VISVESVARAYA TECHNOLOGICAL UNIVERSITY
BELGAUM

ANALYSIS OF STRUCTURES

(Subject Code: BCV401)

LECTURE NOTES

(MODULE-2)

IV-SEMESTER

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Module -2

Deflection of Beams

Deflection of beams: Moment area method: Derivation, Mohr's theorems, sign convention, Application of moment area method to determinate prismatic beams, beams of varying cross section, Use of moment diagram by parts.

Strain Energy: Principle of virtual displacements, principle of virtual forces, Strain energy and complimentary energy, Strain energy due to axial force, bending, shear and torsion (No numerical). Castigliano's theorems, application of Castigliano's theorems to calculate deflection of beams, trusses and frames (No numerical on unit load method).

Structures undergo deformation when subjected to loads. As a result of this deformation, deflection and rotation occur in structures. This deformation will disappear when the loads are removed provided the elastic limit of the material is not exceeded. Deformation in a structure can also occur due to change in temperature & settlement of supports.

Deflection in any structure should be less than specified limits for satisfactory performance. Hence computing deflections is an important aspect of analysis of structures.

The following methods are available for finding the deflections of determinate beams.

1) Double Integration/Macaulay's Method
2) Moment Area Method
3) Conjugate Beam Method
4) Strain Energy Method
5) Castigliano's Method
6) Unit Load Method

In these methods, the geometrical concept is used. These methods are ideal for statically determinate beams. The methods give a very quick solution when the beam is symmetrical.

1. Moment Area Method

The moment area method is based on the following two theorems:

Theorem 1: The change in the slope between two points on a straight member under flexure is equal to the area of $\frac{M}{EI}$ diagram between those two points.

Where M is the Bending Moment

E is the Young's Modulus

and I is the Moment of Inertia

Consider the beam AB (Figure 2.1(a)). Let C and D be any two points on this beam. The $\left(\frac{M}{EI}\right)$diagram is also shown in the figure. Figure 2.1(b) shows the elastic curve of the beam after loading. According to this theorem, θ_{CD} which is the angle between the tangents at C and D is equal to the area of $\left(\frac{M}{EI}\right)$ diagram between C and D (shaded portion). Thus,

$\theta_{CD} = \int_C^D \left(\frac{M}{EI}\right) dx$ ----------(1)

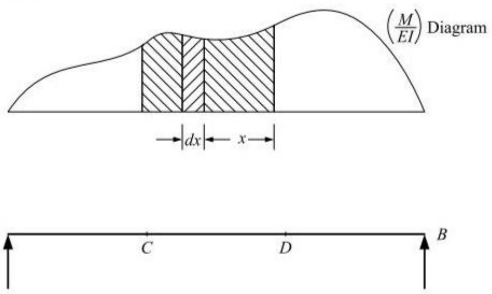


Figure 2.1(a) Beam with *M/EI* diagram as load

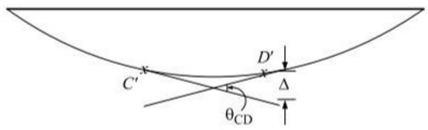


Figure 2.1(b) Angle and deflection between the tangents at *C* and *D*

Theorem 2: Deflection at a point in a beam in the direction perpendicular to its original straight-line position measured from the tangent to the elastic curve at another point is given by the moment of $\left(\frac{M}{EI}\right)$ diagram about the point where the deflection is required.

In Figure 2.1(b), Δ, the vertical (perpendicular to the horizontal position of AB) deflection at point D' from the tangent to the elastic curve at C' is given by the moment of $\left(\frac{M}{EI}\right)$ diagram between C and D about the point D. Thus,

$\Delta = \int_C^D \left(\frac{Mx}{EI}\right) dx$ ---------(2)

Derivation of Moment Area Theorems

Figure 2.2 shows the elemental length 'dx' of Figure 2.1 to an enlarged scale. Let R be the radius of curvature. Then, from flexure formula

$\frac{M}{I} = \frac{E}{R}$ ------(3)

From Figure 2.2,

Rdθ = ds= dx, since, axial deformations are considered negligible.

$\therefore R = \frac{dx}{d\theta}$ ------(4)

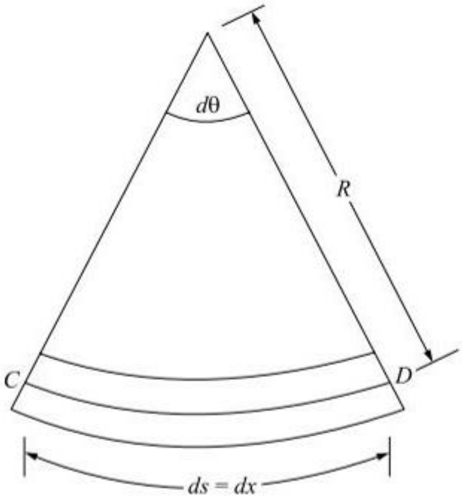
Substituting the value of R in equation 3, we get

$$\frac{M}{I} = \frac{E}{(dx/d\theta)}$$

$$d\theta = \left(\frac{M}{EI}\right) dx$$

$$\therefore \theta_{CD} = \int_C^D d\theta$$

$$= \int_C^D \left(\frac{M}{EI}\right) dx$$



Hence, theorem 1 is proved.

Now consider Figure 2.3 in which portion CD is blown up to an enlarged scale. Let the change of slope in elemental length dx be 'dθ'. Distance of elemental length from D is x (Fig 2.1(a)).

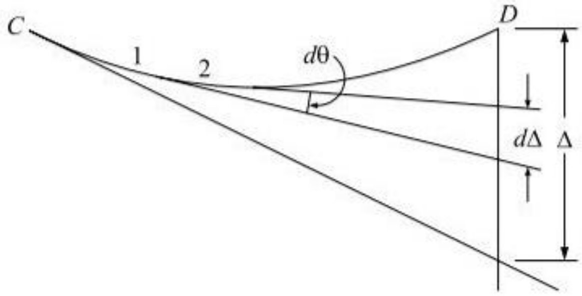


Figure 2.3 Enlarged view of portion CD

Hence, deflection,

$$d\Delta = x\,d\theta = x\left(\frac{M}{EI}\right) dx$$

$$\therefore \ \Delta = \int_C^D \left(\frac{M}{EI}\right) x \, dx$$

Hence, theorem 2 is proved.

Sign Convention in the Moment Area Method Applied to Beams

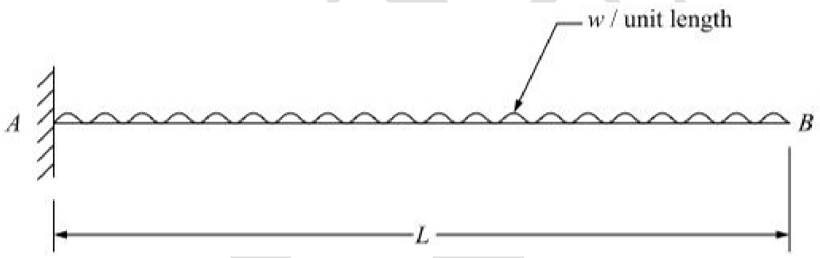
The following sign convention is used in applying moment area theorems to find the deflection of beams:

1. Sagging moment area is positive, which means that, the tangent at D makes an anticlockwise angle with tangent at C.
2. The moment of positive moment gives rise to positive deflection, which implies that the deflected position of a point (D) is above the tangent drawn at the other point (C).

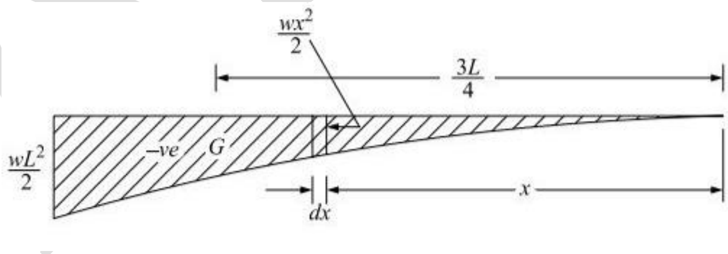
This method is advantageous, if the tangent at a particular point in the beam is along the axis itself, e.g., beam with one end fixed or with a symmetric point.

Problems

1. Determine the rotation and deflection at the free end of the cantilever beam subjected to uniformly distributed load over an entire span as shown in figure.



Solution- The bending moment diagram is shown in figure. At any distance x from free end, bending moment is - $\frac{wx^2}{2}$.



Now,

$\theta_{BA} = \theta_B - \theta_A = \theta_B \quad (\therefore \theta_A = 0)$

Therefore, from the moment area theorem,

$\theta_B = \int_0^L \left(\frac{M}{EI}\right) dx$

$= \int_0^L -\left(\frac{wx^2}{2EI}\right) dx$

$= - \frac{w}{2EI}\left[\frac{x^3}{3}\right]_0^L$

$= \frac{-wL^3}{6EI}$

$= \frac{wL^3}{6EI}$, clockwise with tangent at A

Δ_B = deflection of B with respect to tangent at A

= vertical deflecting, since, tangent at A is horizontal

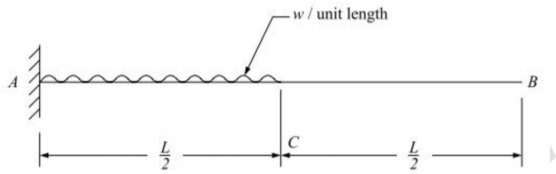
From the second moment area theorem,

$\Delta_B = \int_0^L \left(\frac{M}{EI}\right) x\, dx = \int_0^L -\left(\frac{wx^3}{2EI}\right) dx$

$= - \frac{w}{2EI}\left[\frac{x^4}{4}\right]_0^L = \frac{-wL^4}{8EI}$

$= \frac{wL^4}{8EI}$, downward

2. Find the rotation and deflection at the free end in the cantilever beam shown in figure.

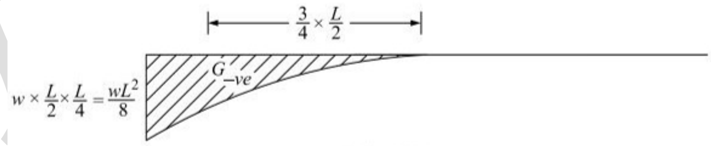


Solution – The bending moment diagram is a parabola as shown in figure with maximum ordinate as $\frac{wL}{4} \times \frac{L}{4} = \frac{wL^2}{8}$. Its centre of gravity from the point C is at a distance $\frac{3}{4} \times \frac{L}{2} = \frac{3}{8}L$. Area of the bending moment diagram is

$$= \frac{1}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{wL^3}{48}$$

and it is negative area, since the bending moment is a hogging moment.

$\theta_{BA} = \theta_B - \theta_A = \theta_B$ $\quad (\because \theta_A = 0)$



From the moment area theorem,

θ_B = Area of $\left(\frac{M}{EI}\right)$ diagram between A and B

$= \left[-\frac{wL^3}{48}\right] \times \frac{1}{EI}$

$= -\frac{wL^3}{48EI}$

$= \frac{wL^3}{48EI}$, clockwise

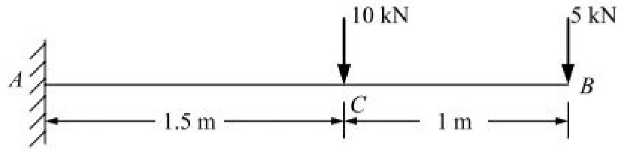
Since, tangent at A is horizontal, vertical deflection at B

$= \text{Moment of } \left(\frac{M}{EI}\right) \text{ diagram about B}$

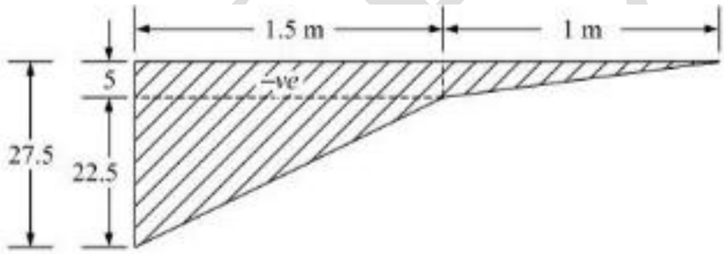
$= \left[-\frac{wL^3}{48EI}\right] \times \left(\frac{3}{8}L + \frac{L}{2}\right)$

$= -\frac{7wL^4}{384EI} = \frac{7wL^4}{384EI}$, downward

3. Determine the slope and deflection at the free end of a cantilever beam as shown in figure by moment area method. (Take EI = 4000 kNm^2)



Solution – The bending moment diagram for this beam is as shown in figure.



$\theta_B = \frac{Area\ of\ bending\ moment\ diagram}{EI}$

$= -\frac{1}{EI}\left[\frac{1}{2} \times 1 \times 5 + 5 \times 1.5 + \frac{1}{2} \times 22.5 \times 1.5\right]$

$= -\frac{26.875}{EI} = -\frac{26.875}{4000}$

$= -\ 6.71875 \times 10^{-3}$ radians

$= 6.71875 \times 10^{-3}$ radians, clockwise

$\Delta_B = \text{Moment of } \left(\frac{M}{EI}\right) \text{ diagram about B}$

$= -\frac{1}{EI}\left[\frac{1}{2} \times 1 \times 5 \times \frac{2}{3} + 5 \times 1.5 \times (1 + 0.75) + \frac{1}{2} \times 22.5 \times 1.5 \times (1 + 1)\right]$

$= -\frac{48.542}{EI} = -\frac{48.542}{4000}$

$= -\ 0.01213$ m

$= -\ 12.13$mm = 12.13mm, downward

STRAIN ENERGY

When an external load acts on a structure, the structure undergoes deformation and hence, the work is done. To resist these external forces, the internal forces develop gradually from zero to their final value and the work is done. This internal work done is stored as energy in the structure and it helps the structure to spring back to the original shape and size, whenever the external loads are removed, provided the material of the structure is still within the elastic limit. This internal work, which is stored as energy is due to the straining of the material and hence, is called strain energy.

When equilibrium is reached, as per the well-known law of conservation of energy, the work done by the external forces must equal the strain energy stored. This concept of energy balance is utilized in structural analysis to develop a number of methods to find the deflections of structures. The following are the methods for finding the deflections of beams and frames:

1. Strain energy/Real work method
2. Virtual work/unit load method
3. Castigliano's method

Principle of Virtual Work

The principle of virtual work states that the stress, body forces (gravity, weight) and traction (area force, wind force, wind pressure, fluid pressure) are in equilibrium if and only if the internal work done (W_i) equal to the external work done (W_e) for every virtual displacement field.

$\delta W = \delta W_e + \delta W_i = 0$ i.e. $\delta W_e = \delta W_i$

Virtual work is defined as work done by real forces acting through virtual displacements. These virtual displacements need not be real and can be virtual (imaginary).

Principle of Virtual Displacements

The principle of virtual displacements for rigid bodies can be stated as follows:

If a rigid body is in equilibrium under a system of forces and if it is subjected to any small virtual displacement, the virtual work done by the external forces is zero.

$W_e = 0$

The principle of virtual displacements for deformable bodies can be stated as follows:

A deformable body is in equilibrium, if the total external virtual work done by the system of true forces moving through the corresponding virtual displacements of the systems is equal to the total internal virtual work for kinematically admissible virtual displacements.

$\sum F_i \, \delta V_i = \int \sigma_{ij} \, \delta \, \varepsilon_{ij} \, \delta v$

Principle of Virtual Forces

For a deformable body, the total external complementary work is equal to the total internal complementary work for every system of virtual forces and stresses that satisfy the equations of equilibrium

$\sum \delta \, F_i \, V_i = \int \delta\sigma_{ij} \, \varepsilon_{ij} \, \delta v$

$\delta\sigma_{ij}$ = virtual stresses due to virtual forces $\delta \, F_i$

ε_{ij} = True strain due to the true displacement V_i

Strain Energy and Complimentary Energy

Consider the general structural system shown in Figure 3.1

Let the cross-sectional area of the element shown in Figure 3.1 be δa and its length δx. The stress in the element gradually increases from zero to its final value p as strain increases from zero to its final value e. Let this stress-strain relation be as shown in Figure 3.2.

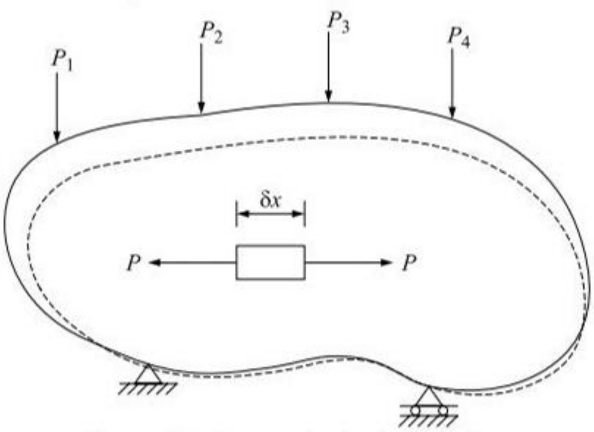


Figure 3.1 A general structural system

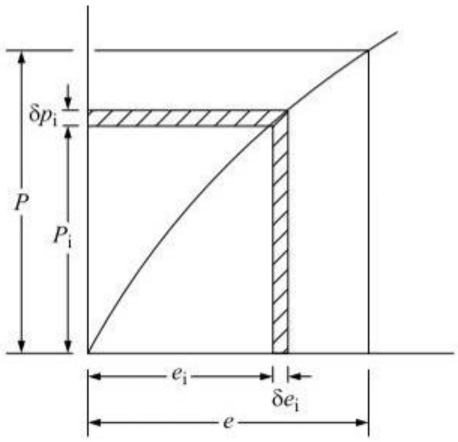


Figure 3.2 Stress-strain relation

Let stress be p_i, hence, work done on the element, when strain δe_i takes place

= Force x Displacement

= $p_i \, \delta a \, \delta e_i \, \delta x$

$= p_i\, \delta e_i\, \delta v$

$\delta a\, \delta x = \delta v$, where v is the volume of the element

Therefore, strain energy stored in the element,

$= \int_0^e p_i \delta e_i \delta v$

= Area under stress-strain curve x dv ---------(3.1)

If the stress-strain curve is linear, strain energy of the element,

$= \frac{1}{2}$ p e dv,

∴ Strain energy stored in the structure

$U = \int \frac{1}{2}$ p e dv

$= \int \frac{1}{2} \times \text{stress} \times \text{strain dv}$ ---------(3.2)

Figure 3.3 shows load versus deformation relation. Let, during deformation the load acting be p_i and deformation be $\delta\Delta_i$. Then, work done by the load under consideration,

$= \int P_i \delta \Delta_i$

= Area under the load deformation curve ---------(3.3)

$= \frac{1}{2} \times P\Delta$, in case of linear elasticity problems

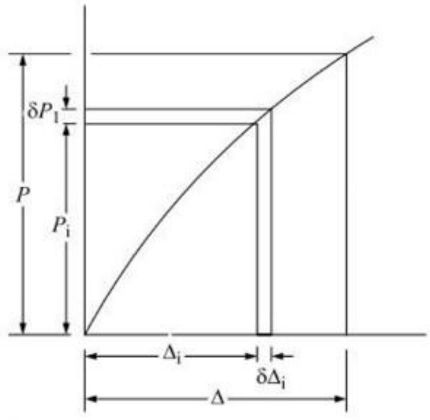


Figure 3.3 Local versus deformation curve

If there are n number of loads, total work done by the external loads is the summation of the expression 3.3 for all the loads.

Hence, in case of a linear elasticity (linear stress-strain curve and material still within elastic limit) work done by external loads,

$$= \sum \frac{1}{2} P\Delta$$

Complementary energy at any instant during deformation of the element is given by e_i δp_i dv.

Hence, the complementary energy of the element when final deformation takes place,

$$= \int_0^{P_i} e_i \, \delta p_i dv$$

= Area above the stress-strain curve x dv

$= \int \frac{1}{2} pedv$, in case of linear elasticity problems

Thus, complementary energy of the entire structure

$U_c = \int \frac{1}{2} pedv$, in case of linear elasticity problems ------(3.5)

From equations (3.2) and (3.5) we conclude, in case of linear elasticity problems, strain energy and complementary energy are equal to each other.

Similarly, complementary work done is given by

$$= \int_0^{P_i} \Delta_i \, \delta P_i$$

In case of linear elasticity problems, work done

$$= \frac{1}{2} \Delta P$$

If there are n number of loads, work done by external loads

$$= \sum \frac{1}{2} \Delta P$$

Strain energy due to axial force

Consider a member of constant cross-sectional area ‘A’ subjected to axial force ‘P’ through the centroid of the cross sectional as shown in above figure.

WKT, Stress, $\sigma = \frac{P}{A}$ ------(1)

Under the action of axial load 'P' applied at one end gradually, the beam get elongated by 'Δ'.

The incremental elongation 'dΔ' of small element of length of beam 'dx' is given by

$\varepsilon = \frac{d\Delta}{dx}$

$\Rightarrow d\Delta = \varepsilon dx$

WKT, $E = \frac{\sigma}{\epsilon} \Rightarrow \varepsilon = \frac{\sigma}{E}$

$\Rightarrow d\Delta = \frac{\sigma}{E} dx$ -----(2)

Substituting eqn (1) in eqn (2)

$\Rightarrow d\Delta = \frac{P}{AE} dx$

Total elongation of the member of length 'L' may be obtained by integration

$\Delta = \int_0^L \frac{P}{AE} dx$ ------(3)

Work done by external loads

$W = \frac{1}{2} P\Delta$

Strain energy = external work done

$U = W = \frac{1}{2} P\Delta$ -----(4)

Sub eqn (3) in eqn (4)

$U = \frac{1}{2} P \left[\int_0^L \frac{P}{AE} dx\right]$

$= \int_0^L \frac{P^2}{2AE} dx \Rightarrow \frac{P^2}{2AE} [x]_0^L \Rightarrow \frac{P^2 L}{2AE}$

$U = \frac{P^2 L}{2AE}$

Strain Energy due to Bending

Consider a small segment of prismatic beam of length 'dx' subjected to Bending moment 'M' as shown in figure.

Now one cross section rotates about another cross section by a small amount dθ.

From fig, dx = R.dθ --------(1)

From bending equation

$\frac{M}{I} = \frac{E}{R}$

$R = \frac{EI}{R}$ -----(2)

Sub (2) in (1)

dx = $\frac{EI}{M}$.dθ

dθ = $\frac{M}{EI}.dx$ ------(3)

where 'R' is the radius of curvature of the bent beam and 'EI' is the flexural rigidity of the beam.

Work done by the moment 'M' while rotating through angle 'dθ' will be stored in the segment of beam as strain energy. Hence

du = $\frac{1}{2}$ M. dθ -----(4)

Sub eqn (3) in eqn (4)

du = $\frac{1}{2}$ M ($\frac{M}{EI}.dx$) $\Rightarrow \frac{1}{2}\frac{M^2}{EI}\ dx$

Energy stored in complete beam of span 'LL

U = $\int_0^L \frac{M^2}{2EI}\ dx$

Strain Energy due to Shear

Consider a cantilever beam of length L as shown in figure.

Let us consider an element of length 'dx'. Let 'V' be the shear force.

Strain energy stored in small elemental length dx

du = $\frac{1}{2}$ x Vx ϕ_s -------(1)

WKT, Modulus of rigidity = $\frac{Shear\ Stress}{Shear\ Strain}$

$\epsilon_1 = \frac{\tau}{\phi}$

$\phi = \frac{\tau}{\epsilon 1}$

But τ = shear stress = $\frac{load}{area} = \frac{V}{A}$

$\phi = \frac{V}{A\epsilon 1}$

For small elemental length 'dx'

$\phi_s = \frac{V}{A\epsilon 1}\ dx$

Eqn (1) becomes

du = $\frac{1}{2}$ x Vx $\frac{V}{A\epsilon 1}\ dx$

du = $\frac{V^2}{2A\epsilon 1}\ dx$

Total Strain energy = U = $\int_0^L \frac{V^2}{2A\epsilon 1}\ dx$

Strain Energy due to Torsion

Consider a circular shaft of length 'L' subjected to a torque T at the end.

Strain energy is stored in an elemental length 'dx'

du = $\frac{1}{2} \times T \times d\theta$ ------(1)

WKT, from torsion equation

$\frac{T}{J} = \frac{\epsilon_1 \theta}{L}$

θ = $\frac{T}{\epsilon_1 J}$ L

For small length 'x'

$\frac{T}{J} = \frac{\epsilon_1 \theta}{x}$

θ = $\frac{T}{\epsilon_1 J}$ x

Diff w.r.t. x

$$\frac{d\theta}{dx} = \frac{T}{\epsilon_1 J}$$

dθ = $\frac{T}{\epsilon_1 J} dx$

Substitute this in eqn (1)

du = $\frac{1}{2} \times T \times \frac{T}{\epsilon_1 J} dx$

du = $\frac{T^2}{2\epsilon_1 J}\ dx$

Total strain energy U = $\int_0^L \frac{T^2}{2\epsilon_1 J}\ dx$

Castigliano's Theorems

Castigliano published two important theorems in structural analysis (1879). The first theorem helps in a determining deflection and the second one in determining redundant reaction component.

First Theorem: In a linearly elastic structure, partial derivative of the strain energy with respect to a load is equal to the deflection of the point where the load is acting, the deflection being measured in the direction of the load.

The load may be a force or a moment. Mathematically, this theorem may be represented by,

$\frac{dU}{dP_i} = \Delta_i, \ \frac{dU}{dM_j} = \theta_j$

where U = total strain energy

P_i, M_j = loads

Δ_i, θ_j = deflections

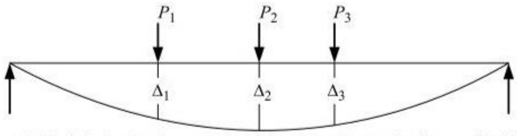


Figure 3.21 (a) A simply supported beam with gradually applied loads

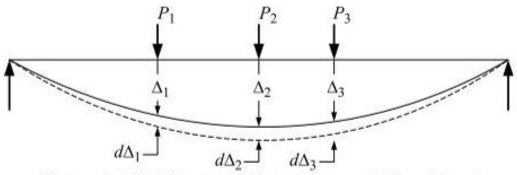


Figure 3.21(b) Beam subjected to a additional load

Consider a simply supported beam shown in figure 3.21(a) on which loads P_1, P_2 and P_3 are applied gradually. Let the deflection under the loads P_1, P_2, and P_3 be Δ_1, Δ_2, Δ_3 respectively.

$\therefore U = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3$ ---------(a)

Let the additional load dp_1 be added after the loads P_1, P_2 and P_3 and applied and let the additional deflections be $d\Delta_1$, $d\Delta_2$, $d\Delta_3$. Then, the additional strain energy stored dU is given by

$dU = \frac{1}{2}dP_1d\Delta_1 + P_1d\Delta_1 + P_2d\Delta_2 + P_3d\Delta_3$ ---------(b)

Therefore, total strain energy of the system is

$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}dP_1d\Delta_1 + P_1d\Delta_1 + P_2d\Delta_2 + P_3d\Delta_3$ ---------(c)

If $(P_1 + dP_1)$, P_2 and P_3 were to be applied simultaneously, strain energy stored is given by,

$= \frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1) + \frac{1}{2}P_2(\Delta_2 + d\Delta_2) + \frac{1}{2}P_3(\Delta_3 + d\Delta_3)$ ---------(d)

Since, the final strain energy in both the cases should be same,

Equation (c) = Equation (d)

i.e. $\frac{1}{2}P_1d\Delta_1 + \frac{1}{2}P_2d\Delta_2 + \frac{1}{2}P_3d\Delta_3 = \frac{1}{2}dP_1\Delta_1$ ----------(e)

But from equation (b), $\frac{1}{2}(P_1d\Delta_1 + P_2d\Delta_2 + P_3d\Delta_3) = \frac{1}{2}(dU - \frac{1}{2}dP_1d\Delta_1)$ -----(f)

From (e) and (f) we get

$\frac{1}{2}$ (dU - $\frac{1}{2}$dP$_1$dΔ_1) = $\frac{1}{2}$dP$_1\Delta_1$ -----(g)

Neglecting small quantity of higher order from equation (g), we get

$\frac{1}{2}$ dU = $\frac{1}{2}$dP$_1\Delta_1$

or $\frac{dU}{dP_i}$ = Δ_1

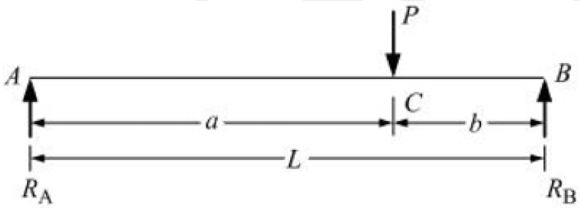
Similarly, if the moments are considered, it may be shown that

$\frac{dU}{dM}$ θ

Second Theorem: The partial derivative of the complimentary strain energy of the structure w.r.t. any particular force gives the displacement of the point of application of that force in the direction of its line of action.

Problems

1. A simply supported beam of span L, carries a concentrated load P at a distance a from left hand side support as shown in figure. Using Castigliano's theorem determine the deflection under the load. Assume uniform flexural rigidity.



Solution- Reaction at A, R$_A$ = $\frac{Pb}{L}$

And Reaction at B, R$_B$ = $\frac{Pa}{L}$

Portion	AC	CB
Origin	A	B
Limit	0-a	0-b
M	$\frac{Pb}{L}x$	$\frac{Pa}{L}x$
Flexural Rigidity	EI	EI

Therefore, Shear Energy of the beam

U = $\int_0^a \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx$ + $\int_0^b \left(\frac{Pa}{L}x\right)^2 \times \frac{1}{2EI}dx$

= $\left[\frac{P^2b^2}{L^2} \times \frac{1}{6EI}x^3\right]_0^a$+ $\left[\frac{P^2a^2}{L^2} \times \frac{1}{6EI}x^3\right]_0^b$

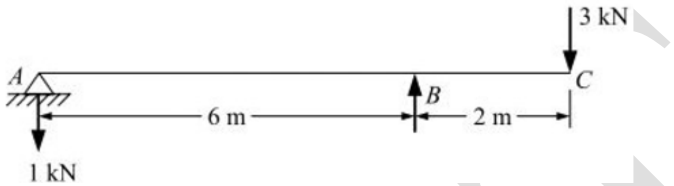
= $\frac{P^2b^2a^3}{6EIL^2} + \frac{P^2a^2b^3}{6EIL^2}$

$= \frac{P^2a^2b^2}{6EIL^2}$(a+b)

$= \frac{P^2a^2b^2}{6EIL}$, since, a+b = L

$\Delta_C = \frac{\delta U}{\delta P} = \frac{P^2a^2b^2}{3EIL}$

2. Determine the vertical deflection at the free end and rotation at A in the overhanging beam shown in figure. Assume constant EI. Use Castigliano's method.



Solution-

Deflection at C: Taking 3kN force as p,

$R_B \times 6 = P \times 8$

$R_B = \frac{4}{3}P \uparrow$

$R_A = \frac{P}{3} \downarrow$

Bending moment expressions are noted, in the tabular form.

Portion	AB	BC
Origin	A	C
Limit	0-6	0-2
M	$\frac{-P}{3}x$	$-Px$
Flexural Rigidity	EI	EI

$U = \int \frac{M^2}{2EI}dx$

$= \int_0^6 \frac{P^2x^2}{9} \times \frac{1}{2EI}dx + \int_0^2 \frac{P^2x^2}{2EI}dx$

$= \frac{P^2}{18EI}\left[\frac{x^3}{3}\right]_0^6 + \left[\frac{P^2x^3}{6EI}\right]_0^2$

$= \frac{4P^2}{EI} + \frac{4}{3} \times \frac{P^2}{EI}$

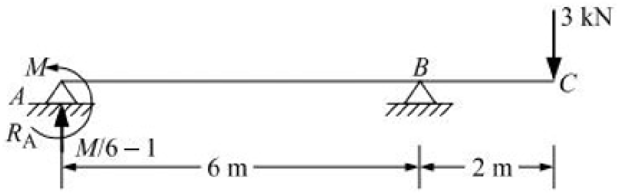
$= \frac{5.333P^2}{EI}$

$\Delta_C = \frac{dU}{dP} = \frac{10.667P}{EI}$

Substituting P = 3 kN, we get

$\Delta_C = \frac{32}{EI}$

Reaction at A: Apply a dummy moment M at A as shown in figure.



$\sum M_B = 0$, gives

$R_A = \frac{M-6}{6} = \frac{M}{6} - 1$

Portion	AB	BC
Origin	A	C
Limit	0-6	0-2
M	$(\frac{M}{6} - 1)x - M$	$-3x$

U = $\int_0^6 \left[\left(\frac{M}{6} - 1\right) x - M\right]_0^2 \frac{1}{2EI} dx$ + $\int_0^2 \frac{(-3x)^2}{2EI} dx$

$\frac{dU}{dM}$ = $\int_0^6 2\left[\left(\frac{M}{6} - 1\right) x - M\right]\left(\frac{x}{6} - 1\right)\frac{dx}{2EI}$ + 0

Since, M is a dummy moment, its value is substituted as zero and then integrated.

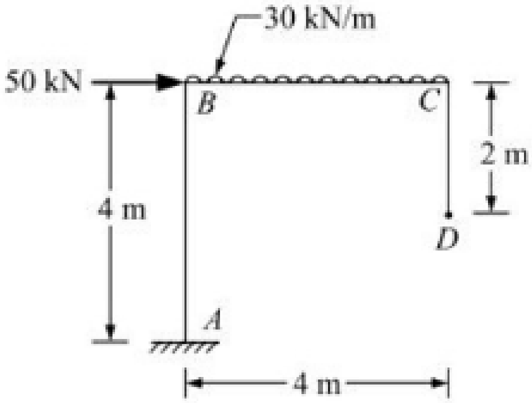
$\frac{dU}{dM}$ = $\theta_A = \frac{1}{EI}\int_0^6 (-x)\left(\frac{x}{6} - 1\right) dx$

$= \frac{1}{EI}\int_0^6 \left(-\frac{x^2}{6} + x\right) dx$

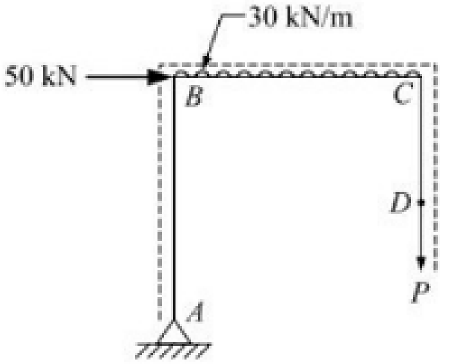
$= \frac{1}{EI}\left(-\frac{x^3}{18} + \frac{x^2}{2}\right)_0^6 dx$

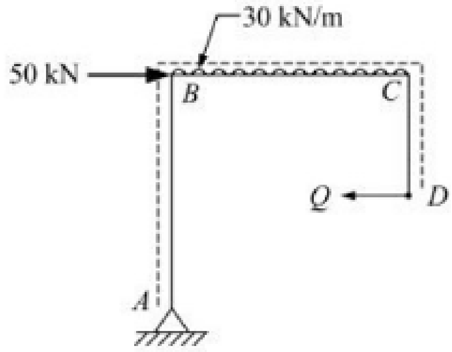
$= \frac{6}{EI}$

3. Determine the vertical and horizontal displacement at the free end D in the frame shown in figure. Take EI = 12 × 10^{13} Nmm^2. Use Castigliano’s theorem.



Solution-





Vertical deflection

Since, there is no load at D in vertical direction, a dummy load P is applied at D in vertical direction, in addition to given loads as shown in figure and the moment expressions are noted down.

Portion	AB	BC	CD
Origin	B	C	D
Limit	0-4	0-4	0-2
M	- (4P+240+50x)	-(Px + 15x^2)	0
Flexural Rigidity	EI	EI	EI

Strain energy U = $\int \frac{M^2}{2EI} dx$

$$= \int_0^4 \frac{(4P+240+50x)^2}{2EI} dx + \int_0^4 \frac{\left(Px+15x^2\right)^2}{2EI} dx + 0$$

$$\therefore \Delta = \frac{\delta U}{\delta P} = \int_0^4 2\,\frac{(4P+240+50x)}{2EI}(4)\,dx + \int_0^4 2\,\frac{\left(Px+15x^2\right)x}{2EI} dx$$

Since, P is dummy load, substitute P = 0

$\Delta_D = \int_0^4 \frac{4(240+50x)}{EI} dx + \int_0^4 \frac{\left(15x^3\right)}{EI} dx$

$= \frac{4}{EI}\left[240x + 25x^2\right]_0^4 + \frac{15}{EI}\left[\frac{x^4}{4}\right]_0^4$

$= \frac{6400}{EI}$

Now, EI = 12×10^{13} Nmm^2

= 12×10^4 kNm^2

$\therefore \Delta_{DV} = \frac{6400}{12\times10^4} = 0.533$m

= 53.33mm

Horizontal deflection

Since, there is no load in the horizontal direction at D, a dummy load is applied shown in figure and the moment expressions are noted down.

Portion	AB	BC	CD
Origin	B	C	D
Limit	0-4	0-4	0-2
M	- [Q(2-x) +240+50x)]	-(2Q + 15x^2)	Qx
Flexural Rigidity	EI	EI	EI

$U = \int_0^4 \frac{(Q(2-x)+240+50x)^2}{2EI}dx + \int_0^4 \frac{\left(2Q+15x^2\right)^2}{2EI}dx + \int_0^2 \frac{Q^2x^2}{2EI}dx$

$\Delta_{DH} = \frac{\delta U}{\delta Q} = \int_0^4 \frac{2[Q(2-x)+240+50x](2-x)}{2EI}dx + \int_0^4 \frac{2\left[2Q+15x^2\right]2}{2EI}dx + \int_0^2 \frac{2Qx^2}{2EI}dx$

Substituting Q = 0

$\Delta_{DH} = \int_0^4 \frac{(240+50x)(2-x)}{EI}dx + \int_0^4 \frac{\left[30x^2\right]}{EI}dx + 0$

$= \int_0^4 \frac{(480-140x-50x^2)}{EI}dx + \int_0^4 \frac{\left[30x^2\right]}{EI}dx$

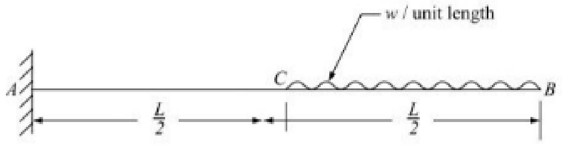
$= \frac{1}{EI}\left[480x - 70x^2 - \frac{50x^3}{3}\right]_0^4 + \frac{1}{EI}\left[10x^3\right]_0^4$

$= \frac{373.33}{EI} = \frac{373.33}{12 \times 10^4} = 0.0031$m

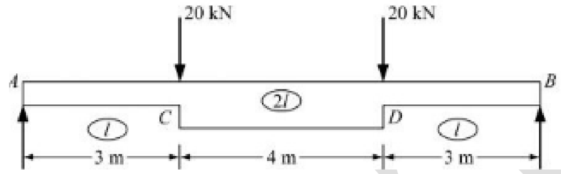
= 3.1mm

Review Questions

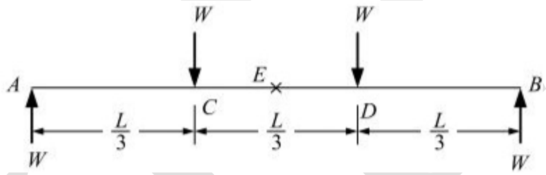
1. Determine the rotation and deflections at B and C in the cantilever beam shown in figure given below by moment area method.



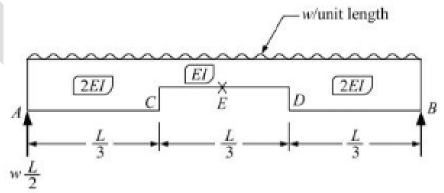
2. Determine the rotation at A and deflections under concentrated load and at mid-span in the beam shown in figure given below by moment area method.



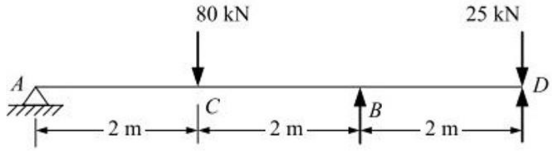
3. Determine the rotation at supports and deflection at mid-span and under the loads in the simply supported beam as shown in figure.



4. Determine the slope at A, deflection at C and mid-span E in the beam shown in figure.



5. Determine the slope and deflection at the end of the beam shown in figure. EI is constant throughout.



6. A cantilever beam is in the form oa quarter of a circle in the vertical plane and is subjected to a vertical load P at its free end as shown in figure. Find the vertical and horizontal displacements at the free end. Assume constant flexural rigidity.