superposed states gives probabilistic answers)
Gates used for processing	Logic gates process the information sequentially, i.e. AND, OR, NOT, etc.	Quantum logic gates process the information parallel

Scope of possible solutions	Defined and limited answers due to the algorithm's design	probabilistic and multiple answers are considered due to superposition and entanglement properties
Operations	Operations use Boolean Algebra	Operations use linear algebra and are represented with unitary matrices.
Circuit implementation	Circuits implemented in macroscopic technologies (e.g. CMOS) that are fast and scalable	Circuits implemented in microscopic technologies (e.g. nuclear magnetic resonance) that are slow and delicate

Concept of Qubit and its properties

Quantum Bit or Qubit is the fundamental unit of quantum information that represents subatomic particles such as atoms, electrons, etc. as a computer's memory while their control mechanisms work as a computer's processor. It can take the value of 0, 1, or both simultaneously. They acquire both, digital as well as analog nature which gives the quantum computer their computational power. The qubit can be in either state (0 and 1) as well as in the superposed state of both states simultaneously. There is a representation of these quantum states also known as Dirac notation. Two possible states for a qubit are the states $|0\rangle$ and $|1\rangle$. Notation like $|$ and $\rangle$ called the Dirac notation, and we'll be seeing it often, as it's the standard notation for states in quantum mechanics. The difference between bits and qubits is that a qubit can be in a state other than $|0\rangle$ and $|1\rangle$. It is also possible to form linear combinations of states, often called superposition

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

The numbers α and β are complex numbers, although for many purposes not much is lost by thinking of them as real numbers. Put another way, the state of a qubit is a vector in a two-dimensional complex vector space. The special states $|0\rangle$ and $|1\rangle$ are known as computational basis states, and form an orthonormal basis for this vector space.

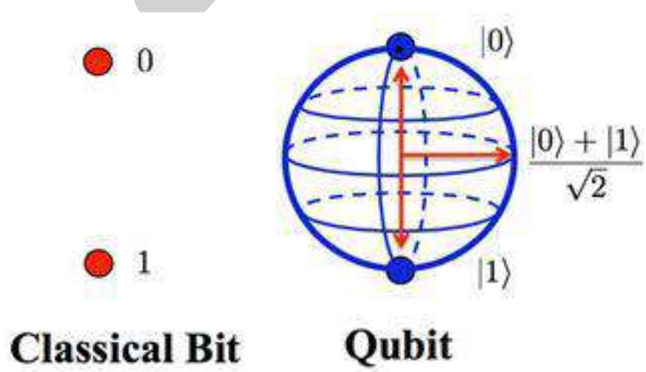


Figure: Classical bit and Qubit

Dirac Representation of Qubit

In this notation, the state label is kept between two symbols | and ⟩ is known as Dirac representation. Therefore, states are written as $|0\rangle$ and $|1\rangle$ which are literally having analog values and both are participating to give any value between 0 and 1 given that sum of probability of occurrence of each state must be 1. Thus any quantum bit wave function can be expressed as a two-state linear combination each with its own complex coefficient i.e. $|w\rangle = x\,|0\rangle + y\,|1\rangle$ where x and y are coefficients of both the states. The probability of the state is directly proportional to the square of the magnitude of its coefficient. $|x|^2$ is the probability of identifying the qubit state 0 and $|y|^2$ is the probability of identifying the qubit state 1. These probabilities when summed up must give a total of 1 or say 100% mathematically, i.e. $|x|^2 + |y|^2 = 1$.

Properties of Qubit

1. A qubit can be in a superposed state of the two states 0 and 1.
2. If measurements are carried out with a qubit in superposed state, then the results that we get will be probabilistic unlike how it's deterministic in a classical computer.
3. Owing to the quantum nature, the qubit changes its state at once when subjected to measurement. This means, one cannot copy information from qubits the way we do in the present computers, as there will be no similarity between the copy and the original. This is known as "no cloning principle".
4. A Qubit can be physically implemented by the two states of an electron or horizontal and vertical polarizations of photons as $|\downarrow\rangle$ and $|\uparrow\rangle$

Representation of Qubit by Bloch sphere

The Bloch sphere is a geometrical representation of pure single-qubit states as a point on the unit sphere. Operations on single qubits commonly used in quantum information processing can be represented on the Bloch sphere. The north pole and the south pole of the Bloch sphere are defined as the orthonormal computational basis states $|0\rangle$ and $|1\rangle$, respectively. It helps one visualize the superposition of quantum states in terms of the angular coordinates and the unitary operations on the state as rotations on the unit sphere.

The geometric representation of qubit is, from the expression, $|\alpha|^2 + |\beta|^2 = 1$

Writing the equation in spherical coordinates form, $|\Psi\rangle = e^{i\gamma}(\cos\frac{\theta}{2}|0\rangle + e^{i\Phi}\sin\frac{\theta}{2}\,|1\rangle)$

Where Θ and ϕ represents is polar angle and azimuthal angle respectively
γ is real number

The term in above equation $e^{i\gamma}$ can be ignore the factor since it has no observable effects,
and for that reason the equation can be written as

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\Phi}\sin\frac{\theta}{2}|1\rangle$$

The normalization constraint is given by

$$\left|\cos\frac{\theta}{2}\right|^2 + \left|\sin\frac{\theta}{2}\right|^2 = 1$$

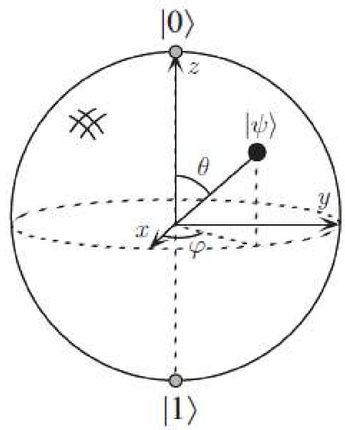


Figure: Bloch sphere representation of Qubit

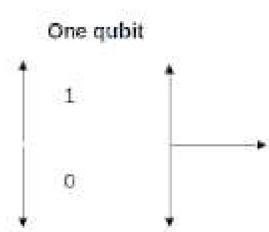
Physical Qubits

Photons or quantum particles of light have a property called polarization. A photon polarization can be vertical or horizontal or a superposition of both and is referred as Qubit.

Single Qubit

The state space of single qubit is a 1-dimensional vector space over C. Single qubit can take the value of two bits i.e. |0⟩ and |1⟩. When measuring a single qubit in an arbitrary state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, where α, β ϵ C, such that $|\alpha|^2 + |\beta|^2 = 1$ and the probability of the outcome 0 is $|\alpha|^2$ and the probability of outcome 1 is $|\beta|^2$. Here, |0⟩ and |1⟩ are orthonormal base vectors

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \quad and \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

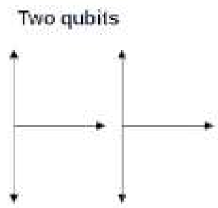


Two Qubits

In a quantum computer, two qubits can also represent the exact same **four states** (00, 01, 10, or 11). Correspondingly, a two qubit system has four computational basis states denoted |00⟩, |01⟩, |10⟩ and

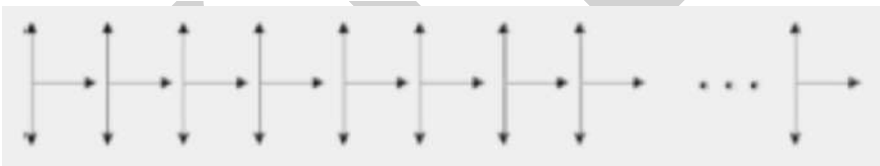
|11⟩. A pair of qubits can also exist in superposition of these four states, such that the state vector describing the two qubits is

$$|\psi\rangle = \alpha_{00}\ |00\rangle + \alpha_{01}\ |01\rangle + \alpha_{10}\ |10\rangle + \alpha_{11}\ |11\rangle$$



Extension to N qubits

In general, for multi-qubit system the extension to N qubits, can take the values of $\mathbf{2^N}$ computational basis states.
Ex: a state with 3 qubits has 23 computational basis states.
Thus for N qubits the computational basis states are denoted as |00 · · · 00⟩, |00 · · · 01⟩, |00 · · · 10⟩, |00 · · · 11⟩ · · · |11 · · · 11⟩.
The block diagram of representation of N qubits is as follows.



Superposition

Superposition in quantum mechanics states that any two quantum states can be summed up (superposed) resulting in another valid quantum state. Superposition in quantum computing refers to the ability of a quantum system where quantum particle or qubit can exist in two different positions or in multiple states at the same time. The quantum computer system holds the information that exists in two states simultaneously.

A qubit can be denoted in an exceedingly linear combination of states : $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, where numbers α and β are complex numbers. The property of having the ability to exist in multiple states is termed superposition. Quantum mechanics does not allow the view of amplitudes, ie. α and β, of the two base vectors. Instead, when we measure a qubit, we get the state |0⟩ with probability $|\alpha|^2$ and the state |1⟩ with probability $|\beta|^2$. The addition of these probabilities must be up to 1. If a quantum operation is performed on a qubit in multiple states, then the operation is performed on all states at the same time.

i.e., $|\alpha|^2 + |\beta|^2 = 1$

Superposition and Entanglement are the properties that enables the Quantum Computing paradigm to supersede classical computing. A two qubit system has four computational basis states, which are $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. The two qubit system may be in any superposition of these states. There are four very fascinating states that such a system can be prepared. These states are referred to as the Bell States or EPR states.

$$\text{An example of Bell State : } |\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$$

When one measures the first qubit in this state, there are two possible results; $|00\rangle$ with probability 1/2 leaving the other qubit in the state $|00\rangle$ and $|11\rangle$ with probability 1/2, leaving the other qubit in the state $|11\rangle$. This suggests that once the second qubit is measured it will forever be in the same state as the first qubit. This correlation between the qubits is assumed as entanglement.

Linear Algebra

Linear Algebra is the study of vector spaces and of linear operations on the vector spaces. A deeper understanding of linear algebra involves to understanding of quantum mechanics. The standard quantum mechanical notation for a quantum state Ψ in vector space is $|\Psi\rangle$. The notation $|\ \rangle$, indicates that the object is a vector and is called Ket vector. Examples of ket vectors are $|\Psi\rangle$, $|u\rangle$, $|v\rangle$, $|\Theta\rangle$, …

Basic and Linear Independence

A spanning set for a vector space is a set of vectors $|v_1\rangle$, $|v_2\rangle$,$|v_3\rangle$,.........$|v_n\rangle$, such that any vector $|v\rangle$ in the vector space can be written as linear combination of $|v\rangle=\sum_i \alpha_i |v_i\rangle$ of vectors in that set.

$$|v\rangle=\sum_i \alpha_i |v_i\rangle \quad \text{--------------(1)}$$

$$\text{Or} \quad |v\rangle= |v_1\rangle+|v_2\rangle+.........$$

Here, the matrix representation of vector plays an important role and becomes significant.

Matrix Representation

We can write $|0\rangle = \begin{vmatrix}1\\0\end{vmatrix}$ and $|1\rangle = \begin{vmatrix}0\\1\end{vmatrix}$ ----------------(2)

These are called column vectors as they have single column.

From eq.(1) for n=2

$|v\rangle= \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle$ where $|v\rangle$, $|v_1\rangle$, and $|v_2\rangle$ are ket vectors which can be represented as column matrices as

$$|v_1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } |v_2\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\text{Hence, } |v\rangle = \alpha_1 \begin{bmatrix}1\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\end{bmatrix} \text{ or } |v\rangle = \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$$

Thus, any vector $|v\rangle$ can be written as a linear combination of vectors $|v_1\rangle$, and $|v_2\rangle$ respectively

Linear Operators

A linear operator is defined to be function A which is linear in its inputs. The action of an operator that turns the vector space $|v\rangle$ into the other vector space $\sum_i \alpha_i |v_i\rangle$

Or $A|v\rangle = \sum_i \alpha_i |v_i\rangle$

An example of linear operator is the

(i) Identity operator I: The identity operator leaves the element on which it operates unchanged ie., $I|v\rangle = |v\rangle$

(ii) Zero operator: The zero operator makes the element zero on which it operates ie., $0|v\rangle = 0$

Consider a column vector $|V\rangle = \begin{bmatrix}1\\-1\end{bmatrix}$ and when multiplied it with identity matrix $I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Then, $I|v\rangle = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$ Keeps the state $|V\rangle$ intact and thus, $I|v\rangle = |v\rangle$

If A is the linear matrix, then

$A(\sum_i \alpha_i |v_i\rangle) = \sum_i \alpha_i A |v_i\rangle$ which is a matrix multiplication of A and the column matrix $|v_i\rangle$.

Pauli matrices

These are 2x2 matrices which imply variety of rotations. The Pauli matrices are very significant in the study of quantum computation and quantum information.

$\sigma_0 = I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\sigma_1 = \sigma_x = X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $\sigma_2 = \sigma_y = Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$ $\sigma_3 = \sigma_z = Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

Hermitian Conjugate

In a matrix representation of an operator A, the action of Hermitian conjugate is to take A to the conjugate-transpose matrix of A.

$$\text{i.e., } A^\dagger = (A^*)^T$$

Ex: $A = \begin{bmatrix}1\\i\end{bmatrix}$ $and\ A^* = \begin{bmatrix}1\\-i\end{bmatrix}$ hence $(A^*)^T = [1 \quad -i]$

A is said to be a Hermitian matrix if $A^\dagger = A$

Unitary Matrix

A matrix U is said to be unitary if $U^\dagger U = I$. It is easily checked that an operator is unitary if and only if each of its matrix representation is unitary. Unitary operators are important because they preserve the inner product between vectors

Row Vectors

We know that represented a ket vector in terms of a column matrix,

$$|V\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Similarly, a row matrix can also be identified with a vector called Bra vector.

$$\langle V| = (\propto_1^*,\ \alpha_2^*) \qquad \text{where } \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}^\dagger = (\propto_1^*,\ \alpha_2^*)$$

here † represents the complex conjugate.

Thus, Bra is the complex conjugate of Ket and conversely Ket is the complex conjugate of Bra. Flipping between kets and bras are called "taking the dual".

Ex: if $|Q_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$ then $\langle Q_1| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$

Inner Product

We have two states,

$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ to multiply $|\psi\rangle$ and $|\phi\rangle$ is by taking inner product as

$$\langle\psi|\phi\rangle = \langle\psi| * |\phi\rangle$$

$$= (\propto_1^* \quad \alpha_2^*) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= \alpha_1^*\beta_1 + \alpha_2^*\beta_2$$

Thus, $\langle\psi|\phi\rangle$ is called Inner Product and the result is always scalar product.

Many properties of quantum computing can be understood by the means of inner product.

Probability

A probability is a complex term used for describing the behavior of systems. The modulus squared of this quantity represents a probability density.

Consider a quantum state, $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

$$|\psi\rangle = \alpha \begin{pmatrix}1\\0\end{pmatrix} + \beta \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

Now the inner product of same quantum state with itself is,

$$\langle\psi|\psi\rangle = (\alpha \;\; \alpha^{*}) \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\alpha^{*} + \beta\beta^{*} = |\alpha|^2 + |\beta|^2$$

Thus, $\langle\psi|\psi\rangle = \psi\psi^{*} = |\psi|^2$ which represents probability density, i.e., the probability of finding a particle in space.

If $\langle\psi|\psi\rangle = \psi\psi^{*} = |\psi|^2 = |\alpha|^2 + |\beta|^2 = 1$,which represents the state is normalized.

Orthogonality

Two states $|\psi\rangle$ and $|\phi\rangle$ are said to be orthogonal if their inner product is zero. $\langle\psi|\phi\rangle = 0$

Consider the inner product of $|0\rangle$ and $|1\rangle$.

$\langle 0|1\rangle = (1 \;\; 0) \begin{pmatrix}0\\1\end{pmatrix} = 1.\,0 + 0.\,1 = 0 + 0 = 0$

Hence, $\langle 0|1\rangle$ are said to be orthogonal.

Orthonormal

Two states $|\psi\rangle$ and $|\phi\rangle$ are said to be orthonormal if,

1. $|\psi\rangle$ and $|\phi\rangle$ are normalized.
2. $|\psi\rangle$ and $|\phi\rangle$ are orthogonal to each other.

Quantum Gates

Introduction to Quantum Gates

In quantum computing a quantum logic gate is a basic quantum circuit operating on a small number of qubits. A qubit is useless unless it is used to carry out a quantum calculation. The quantum calculations are achieved by performing a series of fundamental operations, known as quantum logic gates. They are the building blocks of quantum circuits similar to the classical logic gates in conventional digital circuits.

Single Qubit Gates

Single qubit gates correspond to rotations of a spin about some axis. The simplest gates are rotations about axes in the xy-plane, as these can be implemented using resonant RF pulses.

Quantum Not Gate

In Quantum Computing the quantum NOT gate for qubits takes the state $|0\rangle$ to $|1\rangle$ and vice versa. It is analogous to the classical not gate.

The Matrix representation of Quantum Not Gate is given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

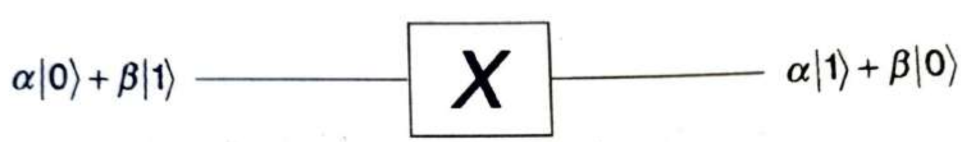
Then $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

Similarly, $X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

A quantum State is given by $\alpha\,|0\rangle + \beta\,|1\rangle$ and its matrix representation is given by $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ Hence the operation of Quantum Not Gate on quantum state is given by

$$X \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

Thus the quantum state becomes $\alpha\,|1\rangle + \beta\,|0\rangle$. Similarly, the input $\alpha\,|1\rangle + \beta\,|0\rangle$ to the quantum not gates change the state to $\alpha\,|0\rangle + \beta\,|1\rangle$. The quantum not gate circuit and the truth table are as shown below.



Quantum NOT gate

Truth table of NOT gate

Input	*Output*
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|1\rangle + \beta\|0\rangle$

Pauli –I, X, Y and Z Gates

a) Pauli-I Gate

From Pauli matrix, $\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the trivial no-operation gate on 1-qubit, represented by the identity matrix. Acting on any arbitrary state, the gate leaves the state unchanged. $I|0\rangle = |0\rangle$ and $I|1\rangle = |1\rangle$

b) Pauli-X gate (X gate, bit flip)

The Pauli-X Gate is nothing but Quantum Not Gate.

The Matrix representation of Pauli-X gate is given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

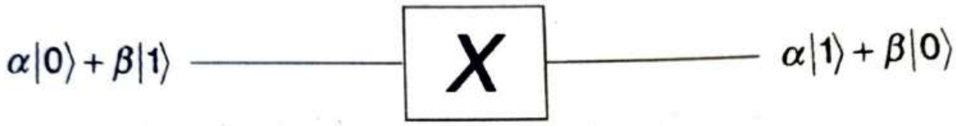
Similarly, $X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

A quantum State is given by $\alpha|0\rangle + \beta|1\rangle$ and its matrix representation is given by $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ Hence the operation of Pauli-X gate on quantum state is given by

$$X\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \alpha|1\rangle + \beta|0\rangle$$

Thus the quantum state becomes $\alpha|1\rangle + \beta|0\rangle$. Similarly, the input $\alpha|1\rangle + \beta|0\rangle$ to the quantum not gates change the state to $\alpha|0\rangle + \beta|1\rangle$.

The Pauli-X gate circuit and the truth table are as shown below.



Input	Output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|1\rangle + \beta\|0\rangle$

Truth Table of Pauli's X-gate

c. Pauli-Y gate (Y-gate)

Y Gate is represented by Pauli matrix σ_y or $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

This gate Maps $|0\rangle$ state to $i\,|1\rangle$ state and $|1\rangle$ state to $-i\,|0\rangle$ state. The Y Gate and its operation is as given below

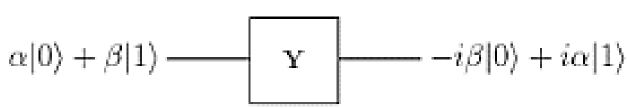
$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = 0|0\rangle + i|1\rangle = i|1\rangle$$

$$\text{Similarly, } Y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle + 0|1\rangle = -i|0\rangle$$

Thus the Y-Gate defines the transformation,

$$Y(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\, Y|0\rangle + \beta\, Y|1\rangle = i\alpha\,|1\rangle - i\beta|0\rangle$$

Quantum Y-Gate is represented by



Truth Table of Y-Gate

Input	Output
$\|0\rangle$	$i\,\|1\rangle$
$\|1\rangle$	$-i\,\|0\rangle$
$\alpha\,\|0\rangle + \beta\,\|1\rangle$	$-i\beta\,\|0\rangle + i\alpha\,\|1\rangle$

d. Pauli-Z gate (Z-gate, phase flip)

The Z-gate is represented by Pauli Matrix σ_z or $Z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

Z-Gate maps input state $|k\rangle$ to $(-1)^k\,|k\rangle$.

1. For input $|0\rangle$ the output remains unchanged.

2. For input $|1\rangle$ the output is $-|1\rangle$.

The operation of Z-Gate on $|0\rangle$ and $|1\rangle$ are as follows:

$$Z|0\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} = |0\rangle$$

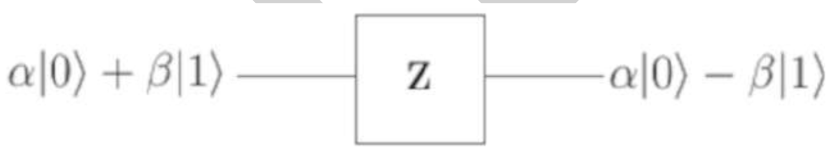
Similarly,

$$Z|1\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ -1\end{pmatrix} = -|1\rangle$$

Thus the Z-Gate defines the transformation,

$$Z(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\, Z|0\rangle + \beta\, Z|1\rangle = \alpha\,|0\rangle - \beta|1\rangle$$

The circuit symbol and the truth table of Z-Gate are as follows,



Truth table of Z gate

Input	*Output*
$|0\rangle$	$|0\rangle$
$|1\rangle$	$-|1\rangle$
$\alpha|0\rangle + \beta|1\rangle$	$\alpha|0\rangle - \beta|1\rangle$

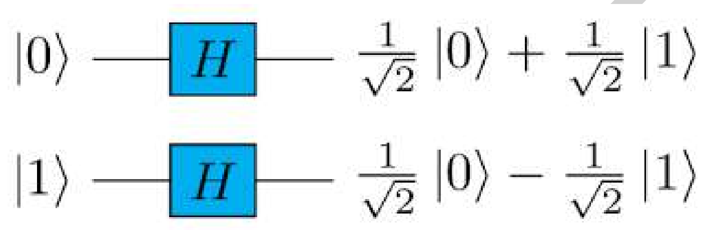
Hadamard Gate

The Hadamard Gate is a truly quantum gate and is one of the most important in Quantum Computing. It is has similar characteristics of $\sqrt{NOT}$ Gate. It is a self-inverse gate. It is used to create the superposition of $|0\rangle$ and $|1\rangle$ states. The Matrix representation of Hadamard Gate is as follows

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

The Hadamard Gate and the output states for the $|0\rangle$ and $|1\rangle$ input states are represented as follows. The Hadamard Gate satisfies Unitary Condition. $H^{\dagger}H=I$

The circuit symbol and the truth table of Hadamard Gate are as follows,



The truth-table for the Hadamard Gate is as follows.

Input	*Action of Hadamard gate*	*Output*
$\vert 0\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$	$\frac{(\vert 0\rangle + \vert 1\rangle)}{\sqrt{2}}$
$\vert 1\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$	$\frac{(\vert 0\rangle - \vert 1\rangle)}{\sqrt{2}}$
$\alpha\vert 0\rangle + \beta\vert 1\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\end{bmatrix}$	$\frac{\alpha+\beta}{\sqrt{2}}\vert 0\rangle + \frac{\alpha-\beta}{\sqrt{2}}\vert 1\rangle$ or $\alpha\frac{\vert 0\rangle + \vert 1\rangle}{\sqrt{2}} + \beta\frac{\vert 0\rangle - \vert 1\rangle}{\sqrt{2}}$

Phase Gate (or S Gate)

The phase gate turns $|0\rangle$ into $|0\rangle$ and $|1\rangle$ into $i\,|1\rangle$ The Matrix representation of the S gate is given by The Phase gate (or S gate) is a single-qubit operation defined by: S= $\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$

The effect of S gate on input |0⟩ is given by

$$S|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

Similarly, the effect of S gate on input |1⟩ is given by

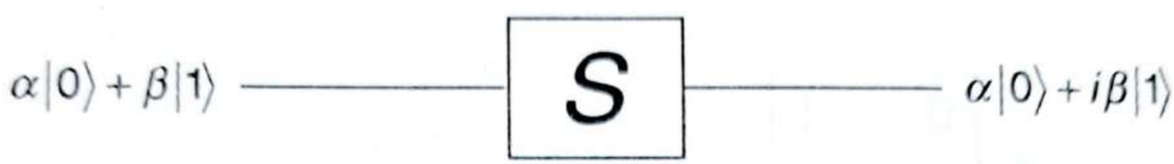
$$S|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

The transformation of state S gate is given by

$$S(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\, S|0\rangle + \beta\, S|1\rangle = \alpha\,|0\rangle + i\,\beta|1\rangle$$

The S Gate and the Truth table are given by for S gate

Input	Output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$i\|1\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|0\rangle + i\beta\|1\rangle$



The S gate is related to the T gate by the relationship $S=T^2$.

The conjugate transpose of the S gate, $S^\dagger$ gate is defined by: $S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$ and $S^\dagger S=I$

T Gate or π/8 Gate

The T Gate is represented by the matrix as follows

$$T= \begin{bmatrix} 1 & 0 \\ 0 & \exp(\frac{i\pi}{4}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1+i}{\sqrt{2}}\right) \end{bmatrix}$$

It is also called $\pi/8$ gate as it could be represented in the following form

$$T = \exp\frac{i\pi}{8}\begin{bmatrix}\exp\frac{-i\pi}{8} & 0 \\ 0 & \exp(\frac{i\pi}{8})\end{bmatrix}$$

The T gate is related to the S gate by the relationship $S=T^2$

The conjugate transpose of the T gate is defined by: $T^{\dagger} = \begin{bmatrix}1 & 0 \\ 0 & \exp(\frac{-i\pi}{4})\end{bmatrix}$

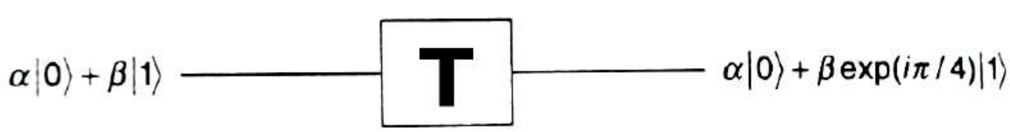
The Operation of T gate on $|0\rangle$ and $|1\rangle$ are given by

$$T|0\rangle = \begin{bmatrix}1 & 0 \\ 0 & \left(\frac{1+i}{\sqrt{2}}\right)\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}1 \\ 0\end{bmatrix} = |0\rangle$$

Similarly, $T|1\rangle = \begin{bmatrix}1 & 0 \\ 0 & \left(\frac{1+i}{\sqrt{2}}\right)\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}0 \\ \left(\frac{1+i}{\sqrt{2}}\right)\end{bmatrix} = \left(\frac{1+i}{\sqrt{2}}\right)|1\rangle$

The T Gate and the Truth Table are as follows.

Input	*Output*
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\exp(i\pi/4)\|1\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|1\rangle + \beta\exp(i\pi/4)\|1\rangle$



Multiple Qubit gates

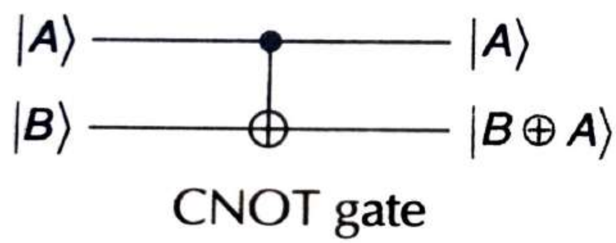
Controlled Gates

A Gate with operation of kind "If 'A' is True then do 'B'" is called Controlled Gate. The '$|A\rangle$' Qubit is called Control qubit and '$|B\rangle$' is the Target qubit. The target qubit is altered only when the control qubit is $|1\rangle$. The control qubit remains unaltered during the transformations

a) Controlled Not Gate or CNOT Gate

The CNOT gate is a typical multi-qubit logic gate and the circuit is as follows. The Matrix representation of CNOT Gate is,

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



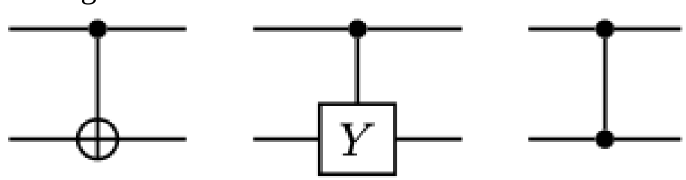
The Transformation could be expressed as $|A, B\rangle \rightarrow |A, B \oplus A\rangle$

Consider the operations of CNOT gate on the four inputs $|00\rangle$,$|01\rangle$,$|10\rangle$ and $|11\rangle$.

The Truth Table of operation of CNOT gate is as follows.

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

The modifications made by the CNOT gate can be represented by the matrix (permutation matrix form)
Circuit representation of CNOT gates:

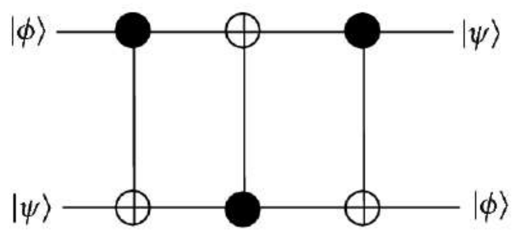


b. SWAP Gate

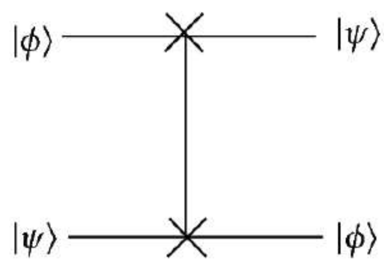
Two bits are swapped by using SWAP gate. With respect to the basis $|00\rangle$,$|01\rangle$,$|10\rangle$ and $|11\rangle$ and it can be represented by the matrix:

$$\text{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The schematic symbol of swap gate circuit is as follows



Equivalent circuit representation of SWAP gate:



The action and truth table of the swap gate is as follows.

Gate	*Input to gate*	*Output of gate*
1	$\vert a, b\rangle$	$\vert a, a \oplus b\rangle$
2	$\vert a, a \oplus b\rangle$	$\vert b, a \oplus b\rangle$
3	$\vert b, a \oplus b\rangle$	$\vert b, a\rangle$

Truth table of SWAP gate

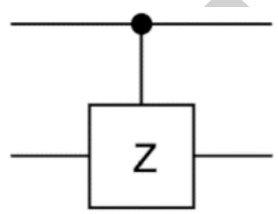
Input	*Output*
$\vert 00\rangle$	$\vert 00\rangle$
$\vert 01\rangle$	$\vert 10\rangle$
$\vert 10\rangle$	$\vert 01\rangle$
$\vert 11\rangle$	$\vert 11\rangle$

Controlled Z Gate

In Controlled Z Gate, the operation of Z Gate is controlled by a Control Qubit. If the control Qubit is $|A\rangle$ = $|1\rangle$ then only the Z gate transforms the Target Qubit $|B\rangle$ as per the Pauli-Z operation. The action of Controlled Z-Gate could is specified by a matrix as follows.

$$U_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The controlled Z gate and the truth table are as follows.



Truth Table of controlled Z gate is

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$-\|11\rangle$

Toffoli gate

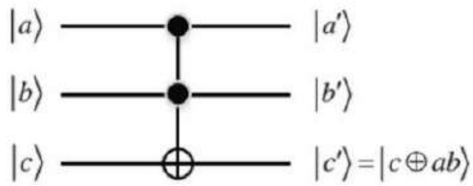
The Toffoli Gate is also known as CCNOT Gate (Controlled-Controlled-Not). It has three inputs out of which two are Control Qubits and one is the Target Qubit. The Target Qubit flips only when both the Control Qubits are $|1\rangle$. The two Control Qubits are not altered during the operation.

The matrix representation, Gate Circuit and the Truth

$$U_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Table of Toffoli Gate are as follows.

Inputs			Ouputs		
a	*b*	*c*	*a′*	*b′*	*c′*
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0



The Toffolli matrix is unitary. The Toffolli Gate is its own inverse. It could be used for NAND Gate Simulation.

QUESTIONS:

1. Define a bit and Qbit and explain the properties of qubit.
2. Discuss the CNOT gate and its operation on four different input states.
3. Discuss the SWAP gate and its operation on four different input states.
4. A Linear Operator 'X' operates such that $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$. Find the matrix representation of 'X'.
5. State the Pauli matrices and apply Pauli matrices on the states $|0\rangle$ and $|1\rangle$
6. Elucidate the differences between classical and quantum computing.
7. Describe the working of controlled-Z gate mentioning its matrix representation and truth-table.
8. Discuss the working of phase gate mentioning its matrix representation and truth table.
9. Explain Orthogonality and Orthonormality with an example for each.
10. Explain the representation of qubit using Bloch Sphere.
11. Explain Single qubit gate and multiple qubit gate with an example for each.
12. Explain the Matrix representation of 0 and 1 States and apply identity operator I to $|0\rangle$ and $|1\rangle$ states.
13. Given $|\psi\rangle = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$ Prove that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$
14. For S gate, prove that $S^{\dagger}S=I$
15. Explain the working of Hadamard Gate mentioning its matrix representation and truth-table.

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MODULE 4

ELECTRICAL PROPERTIES OF MATERIALS AND APPLICATIONS

SYLLABUS

Electrical Conductivity in metals

Resistivity and Mobility, Concept of Phonon, Matheissen's rule, Failures of Classical Free Electron Theory, Assumptions of Quantum Free Electron Theory, Fermi Energy, Density of States, Fermi Factor, Variation of Fermi Factor with Temperature and Energy. Numerical Problems.

Classical free electron theory:(Drude – Lorentz theory)

Postulates:

1. A metal is assumed to possess a three dimensional array of positive ions with randomly moving free electron gas confined to metallic boundary.
2. These free electron gas is treated as equivalent to gas molecules and they are assumed to obey the laws of kinetic energy of gases. In the absence of any electric field the energy associated with electrons is equal to

$$\text{Kinetic energy} = \frac{3}{2}kT$$

3. The electric current in a metal is due to the drift of electrons in a directionopposite to applied Electric field.
4. The electrostatic force of attraction or repulsion between the electron-lattice, lattice-lattice and electron –electron is ignored.

Drift velocity:

The net displacement in the position of electrons per unit time caused by the application of electric field is known as drift velocity.

$$\mathbf{v_d = \frac{eE\tau}{m}}$$

Where e – charge on the electron, E – Electric intensity, τ - Mean collision time.

Mean Collision Time (τ) : It is the average time taken between two consecutive collisions of electrons.

Relaxation time: It is the time taken for the drift velocity to decay to (1/e) times after the removal of electric field.

$$\tau_r = \frac{\tau}{\langle 1 - \cos\theta \rangle}$$

Mean free path (λ) **:** The average distance traveled by the electrons between two successive collisions.

Expression for the Electric current through a metal:

Electric current, $I = neAV_d$

Expression for electrical conductivity(σ):

From Ohms law $J = \sigma E$

Thus $\sigma = \frac{J}{E} = \frac{ne^2\tau}{m}$

Mobility of electrons

Mobility of a charge carrier is the ratio of the drift velocity to the electric field,

i.e, mobility $\mu = \frac{v_d}{E}$

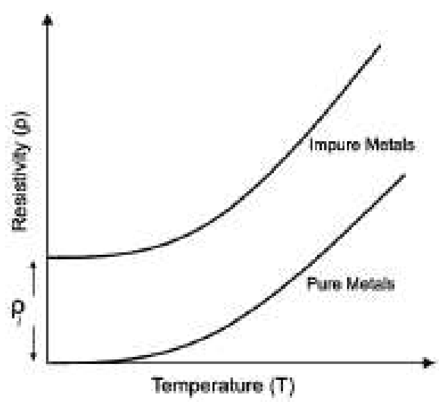
where 'v_d' be the drift velocity and E is the applied electric field.

The unit of mobility is **meter per second per volt per meter** or ($m^2V^{-1}s^{-1}$). The mobility represents the ease with which electrons move in a solid.

Concept of Phonon

A Phonon is a quantum of lattice vibration, the collective motion of atoms constituting a crystal. The Energies and Momenta of Phonons are quantized. It is often characterized as Heat Energy. The study of phonon is an important part of solid state physics. The phonon plays an important role in many of the physical properties of solids such as the thermal conductivity and the electrical conductivity. The conduction electrons in a metal collide against lattice ions during the motion. The interaction is considered to be of type phonon exchange. This results in non-radioactive transitions.

Mathiessen's rule (Effect of temperature and impurity on electrical resistivity of metals)



The variation of electrical resistivity (ρ) with temperature T for a metal is shown below

Resistivity arises due to scattering of conduction electrons. In metals, two types of scattering mechanisms exist.

1. Resistivity ρ_{ph} due to scattering of electrons by lattice vibrations (phonons) which is temperature dependent and is called ideal resistivity.
2. Resistivity ρ_i due to the scattering of the electrons by the presence of impurities and imperfections. This

resistivity is temperature independent and exists even at 0K. Hence it is called residual resistivity.

The total resistivity ρ of a material is given by,

$$\rho = \rho_{ph} + \rho_{i} = \frac{m}{ne^2\tau_{ph}} + \frac{m}{ne^2\tau_{ph}}$$

This is called Mathiessen's rule.

Mathiessen's rule states that the total resistivity of a metal is the sum of the resistivity due to phonon scattering (temperature dependent) and the resistivity due to scattering by impurities (temperature independent).

At low temperatures, lattice vibration is negligible and phonon scattering is very less.

$$\therefore \text{At low temperatures, } \rho \approx \rho_i$$

At high temperatures, lattice vibration becomes very significant and resistivity becomes linearly dependent on temperature.

$$\therefore \text{At high temperatures, } \rho \approx \rho_{ph}$$

Failures of Classical free electron theory:

1. **Temperature dependence of electrical conductivity**:

From the assumption of kinetic theory of gases

$$\frac{3}{2}kT = \frac{1}{2}mv^2$$

$$\therefore v \propto \sqrt{T}$$

Also mean collision time τ is inversely proportional to velocity,

$$\tau \propto \frac{1}{v}$$

$$\tau \propto \frac{1}{\sqrt{T}}$$

$$\therefore \sigma = \frac{ne^2\tau}{m} \Rightarrow \sigma \propto \frac{1}{\sqrt{T}}$$

However experimental studies show that $\sigma \propto \frac{1}{T}$

2.**Specific heat:**

The theoretically predicted value of specific heat ($C_v = {}^3/_2 R$) of a metal does not agree with the experimentally obtained value (10^{-4} RT). Experimentally observed value of specific heat is far lower than expected value.

3. **Dependence of electrical conductivity on electron concentration:**

As per free electron theory, $\sigma \propto n$

The theory predicts the direct dependence of electrical conductivity (σ) on number of free electrons per unit volume (n) called number density. But experiments have revealed different with $\sigma_{cu} > \sigma_{Al}$ even though the number densities $n_{cu} < n_{Al}$. Hence it fails to explain the dependence of electrical conductivity σ on the number free electrons per unit volume 'n'. The experimental observations are as in the table below

Metal	$\sigma(\Omega^{-1}m^{-1})$	$n(m^{-3})$
Copper	5.88×10^7	8.45×10^{28}
Aluminium	3.65×10^7	18.06×10^{28}

Quantum free electron theory:

Assumptions:

1. The energy of conduction electrons in a metal is quantized.
2. The distribution of electrons amongst various energy levels is according to Pauli's exclusion principle and Fermi – Dirac statistical theory.
3. The average kinetic energy of an electron is equal to $\frac{3}{5}E_F$
4. The electrostatic force of attraction or repulsion between the electron-lattice, lattice-lattice and electron –electron is ignored. The electrons travel in a constant potential inside the metal but stay confined within its boundaries.

Fermi energy (E_F): It is the highest energy possessed by an electron at zero Kelvin.

$$E_F = \frac{h^2}{8m}\left(\frac{3n}{\Pi}\right)^{\frac{2}{3}}$$

Fermi-Dirac distribution and Fermi Factor

According to Fermi-Dirac statistics, the probability that an electron occupies an energy level E at thermal equilibrium is given by f(E) fermi factor,

$$f(E) = \frac{1}{1+e^{\left(\frac{E-E_F}{kT}\right)}}$$

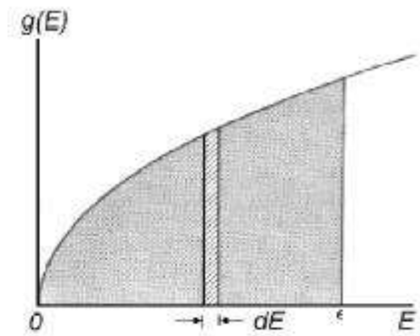
where E_F is called Fermi level,. Fermi level is the highest filled energy level by an electron at 0 K.

The probability f(E) is known as **Fermi factor**.

Density of energy of states g(E)

The Density of States is defined as the number of energystates available per unit volume of the material in the unit energy range in the valence band of the material.It represents the number of energy levels per unit energy range per unit volume.

$$g(E)dE = 8\Pi\sqrt{2}m^{\frac{3}{2}}\frac{E^{\frac{1}{2}}}{h^3}dE$$



The variation of g(E)dE as a function of E

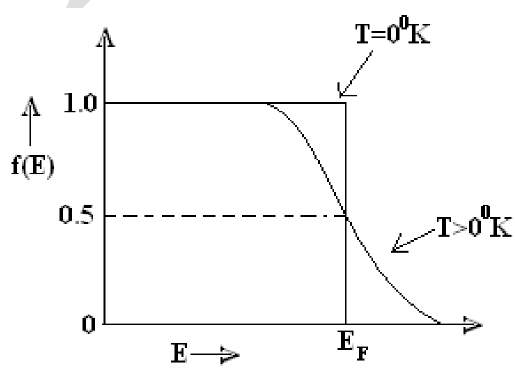
Effect of temperature on Fermi factor f(E):

Probability of occupation of level with energies,

1) For E < E_F, at T = 0, Probability of occupation f (E) = $\frac{1}{e^{(\frac{E-E_F}{kT})}+1}$ = 1

2) For E > E_F, at T=0, Probability of occupation f (E) = $\frac{1}{e^{(\frac{E-E_F}{kT})}+1}$ = 0

3) For E=E_F, at T>0K, Probability of occupation f(E) = $\frac{1}{2}$



Variation of f(E) with E

SUPERCONDUCTIVITY

SYLLABUS

Introduction to Super Conductors, Temperature dependence of resistivity, Meissner's Effect, Critical Field, Temperature dependence of Critical field, Types of Super Conductors, BCS theory (Qualitative), Quantum Tunnelling, High Temperature superconductivity, Josephson Junctions (Qualitative), DC and RF SQUIDs (Qualitative), Applications in Quantum Computing: Charge, Phase and Flux qubits, Numerical Problems.

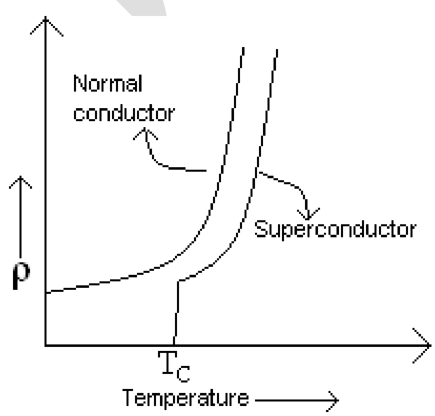
Introduction to Super Conductors

Superconductivity was first observed byDutch physicistKammerlinghOnnes in 1911 while measuring the resistivity of mercury at low temperatures. In the year 1913, he received the Nobel prize for his work.Superconductivity is a physical property observed in certain materials where electrical resistance vanishes and magnetic flux fields are expelled from the material. Any material exhibiting these properties is a superconductor. Unlike an ordinary metallic conductor, whose resistance decreases gradually as its temperature is lowered, even down to near absolute zero, a superconductor has a characteristic critical temperature below which the resistance drops abruptly to zero.

The phenomenon in which resistance of certain metals, alloys and compounds drops to zero abruptly, below certaintemperature is called **superconductivity**

Temperature dependence of resistivity in superconducting materials:

Unlike an ordinary metallic conductor, whose resistance decreases gradually as its temperature is lowered, even down to near absolute zero. The electrical resistivity of many metals and alloys drops suddenly to zero when their specimens are cooled to a sufficiently low temperature, often a temperature in the liquid Helium range (4 K). This phenomenon is known as superconductivity. Materials which show superconductivity property are called superconducting materials.



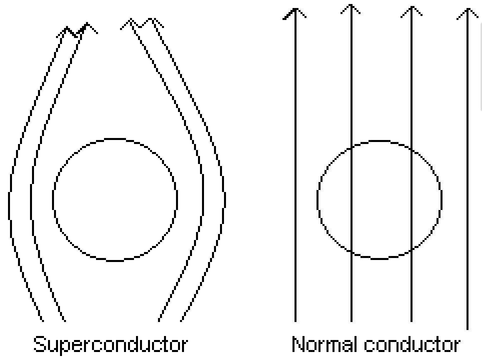
The temperature at which the resistivity of the material suddenly changes to zero is called critical temperature or superconducting transition temperature (T_c). The transition temperature of mercury is 4.15 K. The transition temperatures of some superconducting materials are given below:

Material	T_c(K)	Material	T_c(K)	Material	T_c(K)
Hafnium (Hf)	0.12	Tin (Sn)	3.72	Nb_3Sn	18

Titanium (Ti)	0.39	Mercury (Hg)	4.15	Nb_3Ge	23
Cadmium (Cd)	0.56	Lead (Pb)	7.19	$LaBa_2Cu_3O_7$	30
Zinc (Zn)	0.88	Technetium (Tc)	7.8	$YBa_2Cu_3O_7$	90
Aluminium (Al)	1.14	Niobium (Nb)	9.5	$Tl_2Ba_2Ca_2Cu_3O_{10}$	120
Indium (In)	3.40	Nb_3Al	17.5	$HgBa_2Ca_2Cu_3O_8$	140

Meissner effect:

Meissner and Ochsenfeld in 1933 found that if a superconductor is cooled in a magnetic field below the transition temperature, the magnetic flux lines are pushed out of the body of the superconductor as shown:



This phenomenon is called Meissner effect which establishes that a superconductor is a perfect diamagnetic.

$\therefore$ Inside the specimen, the magnetic field, B=0

$$\text{But } B = \mu(H + M)$$

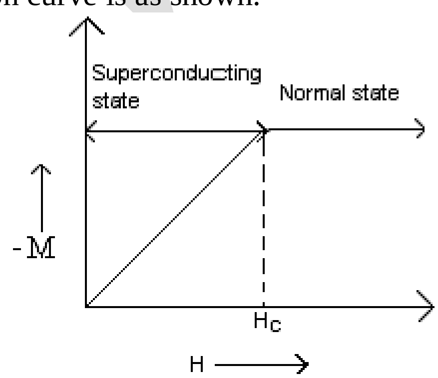
$\therefore$ H+M = 0 (since B=0)

$$\text{Or } H = -M$$

Thus Magnetic susceptibility $\chi = \frac{M}{H} = -1$

Types of Superconductors

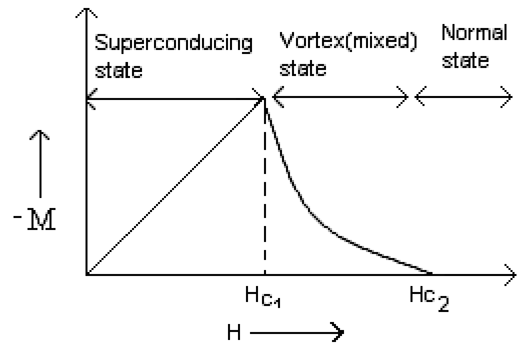
In type I superconductors, magnetization curve is as shown:



They are completely diamagnetic or exhibits complete Meissener effect up to critical field H_c. They are also called **soft superconductors**. The H_c value for Type I superconductors are found to be very low. Hence it is not used for the construction of superconducting magnets.

Eg: Al, In, Sn, Pb etc.

In type II superconductors, magnetization curve is as shown:

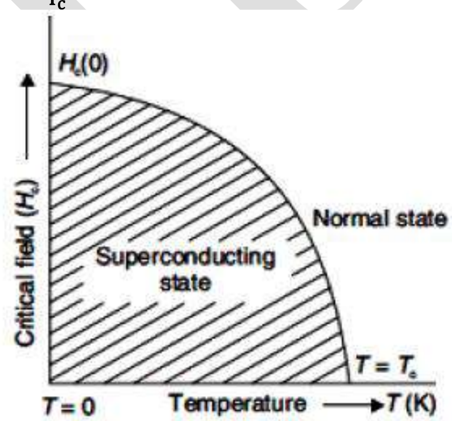


For applied fields below Hc_1 the specimen is diamagnetic, exhibiting complete Meissner effect. At Hc_1, the flux begins to penetrate the specimen and the penetration increases until Hc_2 is reached. Here Meissner effect is incomplete and the specimen is said to be in a vortex (mixed) state. At Hc_2, the specimen becomes a normal conductor. Hc_2 is called upper critical field. They are also called hard superconductors. Hc_2 value is larger than (may be even 100 times) the H_c value for type I superconductors. Hence they are used in the construction of superconducting magnets.
Eg: Nb_3Sn, Nb_3Ge etc.

Temperature dependence of critical field

When the superconducting materials are subjected to a strong magnetic field, it will result in the destruction of the superconducting property. I.e. they return to the normal state. The minimum magnetic field required to destroy the superconducting property is called the critical field (H_c). The variation of H_c with temperature is shown in the figure.The dependence of critical field on temperature is governed by the following relation,

$$\mathbf{H_c(T) = H_c(0)[1-(\frac{T}{T_c})^2]}$$ Where $H_c(0)$ is the critical field at 0 K.

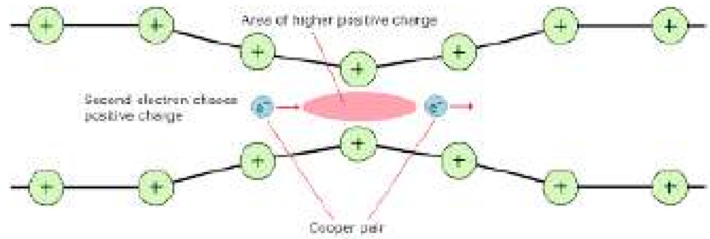


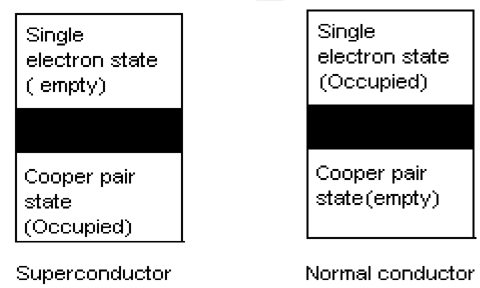
BCS Theory of Superconductivity

This theory was developed by Bardeen, Cooper and Schrieffer in 1957 based on electron- lattice- electron interaction. According to this theory, an electron attracts lattice ions towards itself, so that it is surrounded by a region of positive charges. Another electron gets attracted to this region of high positive ion concentration. Thus an electron- lattice- electron interaction results in an electron pair formation. These pairs are called Cooper pairs. They can be scattered only if the energy involved is sufficient to break it up into two single electrons.

Cooper pair electrons possess opposite momenta and spin (K↑ and –K↓). In addition, a Cooper pair does not obey Pauli's exclusion principle and hence any number of Cooper pairs can be accommodated into a single quantum state.

Since an electron pair has a lower energy than the two normal electrons, there is an energy gap between the paired (Cooper pair) and the two single electrons.





As long as Cooper pair electrons remain in Cooper pair states, they do not suffer scattering and hence resistivity will be zero. When the temperature is raised, to overcome the energy gap, Cooper pair electrons gets separated to normal single electrons which may undergo scattering due to the presence of imperfections in the crystal or lattice vibrations, which leads to a finite resistivity.

The idea that the electron interaction plays a crucial role in superconductivity is supported by the fact that the best of the conductors such as gold, silver and copper do not exhibit superconductivity. The reason attributed is that the electrons in those metals move so freely in the lattice that, the electron-lattice interaction is virtually absent. This rules out the possibility of formation of Cooper pairs, and also that of occurrence of superconductivity in the material.

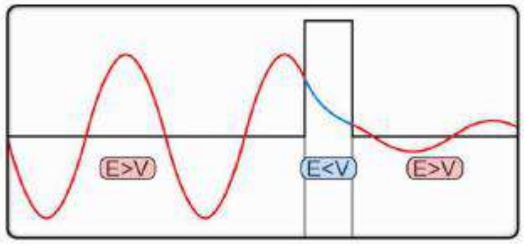
High temperature superconductors

In 1986, Bednorz and Muller synthesized a particular type of ceramic material ($LaBa_2Cu_3O_7$)whose transition temperature was 30 K. For this they were awarded the Nobel Prize in Physics in the year 1987. Later researchers synthesized $YBa_2Cu_3O_7$ with T_c around 90 K. The success broke the barrier of liquid nitrogen temperature of 77 K, and was a sort of dream come true for many scientists. It is because liquid nitrogen is readily available in most of the places and inexpensive.

All high temperature superconductors are different types of oxides of copper assuming perovskite crystal structure. The critical temperature is higher for those materials which have more number of copper-oxygen layers. The formation of supercurrents in high temperature superconductors is direction dependent. The supercurrents are strong in the copper-oxygen planes and weak in a direction perpendicular to the planes. In bulk materials, the grain boundary effects decrease the critical current value.

Quantum Tunneling

In classical mechanics, when a particle has insufficient energy, it would not be able to overcome a potential barrier. In the quantum world the particles can often behave like waves. On encountering a barrier, a quantum wave will not end abruptly. Rather its amplitude deceases exponentially. This drop in amplitude corresponds to a drop in the probability of finding a particle further into the barrier. If the barrier is thin enough, then the amplitude may be non-zero on the other side. This would imply that there is a finite probability that some of the particles will tunnel through the barrier.

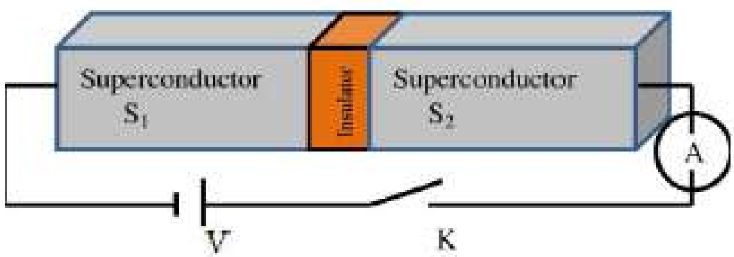


Josephson Junction

When an insulator is placed between two superconductors, the insulator behaves like a superconductor. This effect is known as Josephson Effect and the junction between the two superconductors is known as Josephson junction. This effect was first predicted by Josephson in 1962.

1. DC Josephson Effect

The experimental arrangement was a Josephson junction which consists of a thin insulator sandwiched between two superconductors as shown:



If the insulator layer is very thin, of the order of 10-50 Å in thickness, a tunneling phenomenon called Josephson tunneling (Josephson effect) takes place through the insulator. Thus the insulator turns into a superconductor. Thus the insulating layer introduces a phase difference between the wave function of copper pairs. This phase differences induces the supercurrent across the junction even though the applied voltage is zero. This is called DC Josephson effect. The super current across the junction is given by

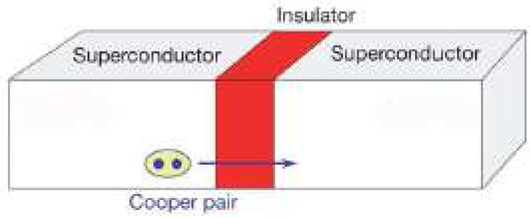
$$I_S = I_C \sin\phi_0$$

where ϕ_0 phase difference between the wave function of copper pairs and I_C is the critical current at zero voltage condition.

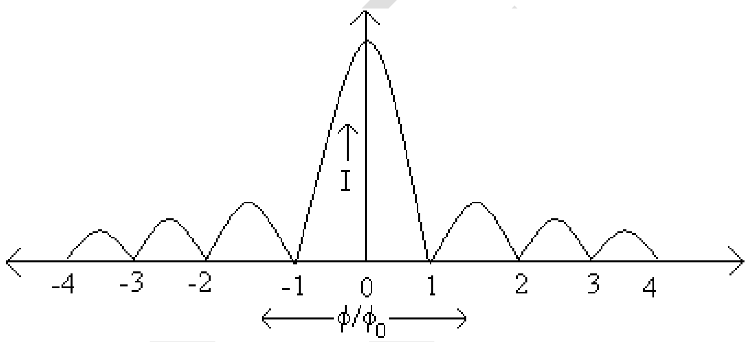
2. AC Josephson Effect

In the A.C Josephson effect, a constant chemical potential difference (voltage) is applied, which causes an oscillating current to flow through the barrier. When a DC Voltage is applied across the Josephson junction it introduces an additional phase on copper pairs during tunneling. Thus DC voltage generates an alternating current (AC) I given by,

$$I_S = I_C \sin(\phi_0 + \Delta\phi)$$

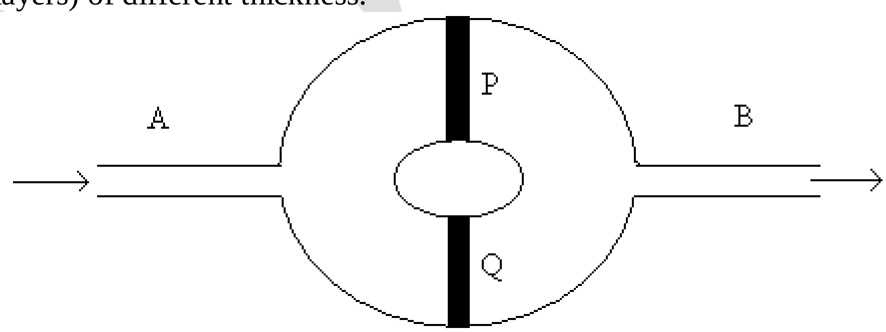


Squids
If a magnetic field is applied perpendicular to the Josephson junction, the value of critical supercurrent drops to zero whenever flux through the junction is a multiple of flux quantum ϕ_0(flux quantum ϕ_0=h/2e). This property is used in SQUID (**S**uperconducting **Qu**antum **I**nterference **D**evice).



DC Squid

It consists of a ring of superconducting material with two side arms A and B. P and Q are the Josephson junctions (insulating layers) of different thickness.



Let δ_p and δ_q represent the phase difference between the input current and output current while passing through the insulator junctions P and Q respectively. In the absence of magnetic field, these two phases are equal. i.e. $\delta_p = \delta_q = \delta_0$.
When a magnetic field is applied, the phase difference between the reunited currents is directly proportional to the magnetic flux ϕ passing through the ring. It can be shown that total current coming out of the ring,

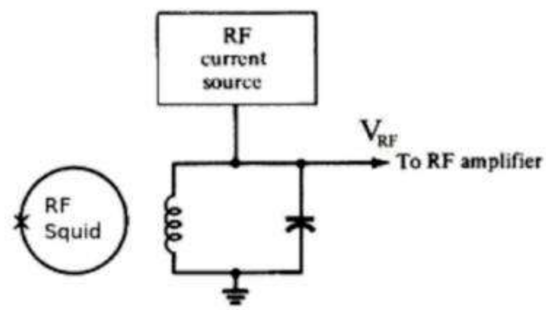
$$\mathbf{I = 2I_0 \sin\delta_0 \cos\left(\frac{e\Phi}{hc}\right)}$$

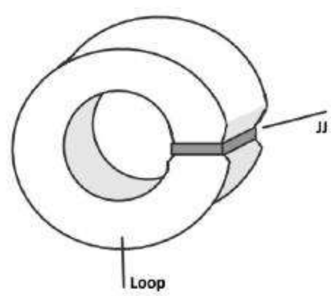
This expression indicates that the output current varies with the applied magnetic flux and shows oscillations.

SQUID is used as a very sensitive magnetometer which can measure very weak magnetic fields of the order of 10^{-13} Tesla.

RF Squid

It is based on the AC Josephson effect and uses only one Josephson junction. It is less sensitive compared to DC squid but is cheaper and easier to manufacture in smaller quantities. In RF squid the flux is coupled into a loop containing a single Josephson junction through an input coil and an RF col. RF coil is part of high Q resonant circuit to read out current changes due to induced flux in the SQUID loop. The tuned circuit is driven by a constant RF oscillator which is weakly coupled to the loop. Measuring the change in the input coil current is done by counting the number of periods the coil produces in the detected RF output, because the detected RF output is a periodic function

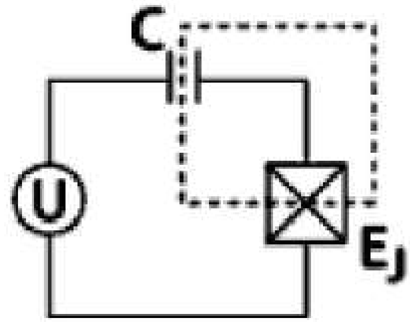




The DC SQUIDs offer higher sensitivity, but RF SQUIDs have lower sensitivity. RF SQUIDs are commonly used form of the sensor, because of their ease and low price of manufacturing in small batches

Charge Qubit

In quantum computing, a charge qubit is also known as cooper-pair box. It is a qubit whose basis states are charge states. The states represent the presence or absence of excess cooper pairs in the island (dotted region in the figure). In superconducting quantum computing, a charge qubit is formed by a tiny superconducting island coupled by Josephson Junction to a superconducting reservoir.

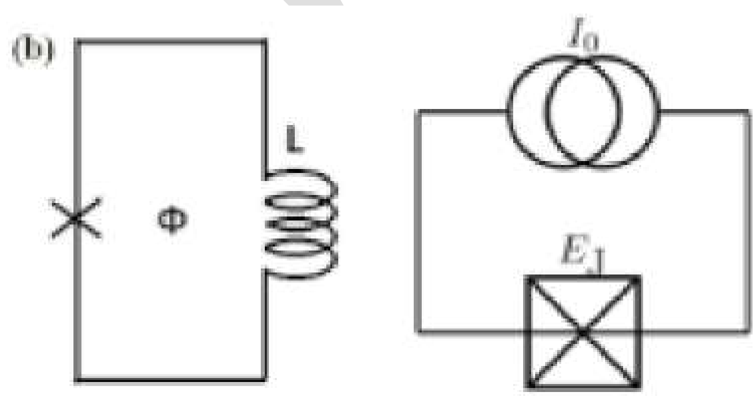


Flux Qubit

Flux Qubit (also known as persistent current qubits) are micrometer sized loops of superconducting metal that is interrupted by a number of Josephson Junctions. These devices function as quantum bits. The Josephson Junctions are designed so that a persistent current will flow continuously when an external magnetic flux is applied. Only an integer number of flux quanta are allowed to penetrate the superconducting ring.

Phase Qubit

A Phase qubit is a current-biased Josephson junction, operated in the zero voltage state with a non-zero current bias. This employs a single Josephson junction and the two levels are defined by quantum oscillations of the phase difference between the electrodes of the junction. DC SQUID is a type of phase qubit.



QUESTIONS

1.Define Fermi energy. Discuss the variation of Fermi factor with temperature.

2. Explain the failures of CFT

3. Mention the assumptions of QFT.

4. Define Phonon. Describe Mathiessen's Rule.
5. Explain the concept of Fermi energy, Fermi Level, Fermi Factor and Density of states.
6. Define superconductivity and hence discuss the variation of resistivity with temperature in superconductor with critical temperature as reference.
7. State and explain Meissner's Effect.
8. Define critical field and hence explain its variation with temperature below critical temperature
9. Distinguish between Type-1 and Type2 superconductors.
10. Write a note on high temperature superconductors
11. Elucidate the BCS theory of Superconductivity
12. Explain the phenomenon of quantum tunneling.
13. Define SQUID and describe DC and RF squids.
14. Brief the application of superconductivity in quantum computing.
15. Describe Meissner's Effect and hence classify superconductors into Soft and Hard superconductors using M-H graphs.
16. Explain DC and AC Josephson effects and mention the applications of superconductivity in quantum computing.

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APPLICATIONS OF PHYSICS IN COMPUTING

Chapter : Physics of Animation and

Animation

Animation is a method of photographing successive drawings, models, or even puppets, to create an illusion of movement in a sequence. Because our eyes can only retain an image for approx. $1/10^{th}$ of a second, when multiple images appear in fast succession, the brain blends them into a single moving image. Animation is t he process of displaying still images in a rapid sequence to create the illusion of movement.

The Taxonomy of Physics-Based Animation Methods

At the highest level, the field of physics-based animation and simulation can roughly be subdivided into two large groups:

1. Kinematics is the study of motion without consideration of mass or forces.
2. Dynamics is the study of motion taking mass and forces into consideration.

kinematics and dynamics come in two flavors or subgroups:

1. Inverse is the study of motion knowing the starting and ending points.
2. Forward is the study of motion solely given the starting point.

Frames

A frame is a single image in a sequence of pictures. A frame contains the image to be displayed at a unique time in the animation. In general, one second of a video is comprised of 24 or 30 frames per second also known as FPS. The frame is a combination of the image and the time of the image when exposed to the view. An extract of frames in a row makes the animation.

Frames per Second

Animation shot on film and projected is played at 24 frames per second. Animation for televisionin Europe, Africa, the Middle East and Australia is played at 25 frames per second.

Sl. No.	System	Frames Per Second
1	PAL (Australia, Middle East, Africa)	25
2	NTSC(America, West Indies, Specific Rim Countries)	30

An animated film with 25 frames per second is played on television at 24 frames per second would result in a black bar rolling up the screen. Then Digital Converts are to be used to transfer one speed of film to another speed of video. The most important thing to find out when animating something is what speed the animation will be played back at.

Size and Scale

The size and scale of characters often play a central role in a story's plot. What would Superman be without his height and bulging biceps? Some characters, like the Incredible Hulk, are even named after their body types.

We often equate large characters with weight and strength, and smaller characters with agility and speed. There is a reason for this. In real life, larger people and animals do have a larger capacity for strength, while smaller critters can move and maneuver faster than their large counterparts. When designing characters, you can run into different situations having to do with size and scale, such as:

1. Human or animal-based characters that are much larger than we see in our everyday experience. Superheroes, Greek gods, monsters,
2. Human or animal-based characters that are much smaller than we are accustomed to, such as fairies and elves.
3. Characters that need to be noticeably larger, smaller, older, heavier, lighter, or more energetic than other characters.
4. Characters that are child versions of older characters. An example would be an animation featuring a mother cat and her kittens. If the kittens are created and animated with the same proportions and timing as the mother cat, they won't look like kittens; they'll just look like very small adult cats.

Proportion and Scale

Creating a larger or smaller character is not just a matter of scaling everything about the character uniformly. To understand this, let's look at a simple cube. When you scale a cube, its volume changes much more dramatically than its surface area. Let us say each edge of the cube is 1 unit length. The area of one side of the cube is 1 square unit, and the volume of the cube is 1 cubed unit. If you double the size of the cube along each dimension, its height increases by 2 times, the surface area increases by 4 times, and its volume increases by 8 times. While the area increases by squares as you scale the object, the volume changes by cubes.

Wight and strength

Body weight is proportional to volume. The abilities of your muscles and bones, however, increase by area because their abilities depend more on cross-sectional area than volume. To increase a muscle or bone's strength, you need to increase its cross- sectional area. To double a muscle's strength, for example, you would multiply its width by $\sqrt{2}$. To triple the strength, multiply the width by $\sqrt{3}$. Since strength increases by squares and weight increases by cubes, the proportion of a character's weight that it can lift does not scale proportionally to its size.

Let us look at an example of a somewhat average human man. At 6 feet tall, he weighs 180 pounds and can lift 90 pounds. In other words, he can lift half his body weight. If you scale up the body size by a factor of 2, the weight increases by a factor of 8. Such a character could then lift more weight. But since he weighs more than 8 times more than he did before, he can not lift his arms and legs as easily as a normal man. Such a giant gains strength, but loses agility.

Motion and Timing in animations

Introduction to Motion :

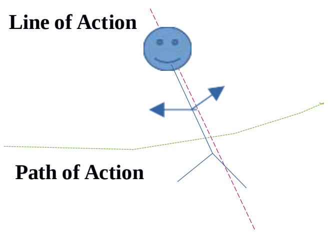
Motion is an essential component in games and animations. The motion is governed by the newtons laws and kinematic equations. When animating a scene, there are several types of motion to consider. These are the most common types of motion:

1. Linear
2. Parabolic
3. Circular
4. Wave

Motion and timing go hand in hand in animation.

Motion Lines and Paths

Individual drawings or poses have a line of action, which indicates the visual flow of action at that single image. Motion has a path of action, which indicates the path along which the object or character moves. The path of action refers to the object's motion in space. While it can help show timing, its primary function is to see the direction and path of the motion, and not necessarily its timing.



Timing

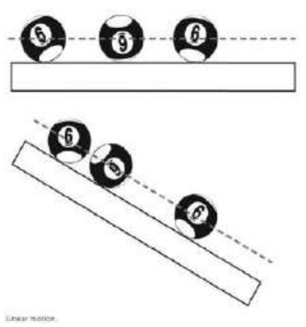
The timing is the choice of when something should be done; the regulation of occurrence and pace to achieve a desired effect. Animators have the ability to move forward and backward in time to place objects when and where they are to be.

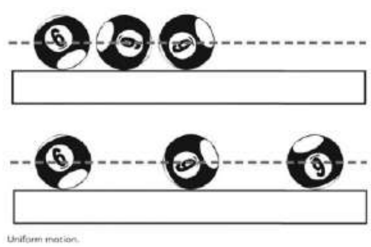
Timing Tools

In animation, timing of action consists of placing objects or characters in particular locations at specific frames to give the illusion of motion. Animators work with very small intervals of time; most motion sequences can be measured in seconds or fractions of seconds. Frame intervals between keys are usually smaller than one second.

Linear Motion Timing

Linear motion refers to motion in a straight line, always in the same direction. An object moving with linear motion might speed up or slow down as it follows a linear path. A heavy ball rolling on a table or incline is an example of linear motion. The all is rotating, but its center of gravity follows a linear path. A heavy ball rolling on a table or incline is an example of linear motion. The ball is rotating, but its center of gravity follows a linear path.





Uniform Motion Timing

When uniform motion occurs, the net force on the object is zero. Net force is the total of all forces added up. There might be several forces acting on the object, but when both the magnitude and direction of the forces are added up, they add up to zero. Uniform motion is the easiest to animate because the distance the object travels between frames is always the same. Uniform motion is a type of linear motion with constant speed and no acceleration or deceleration. The object moves the same distance between consecutive frames. The longer the distance between frames, the higher the speed.

Slow in and Slow out

When motion is accelerating or decelerating, we refer to this type of motion as a slow in or slow out. This type of motion is sometimes called ease in or ease out. In this book, we use the hyphenated forms slow-in and slow-out for easier understanding.

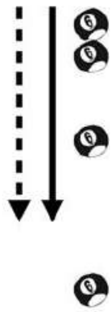
1. Slow in, ease in—The object is slowing down, often in preparation for stopping.
2. Slow out, ease out—The object is speeding up, often from a still position.

The term slow out can be confusing, since it essentially means “speed up.” one can think of slow out as the same as ease out, as in easing out of a still position and speeding up to full speed.

For example, a ball rolling down an incline or dropping straight down is slowing out, as it goes from a still position or slow speed to a fast speed. A ball rolling up an incline is slowing in.


Slowing out: speeding up.


Slowing in: slowing down.



Acceleration Timing

Timing for acceleration can be calculated very accurately when the net force being exerted is constant. Let’s take a look at the forces and how they can be used to calculate the animation’s timing.

Constant Forces

A constant force is a force that doesn’t vary over time. Examples of constant forces include:

1. Gravity pulling an object to the ground
2. Friction bringing an object to a stop

Constant force and Acceleration

Constant forces result in constant acceleration. Because the acceleration is constant, we can figure out the timing for such sequences using a few principles of physics.

The resulting acceleration depends on the direction of the force and motion, if there is any motion at all to begin with.

1. When constant net force is applied to an unmoving object, the result is acceleration.

2. When constant net force is applied to a moving object in the same direction as the motion, the result is acceleration.
3. When constant net force is applied in the direction opposite the existing motion, the result is deceleration (acceleration in the opposite direction).

Forces Exerted by Characters

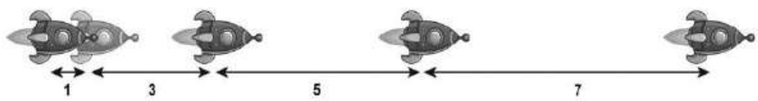
Forces exerted by people's bodies are rarely constant throughout an entire motion. For the purposes of animation, however, one can break the character motion into short time segments and consider each of these segments to be responding to constant net force. This will make it easier for one to calculate the timing for each individual segment.

As an example, let's look at the push for a jump. The force a character exerts during the push is somewhat constant, and the timing is very short (less than half a second). In such a case the timing for a constant force is an excellent starting point, and in most cases will do the job as is.

A character walking and pushing a rock is not exerting a constant force throughout the entire sequence, but during each short part of the walk cycle the net force could be considered to be a different constant value.

The Odd Rule

When acceleration is constant, one can use the Odd Rule to time the frames. With this method, one calculate the distance the object moves between frames using a simple pattern of odd numbers. Between consecutive frames, the distance the object moves is a multiple of an odd number. For acceleration, the distance between frames increases by multiples of 1, 3, 5, 7, etc.



Rocket speeding up using the Odd Rule.

For deceleration, the multiples start at a higher odd number and decrease, for example 7, 5, 3, 1.

Sled coming to a stop using the Odd Rule.

The Odd Rule is a multiplying system based on the smallest distance traveled between two frames in the sequence. For a slow-out, this is the distance between the first two frames; for a slow-in, it's the distance between the last two frames. This distance, the base distance, is used in all Odd Rule calculations.

Odd Rule Multipliers

The Odd Rule in its simplest form, as described above, is just one way to use it. For example, one can instead calculate the distance from the first frame to the current frame and use these distances to place the object on specific frames.

	Multiply by base distance to get distance between:	
Frame #	Consecutive frames	First frame and this frame
1	n/a	0
2	1	1
3	3	4
4	5	9
5	7	16
6	9	25
7	11	36

calculating the distance for a large number of frames and a chart like this isn't practical, one can figure out the odd number multiplier for consecutive frames with this formula:

Odd number multiplier for consecutive frames = ((frame # – 1) 2) – 1

In the charts above, note that the distances in the last column are squared numbers: $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, and so on. One of the benefits of the Odd Rule is one can calculate the total distance traveled from the start point to the current frame with the following formula:

Multiplier for distance from first frame to current frame = $(\text{current frame \#} - 1)^2$

When setting the keys, one can use either the consecutive key multipliers or total distance multipliers but need to Choose the one that's easiest to use for the animated sequence.

Odd Rule Scenarios

Here are a few different scenarios for calculating the distance an object travels between keys in a slow-in or slow-out.

Base Distance Known Speeding up

If the object is speeding up, the first frame distance is the base distance. If one knows the base distance, figuring out the distance the object travels at each frame is pretty straightforward. Just multiply the base distance by 3, 5, 7, etc. to get the distances between consecutive frames, or use squares to multiply the base distance to get the total distance traveled on each frame.

Base Distance Known Slowing Down

Suppose one wants an object to slow down, and one knows the distance between the last two frames before it stops. For slow-ins, the base distance is the distance between the last two frames. The solution is to work backward, as if the object were speeding up in the opposite direction. Working backward, multiply the base distance by 3, 5, 7, etc. to get the distances between each previous frame in the sequence.

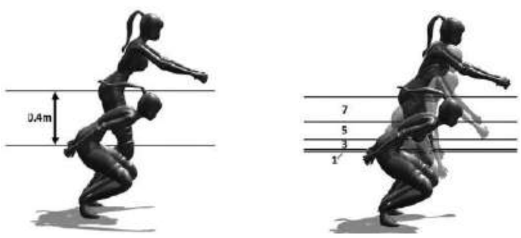
Total Distance and Number of Frames Known, Speeding Up

If one wants know the total distance and the total number of frames, one can find the base distance with this formula:

Base distance = Total distance/$(\text{Last frame number} - 1)^2$

Suppose there is a jump push (takeoff) with constant acceleration over 5 frames, and the total distance traveled is 0.4m. Using the formula above, we find the base distance.

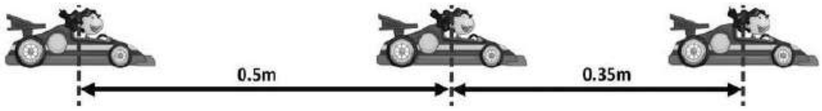
Base distance = $0.4\text{m}/(5 - 1)^2 = 0.4\text{m}/16 = 0.025\text{m}$



Using the base distance, one can calculate the distances between each frame. If one adds up the distances traveled, one will find that they add up to exactly 0.4m.

First Key Distance Known Slowing Down

Suppose one has a moving object that one wants to slow down, and one has set the first frame of the slow-in to give an idea of the pacing for the sequence. In this case, one can consider that the distance the object moved between the last two frames before the slow-in is part of the calculation— the distance between them becomes the first frame distance, and the first slow-in frame becomes the second frame in the sequence.



One feature of the Odd Rule is that the base distance is always half the difference between any two adjacent distances. To find the base distance, one can simply calculate:

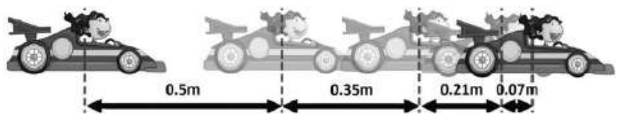
(0.5m – 0.35m)/2 = 0.07m

To figure out how many frames are in the slow-in, divide the first distance by the base distance to find out which odd number it corresponds to.

0.5/0.07 = 7

This means the first distance corresponds to 7 in the 7, 5, 3, 1 sequence, making the sequence four frames long. Now ibe can work back the other way, multiplying the base distance by odd numbers to get the distances for the rest of the slow-in frames.

Frame #	Consecutive frame multiplier	Distance from previous frame
1	7	7 * 0.07m = 0.5m
2	5	5 * 0.07m = 0.35m
3	3	3 * 0.07m = 0.21m
4	1	1 * 0.07m = 0.07m



Motion Graphs

A motion graph plots an object’s position against time. If one is using animation software, understanding and using motion graphs is a key skill in animating anything beyond the simplest of motions. If one is drawing the animation, drawing motion graphs before animating can help one to visualize the motion. On a motion graph, the time goes from left to right across the bottom of the graph, while the object’s position is plotted vertically against the time. Each axis in 3D space (X, Y, Z) has its own line showing the object’s position along that axis. At the very least, one will need to understand the types of lines in a motion graph and what they represent in terms of visible motion. one can also look at motion graphs to get a better understanding of any difficulties one is having with the timing or action.

Examples of Character Animation

Jumping and Walking

Jumping

A jump is an action where the character's entire body is in the air, and both the character's feet leave the ground at roughly the same time. A jump action includes a takeoff, free movement through the air, and a landing.

Parts of Jump

A jump can be divided into several distinct parts:

- **Crouch**—A squatting pose taken as preparation for jumping.
- **Takeoff**—Character pushes up fast and straightens legs with feet still on the ground. The distance from the character's center of gravity (CG) in the crouch to the CG when the character's feet are just about to leave the ground is called the push height. The amount of time (or number of frames) needed for the push is called the push time.
- **In the air**—Both the character's feet are off the ground, and the character's center of gravity (CG) moves in a parabolic arc as any free-falling body would. First it reaches an apex, and then falls back to the ground at the same rate at which it rose. The height to which the character jumps, called the jump height, is measured from the CG at takeoff to the CG at the apex of the jump. The amount of time the character is in the air from takeoff to apex is called the jump time. If the takeoff pose and the landing pose are similar, then the jump height and jump time are about the same going up as they are going down.
- **Landing**—Character touches the ground and bends knees to return to a crouch. The distance from the character's CG when her feet hit to the ground to the point where the character stops crouching is called the stop height. The stop height is not always exactly the same as the push height.

Calculating Jump Actions

When working out the timing for a jump, one will need to first decide on:

1. Jump height or jump time
2. Push height
3. Stop height
4. Horizontal distance the character will travel during the jump

From these factors, one can calculate the timing for the jump sequence.

Calculating Jump Timing

When planning the jump animation, the most likely scenario is that you know the jump height, expressed in the units you are using for the animation (e.g., inch or cm).

Placement and timing for frames while the character is in the air follow the same rules as any object thrown into the air against gravity. Using the tables in the Gravity chapter (or an online calculator), one can figure out the jump time for each frame. Look up the amount of time it takes an object to fall that distance due to gravity, and express the jump time in frames based on the fps one is using.

Example:

Jump height = 1.2m

Jump time for 1.2m = 0. 5 seconds

Jump time at 30fps = 0. 5 * 30 = 15 frames

Jump Magnification

When calculating the remainder of the timing for the entire jump action, you can use a factor called jump magnification (JM). The JM can be used to calculate the push timing and stop timing.

The JM is the ratio of the jump height to the push height.

$$JM = \frac{Jump\ Height}{Push\ Height}$$

Since you already know the jump height and push height, you can calculate the JM. Then you can use the JM to calculate other aspects of the jump.

Example:

Jump Height = 1m

Push Height = 0.33m

JM = Jump Height/Push Height = 3

Jump Magnification and Acceleration

Jump Magnification is in fact an exact ratio that tells one how much the character has to accelerate against gravity to get into the air. The JM, besides being the ratio of jump-to-push vertical height and time, is also the ratio of push-to-jump vertical acceleration. Opposite the other ratios: while a longer jump time means a shorter push time, a higher jump acceleration means a much, much higher push acceleration. Knowing about this can help you make more informed decisions about your push timing.

To see how this works, let's look at the formula for JM and relate it to acceleration:

Jump Time Jump Height

$$JM = \frac{Jump\ Time}{Push\ Time} = \frac{Jump\ Height}{Push\ Height} = \frac{Push\ Acceleration}{Jump\ Acceleration}$$

The magnitude of jump acceleration is always equal to gravitational acceleration, with deceleration as the character rises and acceleration as it falls.

$$JH = \frac{Push\ Acceleration}{Jump\ Acceleration} = \frac{Push\ Acceleration}{Gravitational\ Acceleration}$$

Your landing speed is the same as the velocity of any falling object, which you can easily calculate from the free fall time. Since acceleration due to gravity is 10m/sec2, this means that after one second a falling object is traveling at 10m/sec, after two seconds at 20m/sec, after three seconds at

30m/sec, and so on. Since takeoff speed is the same as landing speed, you need to get up to that same speed when taking off for a jump. If your landing speed is 10m/sec, then during your takeoff you need to get up to a speed of 10m/sec in that little bit of push time.

The general formula for calculating the velocity of an accelerating object is: Velocity = Acceleration * Time

Physics shorthand: v = at

Let's relate this back to our jump. If the landing velocity is the same as the push velocity, we know that:

v = Jump Acceleration * Jump Time

So . . .

Jump Acceleration * Jump Time = Push Acceleration * Push Time

Moving things around with a bit of algebra, we arrive at this equation:

$$\frac{Jump\ Time}{Push\,Time} = \frac{Push\,Acceleration}{Jump\ Acceleration\left(Gravity\right)}$$

Look, it's the JM! And it's equal to the ratio of the push acceleration to gravity. Increase your jump time, and the push acceleration goes up. Decrease your push time, and the push acceleration goes up. Distance (or in this case, jump or push height) is also related to velocity: Distance = Average Velocity * Time

Physics shorthand:

d = vt

With some algebra, we make this into yet another formula for the average

velocity:

v = d/t

Because the average velocity is the same for both the push and jump, we can say that d/t is the same for both jump and **push:**

Jump Height/Jump Time = Push Height/Push Time

And with a little more algebra:

$$\frac{Jump\ Height}{PushTime} = \frac{Push\,Height}{PushTime}$$

Push Time

The JM also gives you the ratio of the jump time to the push time.

JM = Jump Time/Push Time

Working a little algebra, we can express the equation in a way that directly calculates the push time:

Push Time = Jump Time/JM

Example:

JM = 3

Jump Time: 15 frames

Push Time = 15/3 = 5 frames

Landing

The forces on landing are similar to takeoff. If the landing has faster timing, the forces will be larger than for a longer timing.

Stop Time

The stop height is often a bit larger than the push height, but the timing ofthe push and stop are the same in the sense that the CG moves the same distance per frame in the push and stop. If the stop height is larger than the push height, you'll just need more frames for the stop than the push.

Push Height/Push Frames = Stop Height/Stop Frames

This can also be expressed as:

Push Height/Push Time = Stop Distance/Stop Time

You can also flip everything over and express it as:

Push Time/Push Height = Stop Time/Stop Distance

Using algebra, we can get the following equation for stop time:

Stop Time = (Push Time * Stop Distance)/Push Height

Example:

Push Time: 5 frames

Push Height: 0.4m

Stop Height: 0.5m

Stop Time = (5 * 0.5) / 0.4 = 6 frames

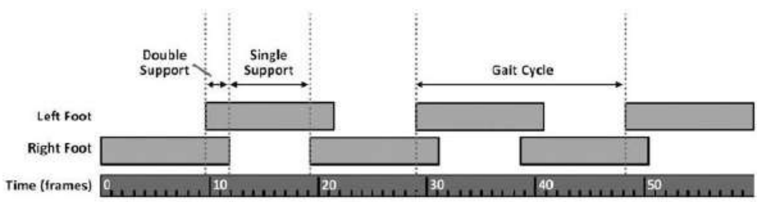
Walking

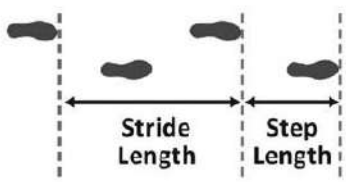
Walks feature all the basics of mechanics while including personality. The ability to animate walk cycles is one of the most important skills a character animator needs to master.

Strides and Steps

A step is one step with one foot. A stride is two steps, one with each foot. Stride length is the distance the character travels in a stride, measured from the same part of the foot. Step and stride length indicate lengthwise spacing for the feet during a walk.

Gait is the timing of the motion for each foot, including how long each foot is on the ground or in the air. During a walk, the number of feet the character has on the ground changes from one foot (single support) to two feet (double support) and then back to one foot. You can plot the time each foot is on the ground to see the single and double support times over time. A normal walking gait ranges from 1/ 3 to 2/3 of a second per step, with 1/2 second being average.





Walk Timing

Walking is sometimes called “controlled falling.” Right after you move past the passing position, your body’s center of gravity is no longer over your base of support, and you begin to tip. Your passing leg moves forward to stop the fall, creating your next step. Then the cycle begins again. The horizontal timing for between the four walk poses is not uniform. The CG slows in going from the contact to passing position, then slows out from passing to contact. The CG also rises and falls, rising to the highest position during passing and the lowest during contact. The head is in the highest position during passing.

Statistical Physics for Computing

Poisson Distribution

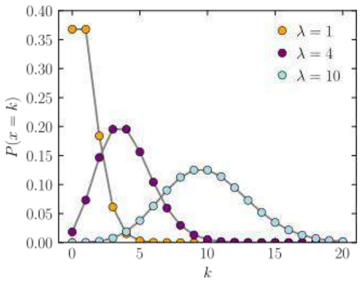
If the probability p is so small that the function has significant value only for very small k, then the distribution of events can be approximated by the Poisson Distribution.

Probability mass function

A discrete Radom variable *X* is said to have a Poisson distribution, with parameter , if it has a probability Mass Function given by

$$f(k;\lambda)=P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$$

Here *k* is the number of occurrences, *e* is Euler’s Number, ! is the factorial function. The positive real number λ is equal to the expected vlaue of *X* and also to its Variance. The Poisson distribution may be used in the design of experiments such as scattering experiments where a small number of events are seen.



Example of probability for Poisson distributions

On a particular river, overflow floods occur once every 100 years on average. Calculate the probability of *k* = 0, 1, 2, 3, 4, 5, or 6 overflow floods in a 100 year interval, assuming the Poisson model is appropriate.

Because the average event rate is one overflow flood per 100 years, λ = 1

$$f(k;\lambda)=P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$$

P(k overflow floods in 100 years) = $\frac{\lambda^k e^{-\lambda}}{k!}=\frac{1^k e^{-1}}{k!}$

P(k=0 overflow floods in 100 years) = $\frac{\lambda^k e^{-\lambda}}{k!}=\frac{1^0 e^{-1}}{0!}=\frac{e^{-1}}{1}=0.368$

P(k=1 overflow floods in 100 years) = $\frac{\lambda^k e^{-\lambda}}{k!}=\frac{1^1 e^{-1}}{1!}=\frac{e^{-1}}{1}=0.368$

P(k=2 overflow floods in 100 years) = $\frac{\lambda^k e^{-\lambda}}{k!} = \frac{1^2 e^{-1}}{2!} = \frac{e^{-1}}{2} = 0.184$

Modeling the Probability for Proton Decay

The experimental search for Proton Decay was undertaken because of the implications of the Grand unification Theories. The lower bound for the lifetime is now projected to be on the order of $\tau = 10^{33}$ Years. The probability for observing a proton decay can be estimated from the nature of particle decay and the application of Poisson Statistics. The number of protons N can be modeled by the decay equation

$$N = N_0 e^{-\lambda t}$$

Here $\lambda = 1/t = 10^{-33}$/ year is the probability that any given proton will decay in a year. Since the decay constant λ is so small, the exponential can be represented by the first two terms of the Exponential Series.

$$e^{-\lambda t} = 1 - \lambda t \text{ , thus } N \approx N_0 (1 - \lambda t)$$

For a small sample, the observation of a proton decay is infinitesmal, but suppose we consider the volume of protons represented by the Super Kameokande neutrino detector in Japan. The number of protons in the detector volume is reported by Ed Kearns of Boston University to be 7.5 x 10^{33}protons. For one year of observation, the number of expected proton decays is then

$$N - N_0 = N_0 \lambda t = (7.5 \times 10^{33} \text{ protons})(10^{-33}/\text{year})(1 \text{ year}) = 7.5$$

About 40% of the area around the detector tank is covered by photo-detector tubes, and if we take that to be the nominal efficiency of detection, we expect about three observations of proton decay events per year based on a 10^{33}year lifetime.

So far, no convincing proton decay events have been seen. Poisson statistics provides a convenient means for assessing the implications of the absence of these observations. If we presume that λ = 3 observed decays per year is the mean, then the Poisson distribution function tells us that the probability for zero observations of a decay is

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad p(k) = \frac{3^0 e^{-3}}{0!} = 0.05$$

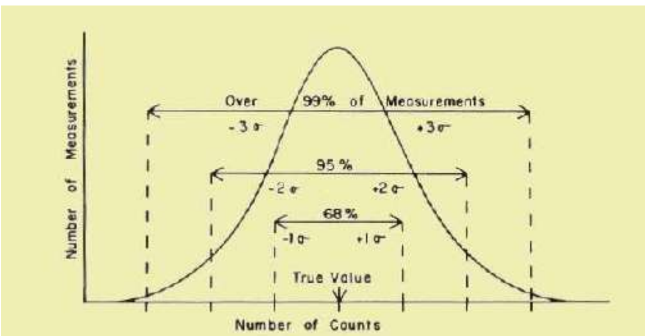
This low probability for a null result suggests that the proposed lifetime of 10^{33} years is too short. While this is not a realistic assessment of the probability of observations because there are a number of possible pathways for decay, it serves to illustrate in principle how even a non-observation can be used to refine a proposed lifetime.

Normal Distribution and Bell Curves

A bell curve is a common type of distribution for a variable, also known as the normal distribution. The term "bell curve" originates from the fact that the graph used to depict a Normal Distribution consists of a symmetrical bell-shaped curve.

The highest point on the curve, or the top of the bell, represents the most probable event in a series of data (its Mean, Mode and Median in this case), while all other possible occurrences are symmetrically distributed around the mean, creating a downward-sloping curve on each side of the peak. The width of the bell curve is described by its Standard Deviation.

The term "bell curve" is used to describe a graphical depiction of a normal probability distribution, whose underlying standard deviations from the mean create the curved bell shape. A standard deviation is a measurement used to quantify the variability of data dispersion, in a set of given values around the mean. The mean, in turn, refers to the average of all data points in the data set or sequence and will be found at the highest point on the bell curve.



Standard Deviations

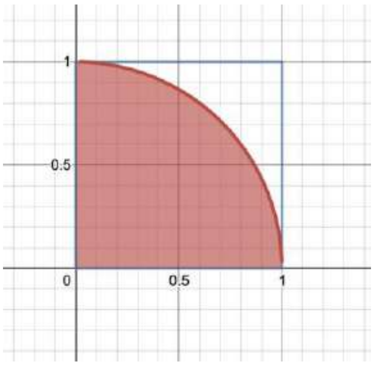
The Standard Deviation is a measure of how spread out numbers are. 68% of values are within 1 standard deviation of the mean. 95% of values are within 2 standard deviations of the mean. 99.7%of values are within 3 standard deviations of the mean

Monte-Carlo Method

Monte Carlo methods vary, but tend to follow a particular pattern:

1. Define a domain of possible inputs
2. Generate inputs randomly from a probability distribution over the domain
3. Perform a deterministic computation on the inputs
4. Aggregate the results

Monte Carlo method applied to approximating the value of π. For example, consider a quadrant inscribed in a unit square. Given that the ratio of their areas is $\pi/4$, the value of π /can be approximated using a Monte Carlo method:



1. Draw a square, then Inscribe a quadrant within it
2. Uniformly scatter a given number of points over the square
3. Count the number of points inside the quadrant, i.e. having a distance from the origin of < 1
4. The ratio of the inside-count and the total-sample-count is an estimate of the ratio of the two areas, $\pi/4$. Multiply the result by 4 to estimate π./

In this procedure the domain of inputs is the square that circumscribes the quadrant. We generate random inputs by scattering grains over the square then perform a computation on each input (test whether it falls within the quadrant). Aggregating the results yields our final result, the approximation of π.

There are two important considerations:

1. If the points are not uniformly distributed, then the approximation will be poor.
2. There are many points. The approximation is generally poor if only a few points are randomly placed in the whole square. On average, the approximation improves as more points are placed.

Uses of Monte Carlo methods require large amounts of random numbers, and their use benefited greatly from Pseudo random number generators, which were far quicker to use than the tables of random numbers that had been previously used for statistical sampling.

QUESTIONS

1. Elucidate the importance of size & scale and weight and strength in animations.
2. Mention the general pattern of Monte Carlo method and hence determine the value of π.
3. Describe the calculation of Push time and stop time with examples.
4. Sketch and explain the motion graphs for linear, easy ease, easy ease in and easy ease out cases of animation.
5. Discuss modeling the probability for proton decay.
6. A slowing-in object in an animation has a first frame distance 0.5m and the first slow in frame 0.35m. Calculate the base distance and the number of frames in sequence.
7. Discuss timing in Linear motion, Uniform motion, slow in and slow out.
8. Distinguish between descriptive and inferential statistics.
9. Illustrate the odd rule and odd rule multipliers with a suitable example.
10. Describe Jumping and parts of jump.
11. Discuss the salient features of Normal distribution using bell curves.
12. The number of particles emitted per second by a random radioactive source has a Poisson's distribution with ë = 4. Calculate the probability of $P(X = 0)$ and $P(X = 1)$.